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 SOME SUMS OF MULTINOMIAL COEFFICIENTS

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1. Recent interest in some lacunary sums of binomial coefficients (see for example [2], [3]) suggests that it may be of interest to consider some simple sums of multinomial coefficients.

Put

$$(i, j, k) = \frac{(i+j+k)!}{i!j!k!},$$

so that

$$(1.1) \quad (x+y+z)^n = \sum_{i+j+k=n} (i, j, k)x^i y^j z^k.$$

Let  $\epsilon_1, \epsilon_2, \epsilon_3 = \pm 1$  and define

$$S_{000} = S_{000}(n) = \sum_{i,j,k \text{ even}} (i, j, k),$$

$$S_{100} = S_{100}(n) = \sum_{\substack{i \text{ odd} \\ j,k \text{ even}}} (i, j, k), \text{ etc.,}$$

where in each case the summation is over all non-negative  $i, j, k$  such that  $i+j+k=n$ . Since

$$S_{100} = S_{010} = S_{001}, S_{011} = S_{101} = S_{110},$$

it is evident from (1.1) that

$$(1.2) \quad S_{000} + S_{100}(\epsilon_1 + \epsilon_2 + \epsilon_3) + S_{011}(\epsilon_2 \epsilon_3 + \epsilon_3 \epsilon_1 + \epsilon_1 \epsilon_2) + S_{111} \epsilon_1 \epsilon_2 \epsilon_3 = (\epsilon_1 + \epsilon_2 + \epsilon_3)^n.$$

Specializing the  $\epsilon_j$  we get

$$\begin{aligned} (1, 1, 1) : S_{000} + 3S_{100} + 3S_{110} + S_{111} &= 3^n \\ (-1, 1, 1) : S_{000} + S_{100} - S_{110} - S_{111} &= 1 \\ (1, -1, -1) : S_{000} - S_{100} - S_{110} + S_{111} &= (-1)^n \\ (-1, -1, -1) : S_{000} - 3S_{100} + 3S_{110} - S_{111} &= (-3)^3. \end{aligned}$$

Solving for the  $S_{ijk}$ , we get

$$(1.3) \quad \begin{cases} 8S_{000} = 3^n + 3 + 3(-1)^n + (-3)^n \\ 8S_{100} = 3^n + 1 - (-1)^n - (-3)^n \\ 8S_{110} = 3^n - 1 - (-1)^n + (-3)^n \\ 8S_{111} = 3^n - 3 + 3(-1)^n - (-3)^n. \end{cases}$$

Tabulating even and odd values of  $n$  separately, (1.3) reduces to

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$$(1.4) \quad \left\{ \begin{array}{l} S_{000}(2n) = \frac{1}{4}(3^{2n} + 3) \\ S_{100}(2n) = 0 \\ S_{110}(2n) = \frac{1}{4}(3^{2n} - 1) \\ S_{111}(2n) = 0 \end{array} \right.$$

$$(1.5) \quad \left\{ \begin{array}{l} S_{000}(2n+1) = 0 \\ S_{100}(2n+1) = \frac{1}{4}(3^{2n+1} + 1) \\ S_{110}(2n+1) = 0 \\ S_{111}(2n+1) = \frac{1}{4}(3^{2n+1} - 3). \end{array} \right.$$

It follows from (1.4) and (1.5) that

$$(1.6) \quad S_{000}(2n) = S_{110}(2n) + 1, \quad S_{100}(2n+1) = S_{111}(2n+1) + 1.$$

We also have the generating functions

$$(1.7) \quad \left\{ \begin{array}{l} \sum_{n=0}^{\infty} S_{000}(n)x^n = \frac{1}{4} \left( \frac{1}{1-9x^2} + \frac{3}{1-x^2} \right) = \frac{1-7x^2}{(1-x^2)(1-9x^2)} \\ \sum_{n=0}^{\infty} S_{100}(n)x^n = \frac{1}{4} \left( \frac{3x}{1-9x^2} + \frac{x}{1-x^2} \right) = \frac{1-3x^2}{(1-x^2)(1-9x^2)} \\ \sum_{n=0}^{\infty} S_{110}(n)x^n = \frac{1}{4} \left( \frac{1}{1-9x^2} - \frac{1}{1-x^2} \right) = \frac{2x^2}{(1-x^2)(1-9x^2)} \\ \sum_{n=0}^{\infty} S_{111}(n)x^n = \frac{1}{4} \left( \frac{3x}{1-9x^2} - \frac{3x}{1-x^2} \right) = \frac{6x^3}{(1-x^2)(1-9x^2)} \end{array} \right.$$

2. Let  $m > 1$  and define

$$(2.1) \quad S_{ijk} = S_{ijk}^{(m)}(n) = \sum_{r+s+t=n} (r, s, t),$$

where the summation is restricted to non-negative  $r, s, t$  such that

$$r \equiv i, \quad s \equiv j, \quad t \equiv k \pmod{m}.$$

We may also assume that

$$0 \leq i < m, \quad 0 \leq j < m, \quad 0 \leq k < m.$$

Clearly  $S_{ijk}$  is symmetric in the three indices  $i, j, k$ . Also it is evident from the definition that

$$(2.2) \quad S_{ijk}^{(m)}(n) = 0 \quad (n \not\equiv i+j+k \pmod{m}).$$

Hence in what follows it will suffice to assume that  $n \equiv i+j+k \pmod{m}$ .

Let  $\zeta$  denote a fixed primitive  $m^{\text{th}}$  root of 1. Then it is clear from (1.1) and (2.1), that for arbitrary integers,  $a, b, c$ ,

$$(2.3) \quad (\zeta^a + \zeta^b + \zeta^c) = \sum_{r+s+t=n} \zeta^{ra+sb+tc} (r, s, t) = \sum_{i,j,k=0}^{m-1} \zeta^{ia+jb+kc} S_{ijk}^{(m)}(n).$$

Since

$$\sum_{a=0}^{m-1} \zeta^{ra} = \begin{cases} m & (r=0) \\ 0 & (0 < r < m), \end{cases}$$

it follows from (2.3) that

$$(2.4) \quad m^3 S_{ijk}^{(m)}(n) = \sum_{a,b,c=0}^{m-1} (\zeta^a + \zeta^b + \zeta^c)^n \zeta^{-ai-bj-ck}.$$

While this theoretically evaluates  $S_{ijk}^{(m)}(n)$ , it is not really satisfactory. For  $m=3$  more explicit results are obtainable without a great deal of computation.

By (2.4) we have

$$27S_{000}^{(3)}(n) = \sum_{a,b,c=0}^2 (\omega^a + \omega^b + \omega^c)^3,$$

where  $\omega^2 + \omega + 1 = 0$ . This reduces to

$$27S_{000}^{(3)}(n) = 3^n + 3(\omega + 2)^n + 3(\omega^2 + 2)^n + 3(2\omega + 1)^n + 3(2\omega^2 + 1)^n \\ + 3(2\omega^2 + \omega)^n + 3(2\omega + \omega^2)^n + 6(\omega^2 + \omega + 1)^n + (3\omega)^n + (3\omega^2)^n.$$

By (2.2),

$$S_{000}^{(3)}(n) = 0 \quad (n \not\equiv 0 \pmod{3}).$$

For  $n$  a multiple of 3 we get, replacing  $n$  by  $3n$ ,

$$27S_{000}^{(3)}(3n) = 3^{3n+1} + 9(2\omega + 1)^{3n} + 9(2\omega^2 + 1)^{3n} \quad (n > 0).$$

This reduces to

$$(2.5) \quad \begin{cases} S_{000}^{(3)}(6n) = 3^{6n-2} + 2(-1)^n 3^{3n-1} & (n > 0) \\ S_{000}^{(3)}(6n+3) = 3^{6n+1} & (n \geq 0). \end{cases}$$

Check.

$$S_{000}^{(3)}(6) = \frac{3 \cdot 6!}{6! 0! 0!} + \frac{3 \cdot 6!}{3! 3! 0!} = 3 + 3 \cdot 60 = 63 = 3^4 - 2 \cdot 3^2, \quad S_{000}^{(3)}(3) = \frac{3 \cdot 3!}{3! 0! 0!} = 3,$$

$$S_{000}^{(3)}(9) = \frac{9!}{3! 3! 3!} + \frac{6 \cdot 9!}{6! 3! 0!} + \frac{3 \cdot 9!}{9! 0! 0!} = 5 \cdot 6 \cdot 7 \cdot 8 + 7 \cdot 8 \cdot 9 + 3 = 3 \cdot 729 = 3^7.$$

Similarly we have

$$27S_{111}^{(3)}(n) = \sum_{a,b,c=0}^2 (\omega^a + \omega^b + \omega^c)^n \omega^{-a-b-c} = 3^n + 3(\omega + 2)^n \omega^{-1} + 3(\omega^2 + 2)^n \omega^{-2} \\ + 3(2\omega + 1)^n \omega^{-2} + 3(2\omega^2 + 1)^n \omega^{-1} + 3(2\omega^2 + \omega)^n \omega^{-2} + 3(2\omega + \omega^2)^n \omega^{-1} \\ + 6(\omega^2 + \omega + 1)^n + (3\omega)^n + (3\omega^2)^n.$$

As in the previous case,

$$S_{111}^{(3)}(n) = 0 \quad (n \not\equiv 0 \pmod{3}),$$

while

$$27S_{111}^{(3)}(3n) = 3 \cdot 3^{3n} + 3(2\omega^2 + 1)^{3n} \omega^{-1} + 3(2\omega + 1)^{3n} \omega^{-2} + 3(2\omega + 1)^{3n} \omega^{-2} + 3(2\omega^3 + 1)^{3n} \omega^{-1} \\ + 3(2\omega + 1)^{3n} \omega^{-2} + 3(2\omega^2 + 1)^{3n} \omega^{-1} + 6(\omega^2 + \omega + 1)^{3n} \\ = 3^{3n+1} + 9(\sqrt{-3})^{3n} \omega^{-2} + 9(-\sqrt{-3})^{3n} \omega^{-1} + 6(\omega^2 + \omega + 1)^{3n}.$$

It follows that

$$(2.6) \quad \begin{cases} S_{111}^{(3)}(6n) = 3^{6n-2} - (-1)^n 3^{3n-1} & (n > 0) \\ S_{111}^{(3)}(6n+3) = 3^{6n+1} + (-1)^n 3^{3n+1} & (n \geq 0). \end{cases}$$

Check.

$$S_{111}^{(3)}(6) = \frac{3 \cdot 6!}{4! 1! 1!} = 3 \cdot 6 \cdot 5 = 90 = 3^4 + 3^2, \quad S_{111}^{(3)}(3) = \frac{3!}{1! 1! 1!} = 6 = 3 + 3,$$

$$S_{111}^{(3)}(9) = \frac{3 \cdot 9!}{7! 1! 1!} + \frac{3 \cdot 9!}{4! 4! 1!} = 3 \cdot 9 + 3 \cdot 9 \cdot 70 = 3^4 \cdot 26 = 3^7 - 3^4.$$

We find also that

$$(2.7) \quad \begin{cases} S_{222}^{(3)}(6n) = 3^{6n-2} - (-1)^n 3^{3n-1} & (n > 0) \\ S_{222}^{(3)}(6n+3) = 3^{6n+1} - (-1)^n 3^{3n+1} & (n \geq 0). \end{cases}$$

Check.

$$S_{222}^{(3)}(6) = \frac{6!}{2! 2! 2!} = 90 = 3^4 + 3^2, \quad S_{222}^{(3)}(9) = \frac{3 \cdot 9!}{5! 2! 2!} = 3^4 \cdot 28 = 3^7 + 3^4.$$

Note that it follows from (2.6) and (2.7) that

$$(2.8) \quad S_{111}^{(3)}(6n) = S_{222}^{(3)}(6n)$$

and from (2.5), (2.6), (2.7),

$$(2.9) \quad S_{111}^{(3)}(6n+3) + S_{222}^{(3)}(6n+3) = 2S_{000}^{(3)}(6n+3).$$

3. Since

$$(r, s, t) = (r-1, s, t) + (r, s-1, t) + (r, s, t-1),$$

it follows from (2.1) that

$$(3.1) \quad S_{i,j,k}^{(m)}(n) = S_{i-1,j,k}^{(m)}(n-1) + S_{i,j-1,k}^{(m)}(n-1) + S_{i,j,k-1}^{(m)}(n-1),$$

where

$$S_{i,j,k}^{(m)}(n) = S_{i',j',k'}^{(m)}(n)$$

when

$$i \equiv i', \quad j \equiv j', \quad k \equiv k' \pmod{m}.$$

In particular

$$(3.2) \quad S_{i,i,i}^{(m)}(n) = 3S_{i,i,i-1}^{(m)}(n-1).$$

For example, when  $m = 2$ , we have

$$\begin{cases} S_{000}^{(2)}(2n) = 3S_{100}^{(2)}(2n-1) \\ S_{111}^{(2)}(2n+1) = 3S_{110}^{(2)}(2n) \\ S_{100}^{(2)}(2n+1) = S_{000}^{(2)}(2n) + 2S_{110}^{(2)}(2n) \\ S_{110}^{(2)}(2n) = S_{111}^{(2)}(2n-1) + 2S_{100}^{(2)}(2n-1) \end{cases}$$

in agreement with previous results.

The case  $m = 3$  is more interesting as well as more involved. We have, to begin with,

$$\begin{cases} S_{000}^{(3)}(3n) = 3S_{200}^{(3)}(3n-1) \\ S_{111}^{(3)}(3n) = 3S_{110}^{(3)}(3n-1) \\ S_{222}^{(3)}(3n) = 3S_{221}^{(3)}(3n-1). \end{cases}$$

It therefore follows from (2.5), (2.6), and (2.7) that

$$(3.3) \quad \begin{cases} S_{200}^{(3)}(6n+5) = 3^{6n+3} - 2(-1)^n 3^{3n+1} \\ S_{200}^{(3)}(6n+2) = 3^{6n} \end{cases} \quad (n \geq 0)$$

$$(3.4) \quad \begin{cases} S_{110}^{(3)}(6n+5) = 3^{6n+3} + (-1)^n 3^{3n+1} \\ S_{110}^{(3)}(6n+2) = 3^{6n} + (-1)^n 3^{3n} \end{cases} \quad (n \geq 0)$$

$$(3.5) \quad \begin{cases} S_{221}^{(3)}(6n+5) = 3^{6n+3} + (-1)^n 3^{3n+1} \\ S_{221}^{(3)}(6n+2) = 3^{6n} - (-1)^n 3^{3n} \end{cases} \quad (n \geq 0).$$

Check.

$$S_{200}^{(3)}(5) = \frac{5!}{5!0!0!} + \frac{2 \cdot 5!}{2!3!0!} = 1 + 20 = 21 = 3^3 - 2 \cdot 3,$$

$$\begin{aligned} S_{200}^{(3)}(8) &= \frac{8!}{8!0!0!} + \frac{2 \cdot 8!}{5!3!0!} + \frac{2 \cdot 8!}{2!6!0!} + \frac{8!}{2!3!3!} \\ &= 1 + 112 + 56 + 560 = 729 = 3^6; \end{aligned}$$

$$S_{110}^{(3)}(5) = \frac{5!}{1!1!3!} + \frac{2 \cdot 5!}{4!1!0!} = 20 + 10 = 30 = 3^3 + 3,$$

$$S_{110}^{(3)}(8) = \frac{2 \cdot 8!}{7!1!0!} + \frac{8!}{4!4!0!} + \frac{2 \cdot 8!}{1!4!3!} + \frac{8!}{1!1!6!} = 16 + 70 + 560 + 56 = 702 = 3^6 - 3^3;$$

$$S_{221}^{(3)}(5) = \frac{5!}{2!2!1!} = 30 = 3^3 + 3,$$

$$S_{221}^{(3)}(8) = \frac{2 \cdot 8!}{5!2!1!} + \frac{8!}{2!2!4!} = 27 \cdot 28 = 3^6 + 3^3.$$

In the next place, it follows from

$$S_{210}^{(3)}(3n) = S_{110}^{(3)}(3n-1) + S_{200}^{(3)}(3n-1) + S_{221}^{(3)}(3n-1)$$

that

$$\begin{aligned} S_{210}^{(3)}(6n) &= S_{110}^{(3)}(6n-1) + S_{200}^{(3)}(6n-1) + S_{221}^{(3)}(6n-1) \\ &= (3^{6n-3} - (-1)^n 3^{3n-2}) + (3^{6n-3} + 2(-1)^n 3^{3n-2}) + (3^{6n-3} - (-1)^n 3^{3n-2}) = 3^{6n-2} \quad (n > 0). \end{aligned}$$

$$\begin{aligned} S_{210}^{(3)}(6n+3) &= S_{110}^{(3)}(6n+2) + S_{200}^{(3)}(6n+2) + S_{221}^{(3)}(6n+2) \\ &= (3^{6n} + (-1)^n 3^{3n}) + 3^{6n} + (3^{6n} - (-1)^n 3^{3n}) = 3^{6n+1} \quad (n \geq 0) \end{aligned}$$

that is,

$$(3.6) \quad \begin{cases} S_{210}^{(3)}(6n) = 3^{6n-2} & (n > 0) \\ S_{210}^{(3)}(6n+3) = 3^{6n+1} & (n \geq 0). \end{cases}$$

Check.

$$S_{210}^{(3)}(6) = \frac{6!}{2!1!3!} + \frac{6!}{5!1!0!} + \frac{6!}{2!4!0!} = 60 + 6 + 15 = 3^4,$$

$$S_{210}^{(3)}(3) = \frac{3!}{2!1!0!} = 3,$$

$$\begin{aligned} S_{210}^{(3)}(9) &= \frac{9!}{2!1!6!} + \frac{9!}{5!1!3!} + \frac{9!}{2!4!3!} + \frac{9!}{8!1!0!} + \frac{9!}{5!4!0!} + \frac{9!}{2!7!0!} \\ &= 9 \cdot 4 \cdot 7 + 9 \cdot 8 \cdot 7 + 9 \cdot 7 \cdot 20 + 9 + 9 \cdot 7 \cdot 2 + 9 \cdot 4 = 9 \cdot 243 = 3^7. \end{aligned}$$

Next it follows in like manner from

$$\begin{cases} S_{211}^{(3)}(3n+1) = S_{111}^{(3)}(3n) + 2S_{210}^{(3)}(3n) \\ S_{220}^{(3)}(3n+1) = S_{222}^{(3)}(3n) + 2S_{210}^{(3)}(3n) \\ S_{100}^{(3)}(3n+1) = S_{000}^{(3)}(3n) + 2S_{210}^{(3)}(3n) \end{cases}$$

that

$$(3.7) \quad \begin{cases} S_{211}^{(3)}(6n+1) = 3^{6n-1} - (-1)^n 3^{3n-1} \\ S_{211}^{(3)}(6n+4) = 3^{6n+2} + (-1)^n 3^{3n+1} \end{cases} \quad (n \geq 0)$$

$$(3.8) \quad \begin{cases} S_{220}^{(3)}(6n+1) = 3^{6n-1} - (-1)^n 3^{3n-1} \\ S_{220}^{(3)}(6n+4) = 3^{6n+2} - (-1)^n 3^{3n+1} \end{cases} \quad (n \geq 0)$$

$$(3.9) \quad \begin{cases} S_{100}^{(3)}(6n+1) = 3^{6n-1} + 2(-1)^n 3^{3n-1} \\ S_{100}^{(3)}(6n+4) = 3^{6n+2} \end{cases} \quad (n \geq 0).$$

Check.

$$S_{211}^{(3)}(7) = \frac{7!}{5!1!1!1!} + \frac{2!7!}{2!4!1!1!} = 42 + 210 = 252 = 3^5 + 3^2, \quad S_{211}^{(3)}(4) = \frac{4!}{2!1!1!1!} = 12 = 3^2 + 3,$$

$$S_{211}^{(3)}(10) = \frac{10!}{8!1!1!1!} + \frac{2 \cdot 10!}{5!4!1!1!} + \frac{10!}{2!4!4!} + \frac{2 \cdot 10!}{2!7!1!1!} \\ = 9 \cdot 10 + 9 \cdot 280 + 9 \cdot 350 + 9 \cdot 80 = 3^4 \cdot 80 = 3^8 - 3^4;$$

$$S_{220}^{(3)}(7) = \frac{7!}{2!2!3!} + \frac{2 \cdot 7!}{5!2!0!} = 3^2 \cdot 28 = 3^5 + 3^2,$$

$$S_{220}^{(3)}(4) = \frac{4!}{2!2!0!} = 6 = 3^2 - 3,$$

$$S_{220}^{(3)}(10) = \frac{10!}{2!2!6!} + \frac{2 \cdot 10!}{5!2!3!} + \frac{10!}{5!5!0!} + \frac{2 \cdot 10!}{8!2!0!} = 9^2 \cdot 82 = 3^8 + 3^4.$$

$$S_{100}^{(3)}(7) = \frac{7!}{7!0!0!} + \frac{2 \cdot 7!}{4!3!0!} + \frac{7!}{1!3!3!} + \frac{2 \cdot 7!}{1!6!0!} = 1 + 70 + 140 + 14 = 3^2 \cdot 25 = 3^5 - 2 \cdot 3^2,$$

$$S_{100}^{(3)}(4) = \frac{4!}{4!0!0!} + \frac{2 \cdot 4!}{1!3!0!} = 1 + 8 = 3^2,$$

$$S_{100}^{(3)}(10) = \frac{10!}{10!0!0!} + \frac{2 \cdot 10!}{7!3!0!} + \frac{10!}{4!3!3!} + \frac{2 \cdot 10!}{4!6!0!} + \frac{2 \cdot 10!}{1!9!0!} + \frac{2 \cdot 10!}{1!6!3!} \\ = 1 + 240 + 4200 + 420 + 20 + 1680 = 3^8.$$

This completes the evaluation of the ten functions  $S_{ijk}^{(3)}(n)$ .

4. The five functions  $S_{ijk\ell}^{(2)}(n)$  can be evaluated without much computation. To begin with, we have

$$2^4 S_{0000}^{(2)}(2n) = (1+1+1+1)^{2n} + 4(1+1+1-1)^{2n} + 6(1+1-1-1)^{2n} \\ + 4(1-1-1-1)^{2n} + (-1-1-1-1)^{2n},$$

which reduces to

$$(4.1) \quad S_{000}^{(2)}(2n) = 2^{4n-3} + 2^{2n-1} \quad (n > 0).$$

Since

$$S_{0000}^{(2)}(2n) = 4S_{1000}^{(2)}(2n-1),$$

we get

$$(4.2) \quad S_{1000}^{(2)}(2n+1) = 2^{4n-1} + 2^{2n-1} \quad (n \geq 0).$$

Next, since

$$S_{1000}^{(2)}(2n+1) = S_{0000}^{(2)}(2n) + 3S_{1100}^{(2)}(2n),$$

it follows that

$$(4.3) \quad S_{1100}^{(2)}(2n) = 2^{4n-3} \quad (n > 0).$$

Similarly, from

$$S_{1100}^{(2)}(2n) = 2S_{1000}^{(2)}(2n-1) + 2S_{1110}^{(2)}(2n-1)$$

we get

$$(4.4) \quad S_{1110}^{(2)}(2n+1) = 2^{4n-1} - 2^{2n-1} \quad (n \geq 0).$$

Finally, it follows from

$$S_{1110}^{(2)}(2n+1) = S_{1111}^{(2)}(2n) + 3S_{1100}^{(2)}(2n)$$

that

$$(4.5) \quad S_{1111}^{(2)}(2n) = 2^{4n-3} - 2^{2n-1} \quad (n \geq 1).$$

For example

$$S_{1111}^{(2)}(6) = \frac{4 \cdot 6!}{3! 1! 1! 1!} = 480 = 2^9 - 2^5.$$

Note that it follows from the above results that

$$(4.6) \quad S_{0000}^{(2)}(2n) + S_{1111}^{(2)}(2n) = 2S_{1100}^{(2)}(2n)$$

and

$$(4.7) \quad S_{1000}^{(2)}(2n+1) + S_{1110}^{(2)}(2n+1) = 8S_{1100}^{(2)}(2n).$$

5. The results of § 4 suggest that it would be of interest to evaluate

$$(5.1) \quad f_{j,k}(n) = S_{\underbrace{1 \dots 1}_j \underbrace{0 \dots 0}_k}(n),$$

where  $j, k$  are arbitrary non-negative integers and the right-hand side of (5.1) has the obvious meaning. Clearly

$$(5.2) \quad f_{j,k}(n) = 0 \quad (n \not\equiv j \pmod{2}).$$

To begin with, we have

$$\begin{aligned} 2^k f_{0,k}(n) &= (1+1+\dots+1)^n + \binom{k}{1} (1+\dots+1-1)^n \\ &\quad + \binom{k}{2} (1+\dots+1-1-1)^n + \dots + \binom{k}{k} (-1-1-\dots-1)^n \\ &= k^n + \binom{k}{1} (k-2)^n + \binom{k}{2} (k-4)^n + \dots + \binom{k}{k} (-k)^n. \end{aligned}$$

Thus

$$(5.3) \quad f_{0,k}(n) = 2^{-k} \sum_{j=0}^k \binom{k}{j} (k-2j)^n.$$

Since

$$(5.4) \quad f_{0,k}(n) = k f_{1,k-1}(n-1),$$

it follows at once that  $S_{1,k-1}$  can be evaluated explicitly by means of (5.3). Next, since

$$f_{1,k-1}(n-1) = f_{0,k}(n-2) + (k-1)f_{2,k-2}(n-2),$$

we get

$$(5.5) \quad k(k-1)f_{2,k-2}(n-2) = f_{0,k}(n) = k f_{0,k}(n-2).$$

Similarly it follows from

$$f_{2,k-2}(n-2) = 2f_{1,k-1}(n-3) + (k-2)f_{3,k-3}(n-3)$$

that

$$(5.6) \quad k(k-1)(k-2)f_{3,k-3}(n-3) = f_{0,k}(n) - (3k-2)f_{0,k}(n-2).$$

We also find that

$$(5.7) \quad k(k-1)(k-2)(k-3)f_{4,k-4}(n-4) = f_{0,k}(n) - 2(3k-4)f_{0,k}(n-2) + 3k(k-2)f_{0,k}(n-4),$$

$$(5.8) \quad k(k-1)(k-2)(k-4)f_{5,k-5}(n-5) = f_{0,k}(n) - 2(5k-10)f_{0,k}(n-2) \\ + (15k^2 - 50k + 24)f_{0,k}(n-4).$$

These results suggest the following general formula:

$$(5.9) \quad \frac{k!}{(k-j)!} f_{j,k-j}(n-j) = \sum_{2s \leq j} (-1)^s P_{j,s}(k) f_{0,k}(n-2s) \quad (0 \leq j \leq k),$$

where  $P_{j,s}(k)$  denotes a polynomial in  $k$  of degree  $s$ . Since

$$f_{j,k-j}(n-j) = j f_{j-1,k-j+1}(n-j-1) + (k-j) f_{j+1,k-j-1}(n-j-1),$$

it follows that

$$\begin{aligned} \frac{(k-j)!}{k!} \sum_{2s \leq j} (-1)^s P_{j,s}(k) f_{0,k}(n-2s) &= j \frac{(k-j+1)!}{k!} \sum_{2s < j} (-1)^s P_{j-1,s}(k) f_{0,k}(n-2s-2) \\ &+ (k-j) \frac{(k-j-1)!}{k!} \sum_{2s \leq j+1} (-1)^s P_{j+1,s}(k) f_{0,k}(n-2s). \end{aligned}$$

Hence we take

$$(5.10) \quad P_{j+1,s}(k) = P_{j,s}(k) + j(k-j+1)P_{j-1,s-1}(k).$$

		$P_{j,s}(k)$			
		$0$	$1$	$2$	$3$
$j \backslash s$	$0$	$1$			
$1$	$1$	$1$			
$2$	$1$	$1$	$k$		
$3$	$1$	$1$	$3k-2$		
$4$	$1$	$1$	$6k-8$	$3k(k-2)$	
$5$	$1$	$1$	$10k-20$	$15k^2-50k+24$	
$6$	$1$	$1$	$15k-40$	$45k^2-210k+184$	$15k(k-2)(k-4)$
$7$	$1$	$1$	$21k-70$	$105k^2-630k+784$	$105k^3-840k^2+1764k-720$

It is evident that

$$(5.11) \quad P_{j,0}(k) = 1 \quad (j \geq 0).$$

Also it follows easily from (5.10) that

$$(5.12) \quad P_{2j,j}(k) = 1 \cdot 3 \cdot 5 \cdot \dots \cdot (2j-1)k(k-2)(k-4) \dots (k-2j+2) \quad (j \geq 0).$$

Since

$$P_{j+1,1}(k) = P_{j,1}(k) + j(k-j+1) \quad (j \geq 1),$$

we get

$$P_{j+1,1}(k) = \sum_{t=1}^j t(k-t+1).$$

This gives

$$(5.13) \quad P_{j,1}(k) = \frac{1}{2} j(j-1)k - \frac{1}{3} j(j-1)(j-2) \quad (j \geq 0).$$

Similarly, since

$$\begin{aligned} P_{j+1,2}(k) &= P_{j,2}(k) + j(k-j+1)P_{j-1,1}(k) = P_{j,2}(k) + \frac{1}{2}j(j-1)(j-2)k^2 - \left\{ \frac{5}{6}j(j-1)(j-2)(j-3) \right. \\ &\quad \left. + j(j-1)(j-2) \right\} k + \frac{1}{3}j(j-1)(j-2)(j-3)(j-4) + j(j-1)(j-2)(j-3), \end{aligned}$$



we find that

$$(5.14) \quad P_{j,2}(k) = 3 \binom{j}{4} k^2 - \left[ 20 \binom{j}{5} + 6 \binom{j}{4} \right] k + \left[ 40 \binom{j}{6} + 24 \binom{j}{5} \right] \\ = \frac{1}{8} j(j-1)(j-2)(j-3)k^2 - \frac{1}{12} j(j-1)(j-2)(j-3)(2j-5)k^2 \\ + \frac{1}{90} j(j-1)(j-2)(j-3)(j-4)(5j-7).$$

For example

$$P_{6,2}(k) = 3 \cdot 15k^2 - (20 \cdot 6 + 6 \cdot 15)k + (40 + 24 \cdot 6) = 45k^2 - 210k + 184.$$

We also find that

$$(5.15) \quad P_{j,3}(k) = 15 \binom{j}{6} k^3 - \left[ 210 \binom{j}{7} + 90 \binom{j}{6} \right] k^2 \\ + \left[ 1120 \binom{j}{8} + 924 \binom{j}{7} + 120 \binom{j}{6} \right] k - \left[ 2240 \binom{j}{9} + 2688 \binom{j}{8} + 720 \binom{j}{7} \right]$$

For example

$$P_{7,3}(k) = 15 \cdot 7k^3 - (210 + 90 \cdot 7)k^2 + (924 + 120 \cdot 7)k - 720 = 105k^3 - 840k^2 + 1764k - 720.$$

We have noted above that  $P_{j,s}(k)$  is a polynomial in  $k$  of degree  $s$ . In addition we can assert that  $P_{j,s}(k)$  is a polynomial in  $j$  of degree  $3s$ . More precisely, if we put

$$(5.16) \quad P_{j,s}(k) = \sum_{t=0}^s (-1)^s c_{s,t}(j) k^{s-t},$$

then  $c_{s,t}(j)$  is a polynomial in  $j$  of degree  $2s+t$ . If we substitute from (4.7) in (4.1) we get

$$\sum_{t=0}^s (-1)^t [c_{s,t}(j+1) - c_{s,t}(j)] k^{s-t} = j(k-j+1) \sum_{t=0}^{s-1} (-1)^t c_{s-1,t}(j-1) k^{s-t-1}.$$

This gives

$$(5.17) \quad c_{s,t}(j+1) - c_{s,t}(j) = jc_{s-1,t}(j-1) + j(j-1)c_{s-1,t-1}(j-1).$$

The table of values of  $P_{j,s}(k)$  suggests that

$$(5.18) \quad \left\{ \begin{array}{l} \sum_{s=0}^j (-1)^{j-s} P_{2j,s}(k) = 1 \cdot 3 \cdot 5 \cdots (2j-1)(k-1)(k-3) \cdots (k-2j+1) \\ \sum_{s=0}^j (-1)^{j-s} P_{2j+1,s}(k) = 1 \cdot 3 \cdot 5 \cdots (2j-1)(2j+1) \cdot (k-1)(k-3) \cdots (k-2j+1) \end{array} \right.$$

These formulas are easily proved by means of (5.10).

The explicit results (5.13), (5.14), (5.15) also suggest that

$$(5.19) \quad P_{j,s}(k) = 0 \quad (j = 0, 1, \dots, 2s-1).$$

This can be proved inductively using (5.10) in the form

$$(5.20) \quad P_{j,s}(k) = P_{j+1,s}(k) - j(k-j+1)P_{j-1,s-1}(k).$$

Thus, to begin with,

$$P_{2s-1,s}(k) = P_{2s,s}(k) - (2s-1)(k-2s+2)P_{2s-2,s-1}(k) = 0,$$

by (5.12). In the next place, taking  $j = 2s-2$ , we get

$$P_{2s-2,s}(k) = P_{2s-1,s}(k) - (2s-2)(k-2s+3)P_{2s-s,s-1}(k) = 0.$$

Continuing in this way, we get

$$P_{j,s}(k) = 0 \quad (1 \leq j \leq 2s-1).$$

Finally, taking  $j = 1$  and replacing  $s$  by  $s + 1$  in (5.10), we have

$$P_{2,s+1}(k) = P_{1,s+1}(k) + kP_{0,s}(k),$$

which gives  $P_{0,s}(k) = 0$ .

6. We now put

$$(6.1) \quad P_j(k, x) = \sum_{2s \leq j} (-1)^s P_{j,s}(k) x^{j-2s}, \quad P_0 = 1, \quad P_1 = x,$$

and

$$(6.2) \quad F(z) = F(k, x, z) = \sum_{j=0}^{\infty} P_j(k, x) \frac{z^j}{j!}.$$

By (5.10),

$$P_{j+1}(k, x) = \sum_{2s \leq j+1} (-1)^s P_{j,s}(k) x^{j-2s+1} + j(k-j+1) \sum_{2s \leq j+1} (-1)^s P_{j-1,s-1}(k) x^{j-2s+1},$$

so that

$$(6.3) \quad P_{j+1}(k, x) = xP_j(k, x) - j(k-j+1)P_{j-1}(k, x).$$

It follows from (6.2) and (6.3) that

$$\begin{aligned} F'(z) &= \sum_{j=0}^{\infty} P_{j+1}(k, x) \frac{z^j}{j!} = x \sum_{j=0}^{\infty} P_j(k, x) \frac{z^j}{j!} - z \sum_{j=0}^{\infty} (k-j)P_j(k, x) \frac{z^j}{j!} \\ &= xF(z) - kzF(z) + z^2F'(z). \end{aligned}$$

Hence

$$\frac{F'(z)}{F(z)} = \frac{x - kz}{1 - z^2},$$

which gives

$$(6.4) \quad F(k, x, z) = (1+z)^{\frac{1}{2}(x+k)} (1-z)^{-\frac{1}{2}(x-k)}.$$

It follows from the recurrence (6.3) that the polynomials

$$(6.5) \quad P_n(k, x) \quad (n = 0, 1, 2, \dots)$$

constitute a set of orthogonal polynomials in  $x$ . The polynomials have been discussed in [1, § 9]; in that paper the relationship with Euler numbers of higher order is stressed. If we put

$$(1+z)^{\frac{1}{2}x} (1-z)^{-\frac{1}{2}x} = \sum_{n=0}^{\infty} A_n(x) \frac{z^n}{n!},$$

so that  $A_n(x) = P_n(0, x)$ , then, by (6.4),

$$(6.6) \quad P_n(k, x) = \sum_{2s \leq n} \frac{(-1)^s}{(n-2s)!} \binom{\frac{1}{2}k}{s} A_{n-2s}(x).$$

Returning to (5.9) and using (5.3), we have

$$\begin{aligned} j! \binom{k}{j} f_{j,k-j}(n) &= \sum_{2s \leq j} (-1)^s P_{j,s}(k) \cdot 2^{-k} \sum_{t=0}^k \binom{k}{t} (k-2t)^{n-2s} \\ &= 2^{-k} \sum_{t=0}^k \binom{k}{t} (k-2t)^n \sum_{2s \leq j} (-1)^s P_{j,s}(k) (k-2t)^{-2s}, \end{aligned}$$

so that

$$(6.7) \quad j! \binom{k}{j} f_{j,k-j}(n) = 2^{-k} \sum_{t=0}^k \binom{k}{t} (k-2t)^n P_j(k, k-2t) \quad (0 \leq j \leq k).$$

We shall show that (6.7) holds for all  $j$ , that is, the right-hand side vanishes identically for  $j > k$ . To prove this, consider the sum

$$\begin{aligned} 2^{-k} \sum_{j=0}^{\infty} \frac{z^j}{j!} \sum_{n=0}^{\infty} \frac{y^n}{n!} \sum_{t=0}^k \binom{k}{t} (k-2t)^n P_j(k, k-2t) &= 2^{-k} \sum_{t=0}^k \binom{k}{t} \sum_{n=0}^{\infty} \frac{y^n}{n!} (k-2t)^n \sum_{j=0}^{\infty} \frac{z^j}{j!} P_j(k, k-2t) \\ &= 2^{-k} \sum_{t=0}^k \binom{k}{t} e^{(k-2t)y} (1+z)^{k-t} (1-z)^t = 2^{-k} e^{-ky} ((1+z)e^{2y} + 1-z)^k \\ &= 2^{-k} (1+z)e^y + (1-z)e^{-y})^k = (\cosh y + z \sinh y)^k. \end{aligned}$$

Since this is a polynomial of degree  $k$  in  $z$ , it follows that the right-hand side of (6.7) does indeed vanish for  $j > k$  and all  $n$ . For example, for  $k=1$ , we get

$$P_j(1,1) + (-1)^n P_j(1,-1) = 0 \quad (j > 1).$$

Since this holds for all  $n$ , we have

$$(6.8) \quad P_j(1,1) = P_j(1,-1) = 0 \quad (j > 1).$$

Indeed, by (6.4),

$$\sum_{j=0}^{\infty} P_j(1,1) \frac{z^j}{j!} = 1+z, \quad \sum_{j=0}^{\infty} P_j(1,-1) \frac{z^j}{j!} = 1-z,$$

in agreement with (6.8).

For  $k=2$  we get

$$2^n P_j(2,2) + 4\delta_{n,0} P_j(2,0) + (-2)^n P_j(2,-2) = 0 \quad (j > 2).$$

This implies

$$P_j(2,2) = P_j(2,0) = P_j(2,-2) = 0 \quad (j > 2).$$

Indeed, by (6.4),

$$\sum_{j=0}^{\infty} P_j(2,2) \frac{z^j}{j!} = (1+z)^2, \quad \sum_{j=0}^{\infty} P_j(2,-2) \frac{z^j}{j!} = (1-z)^2, \quad \sum_{j=0}^{\infty} P_j(2,0) \frac{z^j}{j!} = 1-z^2.$$

Since the determinant

$$\left| \binom{k}{t} (k-2t)^n \right| \neq 0 \quad (t, n = 0, 1, \dots, k),$$

the identical vanishing of the right-hand side of (6.7) implies

$$(6.9) \quad P_j(k, k-2t) = 0 \quad (j > k; 0 \leq t \leq k).$$

This is indeed implied by (6.4), since

$$F(k, k-2t, z) = (1+z)^{k-t} (1-z)^t.$$

It follows from (6.9) and (6.1) that

$$(6.10) \quad \sum_{2s \leq j} (-1)^s P_{j,s}(k) (k-2t)^{j-2s} = 0 \quad (j > k; 0 \leq t \leq k);$$

it evidently suffices to take  $t \leq k/2$ . In particular, for  $j = k+1$ , (6.10) becomes

$$(6.11) \quad \sum_{2s \leq j} (-1)^s P_{j,s} (j-1)(j-2r-1)^{j-2s} = 0 \quad (2t < j)$$

For  $j$  even we consider

$$\sum_{s=0}^j (-1)^s P_{2j,s} (2j-1)(2j-2t-1)^{2j-2s} = 0 \quad (0 \leq t < j).$$

Since  $P_{j,0}(k) = 1$  this may be written in the form

$$(6.12) \quad \sum_{s=1}^j (-1)^{s-1} P_{2j,s} (2j-1)(2r-1)^{2j-2s} = (2r-1)^{2j} \quad (1 \leq r \leq j).$$

By Cramer's rule the system (6.12) has the solution

$$(6.13) \quad P_{2j,s} (2j-1) = \frac{N_s}{D} \quad (1 \leq s \leq j),$$

where

$$D = \det ((2r-1)^{2s-2}) \quad (r, s = 1, 2, \dots, j)$$

and  $N_s$  is obtained from  $D$  by replacing the  $s^{\text{th}}$  column by  $(2r-1)^{2t}$ . Making use of a familiar theorem on the quotient of two alternants [4, Ch. 11], we get

$$(6.14) \quad P_{2j,s} (2j-1) = c_s (1^2, 3^2, 5^2, \dots, (2j-1)^2) \quad (1 \leq s \leq j),$$

where  $c_s(x_1, x_2, \dots, x_j)$  denotes the  $s^{\text{th}}$  elementary symmetric function of the  $x_i$ .

For odd  $j$  in (6.11) we consider

$$\sum_{s=0}^j (-1)^s P_{2j+1,s} (2j)(2j-2t)^{2j-2s+1} = 0 \quad (0 \leq t < j).$$

This may be written in the form

$$(6.15) \quad \sum_{s=1}^j (-1)^{s-1} P_{2j+1,s} (2j)(2r)^{2j-2s+1} = (2r)^{2j+1} \quad (1 \leq r \leq j).$$

Exactly as in the case of (6.12), the solution of the system (6.15) is given by

$$(6.16) \quad P_{2j+1,s} (2j) = c_s (2^2, 4^2, 6^2, \dots, (2j)^2) \quad (1 \leq s \leq j).$$

where again  $c_s$  denotes the  $s^{\text{th}}$  symmetric function of the indicated arguments.

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