The author wishes to thank the referee for suggestions which have been incorporated in the revision made to this paper.

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Riassunto

Sia G un gruppo commutativo compatto e A, A' algebre di Banach commutative con identità e, e'. Hausner [2] *ha discusso gli omomorfismi T di $B^1(G, A)$ e $B^1(G, A')$ tale che $T(ef) = e'f$ per $f \in L^1(G)$, dove $B^1(G, A)$ è costituita da tutte le funzioni di Bochner integrabili definite in G avente valori in A. Lo scopo del presente lavoro è di generalizzare i risultati in* [2] *all'algebra $l_1(S, A)$ discussa in* [1]$_1$, *dove S è un semigruppo commutativo discreto.*

* * *

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Generalized Stirling and related numbers (**)

1. - Introduction.

The Stirling numbers of the first and second kind can be defined by

$$x(x+1)\dots(x+n-1) = \sum_{k=0}^{n} S_1(n, k)x^k \tag{1.1}$$

and

$$x^n = \sum_{k=0}^{n} S(n, k)\, x(x-1)\dots(x-k+1)\,, \tag{1.2}$$

respectively. Since $S_1(n, n-k)$ and $S(n, n-k)$ are polynomials in n of degree k, it follows readily that

$$S_1(n, n-k) = \sum_{j=0}^{k-1} S_1'(k, j)\binom{n}{2k-j} \qquad (k>0) \tag{1.3}$$

and

$$S(n, n-k) = \sum_{j=0}^{k-1} S'(k, j)\binom{n}{2k-j} \qquad (k>0)\,. \tag{1.4}$$

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The coefficients $S_1'(k, j)$, $S'(k, j)$ were introduced by Jordan ([3], Ch. 4) and Ward [9]; the notation used here is that of $[1]_4$. They are closely related to the *associated* Stirling numbers of Riordan ([7], Ch. 4). Indeed

$$(1.5)\qquad S_1'(k+j, j) = d(2k+j, k)\,, \qquad S'(k+j, j) = b(2k+j, k)\,,$$

where $b(n, k)$ is the number of *partitions* of $Z_n = \{1, 2, \ldots, n\}$ into k blocks each of cardinality >1, while $d(n, k)$ is the number of permutations of Z_n with k cycles each of length >1. Moreover

$$1 + \sum_{n=1}^{\infty} \frac{z^n}{n!} \sum_{k=1}^{\infty} b(n, k) x^k = \exp\left\{x \sum_{n=2}^{\infty} \frac{z^n}{n}\right\},$$

$$1 + \sum_{n=1}^{\infty} \frac{z^n}{n!} \sum_{k=1}^{\infty} d(n, k) x^k = \exp\left\{x \sum_{n=2}^{\infty} \frac{z^n}{n!}\right\}.$$

The Stirling numbers and the associated Stirling numbers are related in various ways $[1]_4$. In the first place

$$(1.6)\qquad \begin{aligned} S_1(n, n-k) &= \sum_{j=0}^{k} \binom{k-n}{k+j}\binom{k+n}{k-j} S(j+k, j)\,, \\ S(n, n-k) &= \sum_{j=0}^{k} \binom{k-n}{k+j}\binom{k+n}{k-j} S_1(j+k, j)\,, \end{aligned}$$

while

$$(1.7)\qquad \begin{aligned} S'(n, k) &= \sum_{j=0}^{k} (-1)^j \binom{n-j-1}{k-j} S_1'(n, j)\,, \\ S'(n, k) &= \sum_{j=0}^{k} (-1)^j \binom{n-j-1}{k-j} S'(n, j)\,. \end{aligned}$$

In addition

$$(1.8)\qquad \begin{aligned} S_1(n, n-k) &= \sum_{j=0}^{k-1} (-1)^j \binom{n+k-j-1}{2k-j} S'(k, j)\,, \\ S(n, n-k) &= \sum_{j=0}^{k-1} (-1)^j \binom{n+k-j-1}{2k-j} S_1'(k, j)\,. \end{aligned}$$

The first of (1.6) is due to Schläfli [8]; the second was proved by Gould [2].

Another triangular array of numbers is closely related to $S_1(n, k)$ and $S(n, k)$. Parallel to (1.3) and (1.4) we have $[1]_5$

$$(1.9)\qquad S_1(n, n-k) = \sum_{j=1}^{k} B_1(k, j) \binom{n+j-1}{2k} \qquad (k>0)$$

and

$$(1.10)\qquad S(n, n-k) = \sum_{j=1}^{k} B(k, j) \binom{n+j-1}{2k} \qquad (k>0)\,,$$

where

$$(1.11)\qquad B_1(k, j) = jB_1(k-1, j) + (2k-j)B_1(k-1, j-1)$$

and

$$(1.12)\qquad B(k, j) = (k-j+1)B(k-1, j) + (k+j-1)B(k-1, j-1)\,.$$

Moreover

$$(1.13)\qquad B(k, k-j+1) = B_1(k, j) \equiv a_{k,j}\,,$$

the a_{kj} were defined in $[1]_1$, $[1]_2$ in connection with an asymptotic expansion.

The writer $[1]_3$ proved (1.6)—in a slightly different notation—by making use of the formulas

$$(1.14)\qquad S_1(n, n-k) = \binom{k-n}{k} B_k^{(n)}, S(n, n-k) = \binom{n}{k} B_k^{(-n+k)},$$

where $B_n^{(z)}$ is the Nörlund polynomial ([5], Ch. 6) defined by

$$(1.15)\qquad \left(\frac{x}{e^x-1}\right)^z = \sum_{n=0}^{\infty} B_n^{(z)} \frac{x^n}{n!}\,,$$

where z is an arbitrary complex number. (The polynomial $B_n^{(z)}$ is not to be confused with the Bernoulli polynomial $B_n(z)$ defined by

$$\frac{xe^{zx}}{e^x-1} = \sum_{n=0}^{\infty} B_n(z) \frac{x^n}{n!}\,.$$

The writer also stated that if $\{f_k(z)\}$ denote an arbitrary sequence of polynomials of degree k, such that $f_k(0)=0$ for $k>0$, and we define

$$F_1(n, n-k) = \binom{k-n}{k} f_k(n)\,, \quad F(n, n-k) = \binom{n}{k} f_k(-n+k)\,, \tag{1.16}$$

then (1.6) admits the generalization (2.3) below.

In the present paper we prove (2.3) as well as the corresponding generalizations of (1.7), (1.8) and (1.13). See Theorems 1, 4 below. In proving these results we make use of two functions $G_1(k, j)$, $G(k, j)$ that generalize $B_1(k, j)$, $B(k, j)$. They are defined by

$$\begin{aligned} F_1(n, n-k) &= \sum_{j=1}^{k} G_1(k, j) \binom{n+j-1}{2k}\,, \\ F(n, n-k) &= \sum_{j=1}^{k} G(k, j) \binom{n+j-1}{2k}\,, \end{aligned} \qquad (k \geqslant 1) \tag{1.17}$$

and satisfy the relation

$$G_1(k, j) = G(k, k-j+1) \qquad (1 \leqslant j \leqslant k)\,. \tag{1.18}$$

In order to get a similar generalization of the orthogonality relations

$$\sum_{k=j}^{n} (-1)^{n-k} S_1(n, k) S(k, j) = \sum_{k=j}^{n} (-1)^{k-j} S(n, k) S_1(k, j) = \delta_{n,j}\,, \tag{1.19}$$

additional restrictions seem necessary. The generalized result is contained in Theorem 6 below.

In the final section of the paper several generating functions are obtained by applying the Lagrange expansion ([**6**], p. 125).

2. – Let $\{f_k(z)\}$ denote a sequence of polynomials in z such that

$$\deg f_k(z) = k\,, \qquad f_k(0) = 0 \qquad (k > 0)\,. \tag{2.1}$$

We define two functions $F_1(n, k)$, $F(n, k)$ by means of

$$\begin{aligned} F_1(n, n-k) &= \binom{k-n}{k} f_k(n)\,, \\ F(n, n-k) &= \binom{n}{k} f_k(-n+k) \end{aligned} \qquad (k = 0, 1, 2, \ldots)\,. \tag{2.2}$$

Theorem 1. *The functions* $F_1(n, k)$, $F(n, k)$ *satisfy*

$$\begin{aligned} F_1(n, n-k) &= \sum_{j=0}^{k} \binom{k-n}{k+j}\binom{k+n}{k-j} F(j+k, j)\,, \\ F(n, n-k) &= \sum_{j=0}^{k} \binom{k-n}{k+j}\binom{k+n}{k-j} F_1(j+k, j)\,. \end{aligned} \tag{2.3}$$

Proof. It suffices to prove the identity

$$\binom{z}{k} f_k(k-z) = \sum_{j=0}^{k} \binom{k-z}{k+j}\binom{k+z}{k-j}\binom{-j}{k} f_k(j+k)\,. \tag{2.4}$$

For $z = k-n$, (2.4) reduces to the first of (2.3); for $z = n$, (2.4) reduces to the second of (2.3).

Each side of (2.4) is a polynomial in z of degree $\leqslant 2k$. Hence it is only necessary to show that (2.4) holds for $2k+1$ distinct values of z. For $z = 0, 1, \ldots, k-1$, it is evident that the LHS of (2.4) vanishes; since

$$\binom{k-z}{k+j}\binom{-j}{k} = 0 \qquad (0 \leqslant z < k;\ 0 \leqslant j \leqslant k),$$

it follows that (2.4) holds for these values of z. For $z = k$ we have

$$f_k(0) = \sum_{j=0}^{k} \binom{0}{k+j}\binom{2k}{k-j}\binom{-j}{k} f_k(j+k)\,,$$

which is clearly correct. Finally, for $z = -s$, where $s = 1, 2, \ldots, k$, we note that the RHS of (2.4) reduces to the single term $(j = s)$

$$\binom{k+s}{k+s}\binom{k-s}{k-s}\binom{-s}{k} f_k(s+k) = \binom{-s}{k} f_k(s+k)\,,$$

so that (2.4) holds in this case also. This completes the proof of (2.3).

For brevity we may call $(k = 0, 1, 2, \ldots)$

$$\begin{aligned} g_{1,k}(z) &= F_1(z, z-k) = \binom{k-z}{k} f_k(z)\,, \\ g_k(z) &= F(z, z-k) = \binom{z}{k} f_k(-z+k) \end{aligned} \tag{2.5}$$

a *Stirling pair*—relative to the polynomial $f_k(z)$. Clearly $g_{1,k}(z)$ and $g_k(z)$ are polynomials in z of degree $2k$, such that, for $k \geqslant 1$,

$$g_{1,k}(s) = g_k(s) = 0 \qquad (0 \leqslant s < k) \tag{2.6}$$

and

$$g_{1,k}(k-z) = g_k(z) . \tag{2.7}$$

Conversely, if two polynomials $g_{1,k}(z)$, $g_k(z)$ of degree $2k$, $k \geq 1$, satisfy (2.6) and (2.7), then there exists a polynomial $f_k(z)$ of degree k satisfying (2.5) and such that $f_k(0) = 0$ $(k \geq 1)$.

This proves

Theorem 2. *A pair of polynomials $g_{1,k}(z)$, $g_k(z)$, each of degree $2k$, is a Stirling pair if and only if they satisfy (2.6) and (2.7).*

For example, if $f_k(z) = \binom{z}{k}$, then

$$g_{1,k}(z) = g_k(z) = \binom{z}{k}\binom{-z+k}{k} . \tag{2.8}$$

This, except for a constant factor, is the only case in which the Stirling pair consists of identical polynomials.

If $g_{1,k}(z)$ is an arbitrary polynomial of degree $2k$ such that $g_{1,k}(s) = 0$ $(0 \leq s \leq k)$, then clearly the Stirling pair $g_{1,k}(z)$, $g_k(z)$ is uniquely determined. A similar result holds for an arbitrary $g(z)$ satisfying the same conditions.

3. – It follows from (2.5) that $(k \geq 1)$

$$\begin{aligned} F_1(n, n-k) &= \sum_{j=0}^{k-1} F_1'(k, j) \binom{n}{2k-j} , \\ F(n, n-k) &= \sum_{j=0}^{k-1} F'(k, j) \binom{n}{2k-j} . \end{aligned} \tag{3.1}$$

Thus F_1', F' have the same relationship to F_1, F, respectively, that S_1', S' have to S_1, S.

It follows from the first of (3.1) that

$$\begin{aligned} \sum_{n=k}^{\infty} F_1(n, n-k) x^n &= \sum_{j=0}^{k-1} F_1'(k, j) x^{2k-j} \sum_{n=0}^{\infty} \binom{n+2k-j}{2k-j} x^n \\ &= \sum_{j=0}^{k-1} F_j(k, j) x^{2k-j} (1-x)^{-2k+j-1} . \end{aligned}$$

Put

$$z = \frac{x}{1-x} , \quad x = \frac{z}{1+z} , \quad 1-x = \frac{1}{1+z} \quad \text{and we get}$$

$$\sum_{j=0}^{k-1} F_1'(k, j) z^{2k-j} = \sum_{n=k}^{\infty} F_1(n, n-k) z^n (1+z)^{-n-1} .$$

The right hand side is equal to

$$\sum_{n=k}^{\infty} F_1(n, n-k) z^n \sum_{s=0}^{\infty} (-1)^s \binom{n+s}{s} z^s = \sum_{m=k}^{\infty} z^m \sum_{s=0}^{m-k} (-1)^s \binom{m}{s} F_1(m-s, m-k-s) .$$

Hence

$$F_1'(k, j) = \sum_{s=0}^{k-j} (-1)^s \binom{2k-j}{s} F_1(2k-j-s, k-j-s) ,$$

or equivalently

$$F_1'(k, k-j) = \sum_{s=0}^{j} (-1)^{j-s} \binom{k+j}{k+s} F_1(k+s, s) . \tag{3.2}$$

In exactly the same way, we get

$$F'(k, k-j) = \sum_{s=0}^{j} (-1)^{j-s} \binom{k+j}{k+s} F(k+s, s) . \tag{3.3}$$

Parallel to (3.1) we define $G_1(k, j)$, $G(k, j)$ by means of

$$\begin{aligned} F_1(n, n-k) &= \sum_{j=1}^{k} G_1(k, j) \binom{n+j-1}{2k} , \\ F(n, n-k) &= \sum_{j=1}^{k} G(k, j) \binom{n+j-1}{2k} . \end{aligned} \tag{3.4}$$

It follows from (3.4) that

$$\begin{aligned} G_1(k, k-j+1) &= \sum_{s=0}^{j} (-1)^s \binom{2k+1}{s} F_1(k+j-s, j-s) , \\ G(k, k-j+1) &= \sum_{s=0}^{j} (-1)^s \binom{2k+1}{s} F(k+j-s, j-s) . \end{aligned} \tag{3.5}$$

The proof is similar to the proof of (3.2).

We shall now show that

$$G_1(k,j) = G(k, k-j+1) \qquad (1 \leq j \leq k). \tag{3.6}$$

By (2.2) and (3.4) we have

$$\binom{n}{k} f_k(k-n) = \sum_{j=1}^{k} G(k,j) \binom{n+j-1}{2k}.$$

Since

$$\binom{n+j-1}{2k} = \frac{1}{(2k)!}(n+j-1)\ldots(n+1)\cdot n \ldots (n-k+1)\cdot(n-k)\ldots(n+j-2k),$$

we get

$$f_k(k-n) = \sum_{j=1}^{k} G(k,j)\binom{n+j-1}{j-1}\binom{n-k}{k-j+1}\frac{(j-1)!\,(k-j+1)!\,k!}{(2k)!},$$

so that

$$f_k(k-n) = \sum_{j=1}^{n} G(k, k-j+1)\binom{n+k-j}{k-j}\binom{n-k}{j}\frac{j!\,(k-j)!\,k!}{(2k)!}. \tag{3.7}$$

Similarly, since

$$\binom{k-n}{k} f_k(n) = \sum_{j=1}^{k} G_1(k,j)\binom{n+j-1}{2k}$$

and

$$\binom{n+j-1}{2k} = \frac{1}{(2k)!}(n+j-1)\ldots n\cdot(n-1)\ldots(n-k)\cdot(n-k-1)\ldots(n+j-2k),$$

we get

$$(-1)^k f_k(n) = \sum_{j=1}^{k} G_1(k,j)\binom{n+j-1}{j}\binom{n-k-1}{k-j}\frac{j!(k-j)!\,k!}{(2k)!}. \tag{3.8}$$

Since (3.7) is a polynomial identity in n we may replace n by $k-n$ and get

$$\begin{aligned} f_k(n) &= \sum_{j=1}^{k} G(k, k-j+1)\binom{2k-n-j}{k-j}\binom{-n}{j}\frac{j!\,(k-j)!\,k!}{(2k)!} \\ &= (-1)^k \sum_{j=1}^{k} G(k, k-j+1)\binom{n+j-1}{j}\binom{n-k-1}{k-j}\frac{j!\,(k-j)!\,k!}{(2k)!}. \end{aligned} \tag{3.9}$$

Hence, comparing (3.9) with (3.8), it is clear that (3.6) is implied by the following lemma which has some independent interest.

Lemma. *Every polynomial $\varphi(z)$ of degree $\leq k$ has a unique expansion of the type*

$$\varphi(z) = \sum_{j=0}^{k} C_j \binom{z+j-1}{j}\binom{z-k-1}{k-j}, \tag{3.10}$$

where the C_j are independent of z.

Proof. It is convenient to treat the slightly more general expansion:

$$\varphi(z) = \sum_{j=0}^{k} C_j \binom{z+j-1}{j}\binom{z-m-1}{k-j} \qquad (m \geq k). \tag{3.11}$$

If the C_j in (3.11) are not unique there exist a set of coefficients C_j' not all equal to 0 such that

$$\sum_{j=0}^{k} C_j' \binom{z+j-1}{j}\binom{z-m-1}{k-j} = 0. \tag{3.12}$$

For $z=0$ this implies $C_0' = 0$. Hence

$$\sum_{j=1}^{k} \frac{1}{j} C_j' \binom{z+j-1}{j-1}\binom{z-m-1}{k-j} = 0,$$

or

$$\sum_{j=0}^{k-1} \frac{1}{j+1} C_{j+1}' \binom{z+j}{j}\binom{z-m-1}{k-j-1} = 0. \tag{3.13}$$

We may assume that k in (3.12) is minimal. Then (3.13) furnishes a contradiction.

To find the coefficients in (3.11), multiply both sides by $\binom{z-1}{m}$ and we get

$$\binom{z-1}{m}\varphi(z) = \sum_{j=0}^{k} D_j \binom{z+j-1}{k+m}, \quad D_j = \frac{(k+m)!}{j!\,(k-j)!\,m!}\,C_j. \tag{3.14}$$

It follows from (3.14) that

$$\sum_{n=m+1}^{\infty} \binom{n-1}{m}\varphi(n)x^n = \sum_{j=0}^{k} D_j x^{k+m-j+1}(1-x)^{-k-m-1},$$

$$\sum_{j=0}^{k} D_j x^{k+m-j+1} = (1-x)^{k+m+1}\sum_{n=m+1}^{\infty}\binom{n-1}{m}\varphi(n)\,x^n.$$

This gives

$$\begin{aligned} D_j &= \sum_{n=m+1}^{k+m-j+1}(-1)^{k+m+n+j+1}\binom{k+m+1}{n+j}\binom{n-1}{m}\varphi(n) \\ &= \sum_{n=0}^{k-1}(-1)^{k+j}\binom{k+m+1}{k-n-j}\binom{n+m}{m}\varphi(n+m+1). \end{aligned} \tag{3.15}$$

We may state

THEOREM 3. *The coefficients $G_1(k,j)$, $G(k,j)$ occurring in*

$$\left.\begin{aligned} F_1(n, n-k) &= \sum_{j=1}^{k} G_1(k,j)\binom{n+j-1}{2k}, \\ F(n, n-k) &= \sum_{j=1}^{k} G(k,j)\binom{n+j-1}{2k}, \end{aligned}\right\} \quad (k \geq 1) \tag{3.16}$$

satisfy the relation

$$G_1(k,j) = G(k, k-j+1). \tag{3.17}$$

4. – By making use of (3.16) we are able to prove various relations. In the first place we can obtain another proof of Theorem 1. We shall not take the

space to give this proof, but rather prove some new results. To begin with, by (3.2) and the first of (3.16),

$$\begin{aligned} F_1'(k, k-j) &= \sum_{s=0}^{j}(-1)^s\binom{k+j}{s}F_1(k+j-s, j-s) \\ &= \sum_{s=0}^{j}(-1)^s\binom{k+j}{s}\sum_{t=1}^{k}G_1(k,t)\binom{k+j-s+t-1}{2k} \\ &= \sum_{t=1}^{k}G_1(k,t)\sum_{s=0}^{j}(-1)^s\binom{k+j}{s}\binom{k+j-s+t-1}{2k}. \end{aligned}$$

By Vandermonde's theorem the inner sum reduces to $\binom{t-1}{k-j}$, so that

$$F_1'(k,j) = \sum_{t=j+1}^{k}\binom{t-1}{j}G_1(k,t) \qquad (0 \leq j < k). \tag{4.1}$$

Similarly

$$F'(k,j) = \sum_{t=j+1}^{k}\binom{t-1}{j}G(k,t) \qquad (0 \leq j < k). \tag{4.2}$$

The inverse formulas are

$$G_1(k,t) = \sum_{j+t-1}^{k-1}(-1)^{j-t+1}\binom{j}{t-1}F_1'(k,j) \qquad (1 \leq t \leq k) \tag{4.3}$$

and

$$G(k,t) = \sum_{j=t-1}^{k-1}(-1)^{j-t+1}\binom{j}{t-1}F'(k,j) \qquad (1 \leq t \leq k). \tag{4.4}$$

In the next place, by (3.6) and (4.4)

$$\begin{aligned} F_1(n, n-k) &= \sum_{t=1}^{k}G_1(k,t)\binom{n+t-1}{2k} \\ &= \sum_{t=1}^{k}G(k, k-t+1)\binom{n+t-1}{2k} = \sum_{t=1}^{k}G(k,t)\binom{n+k-t}{2k} \end{aligned}$$

$$= \sum_{t=1}^{k} \binom{n+k-t}{2k} \sum_{j=t-1}^{k-1} (-1)^{j-t+1} \binom{j}{t-1} F'(k, j)$$

$$= \sum_{j=0}^{k-1} F'(k, j) \sum_{t=1}^{j+1} (-1)^{j-t+1} \binom{j}{t-1} \binom{n+k-t}{2k} .$$

The inner sum is equal to

$$\sum_{t=0}^{j} (-1)^{j+t} \binom{j}{t} \binom{n+k-t-1}{2k} = \sum_{t=0}^{j} (-1)^t \binom{j}{t} \binom{n+k-j+t-1}{2k}$$

$$= (-1)^j \binom{n+k-j-1}{2k-j} .$$

Hence

$$F_1(n, n-k) = \sum_{j=0}^{k-1} (-1)^j \binom{n+k-j-1}{2k-j} F'(k, j) \tag{4.5}$$

and similarly

$$F(n, n-k) = \sum_{j=0}^{k-1} (-1)^j \binom{n+k-j-1}{2k-j} F_1'(k, j) . \tag{4.6}$$

Again, by (4.1) and (4.4),

$$F_1'(n, k) = \sum_{j=k+1}^{n} \binom{j-1}{k} G(n, n-j+1) = \sum_{j=1}^{n-k} \binom{n-j}{k} G(n, j)$$

$$= \sum_{j=1}^{n-k} \binom{n-j}{k} \sum_{t=j-1}^{n} (-1)^{t-j+1} \binom{t}{j-1} F'(n, t)$$

$$= \sum_{t=0}^{n-1} F'(n, t) \sum_{j=1}^{t+1} (-1)^{t-j+1} \binom{t}{j-1} \binom{n-j}{k} .$$

The inner sum is equal to

$$\sum_{j=0}^{t} (-1)^{t-j} \binom{t}{j} \binom{n-j-1}{k} = (-1)^t \binom{n-t-1}{k-t} ,$$

so that

$$F_1'(n, k) = \sum_{t=0}^{k} (-1)^t \binom{n-t-1}{k-t} F'(n, t) . \tag{4.7}$$

Similarly

$$F'(n, k) = \sum_{t=0}^{k} (-1)^t \binom{n-t-1}{k-t} F_1'(n, t) . \tag{4.8}$$

To invert (4.5) and (4.6) we use (3.2) and (3.3). It follows from (4.7) and (3.3) that

$$F_1'(n, k) = \sum_{t=0}^{k} (-1)^t \binom{n-t-1}{k-t} F'(n, t)$$

$$= \sum_{t=0}^{k} (-1)^t \binom{n-t-1}{k-t} \sum_{j=0}^{n-t} (-1)^{n-t-j} \binom{2n-t}{n+j} F(n+j, j)$$

$$= \sum_{j=0}^{n} (-1)^{n-j} F(n+j, j) \sum_{t=0}^{n-j} \binom{n-t-1}{k-t} \binom{2n-t}{n+j} .$$

Thus

$$F_1'(n, k) = \sum_{j=0}^{n} (-1)^{n-j} F(n+j, j)\, C_n(k, j) \tag{4.9}$$

and

$$F'(n, k) = \sum_{j=0}^{n} (-1)^{n-j} F_1(n+j, j)\, C_n(k, j) , \tag{4.10}$$

where

$$C_n(k, j) = \sum_{t=0}^{n-j} \binom{n-t-1}{k-t} \binom{2n-t}{n+j} . \tag{4.11}$$

It does not seem possible to simplify $C_n(k, j)$.

To sum up we state

Theorem 4. *The functions* $F_1(n, k)$, $F(n, k)$, $F_1'(n, k)$, $F'(n, k)$ *satisfy the following relations*

$$F_1'(n, n-k) = \sum_{j=0}^{k} (-1)^{k-j} \binom{n+k}{n+j} F_1(n+j, j),$$

$$F'(n, n-k) = \sum_{j=0}^{k} (-1)^{k-j} \binom{n+k}{n+j} F(n+j, j),$$

$$F_1'(n, k) = \sum_{j=0}^{k} (-1)^{j} \binom{n-j-1}{k-j} F'(n, j),$$

$$F'(n, k) = \sum_{j=0}^{k} (-1)^{j} \binom{n-j-1}{k-j} F_1'(n, j),$$

$$F_1(n, n-k) = \sum_{j=0}^{k-1} (-1)^{j} \binom{n+k-j-1}{2k-j} F'(k, j),$$

$$F(n, n-k) = \sum_{j=0}^{k-1} (-1)^{j} \binom{n+k-j-1}{2k-j} F_1'(k, j),$$

$$F_1'(n, k) = \sum_{j=0}^{n} (-1)^{n-j} C_n(k, j) F(n+j, j),$$

$$F'(n, k) = \sum_{j=0}^{n} (-1)^{n-j} C_n(k, j) F_1(n+j, j),$$

where

$$C_n(k, j) = \sum_{t=0}^{\min(k, n-j)} \binom{n-t-1}{k-t} \binom{2n-t}{n+j}.$$

5. – For the results obtained above it sufficed to assume that the $\{f_k(z)\}$ were a sequence of polynomials in z satisfying

(5.1) $$\deg f_k(z) = k; \qquad f_k(0) = 0 \qquad (k \geqslant 1).$$

In order to obtain orthogonality relations we require more.

Let

(5.2) $$\varphi(x) = 1 + \sum_{n=1}^{\infty} c_n x^n / n!$$

denote a function that is analytic in the neighborhood of $x = 0$ and such that $\varphi(0) = 1$. Put

(5.3) $$(\varphi(x))^z = \sum_{k=0}^{\infty} f_k(z) \frac{x^k}{k!}.$$

It is easily verified that the $\{f(z)\}$ are polynomials in z that satisfy (5.1). The Bernoulli polynomials $B_k^{(z)}$ are evidently given by $\varphi(x) = x/(e^x - 1)$.

It follows at once from (5.3) that

(5.4) $$\sum_{j=0}^{k} \binom{k}{j} f_j(y) f_{k-j}(z) = f_k(y+z) \qquad (k = 0, 1, 2, \ldots).$$

We shall show that (5.4) characterizes polynomial sequences defined by (5.2) and (5.3).

Theorem 5. *A sequence of polynomials* $\{f_k(z)\}_{k=0}^{\infty}$ *is defined by* (5.2) *and* (5.3) *for some* $\varphi(x)$ *if and only if they satisfy* (5.4).

Proof. The necessity is clear. To prove the sufficiency let $\{f_k(z)\}$ denote a sequence of polynomials that satisfy (5.4) and put

(5.5) $$\varphi(x) = \sum_{k=0}^{\infty} f_k(1) \frac{x^k}{k!}, \qquad (\varphi(x))^z = \sum_{k=0}^{\infty} \bar{f}_k(z) \frac{x^k}{k!}.$$

Then

(5.6) $$\sum_{j=0}^{k} \binom{k}{j} \bar{f}_j(y) \bar{f}_{k-j}(z) = \bar{f}_k(y+z) \qquad (k = 0, 1, 2, \ldots).$$

We show that

(5.7) $$f_k(n) = \bar{f}_k(n) \qquad (n = 1, 2, 3, \ldots).$$

This clearly holds for $n = 1$. Assume that it holds up to and including the value n. Then, by (5.6),

$$\bar{f}_k(n+1) = \sum_{j=0}^{k} \bar{f}_j(n) \bar{f}_{k-j}(1) = \sum_{j=0}^{k} f_j(n) f_{k-j}(1) = f_k(n+1).$$

This proves (5.6). Since $f_k(z)$, $\bar{f}_k(z)$ are polynomials, it follows from (5.7) that they are equal.

It follows from (5.3) that

$$n\varphi^{n-1}(x)\varphi'(x) = \sum_{k=0}^{\infty} f_{k+1}(n)\, x^k/k!\,,$$

$$(m+n)\varphi^{m+n-1}(x)\varphi'(x) = \sum_{k=0}^{\infty} f_{k+1}(m+n)x^k/k!\,.$$

Since

$$(m+n)\varphi^{m+n-1}(x)\varphi'(x) = \frac{m+n}{n}\,\varphi^m(x)\cdot n\varphi^{n-1}(x)\varphi'(x)\,,$$

we get

$$f_{k+1}(m+n) = \frac{m+n}{n}\sum_{j=0}^{k}\binom{k}{j} f_{k-j}(m)\, f_{j+1}(n)\,. \tag{5.8}$$

We now consider the sum

$$H(n,j) = \sum_{k=j}^{n} (-1)^{n-k} F_1(n,k) F(k,j), \tag{5.9}$$

where, by (1.16),

$$F_1(n,k) = \binom{-k}{n-k} f_{n-k}(n)\,, \qquad F(n,k) = \binom{n}{n-k} f_{n-k}(-k)\,. \tag{5.10}$$

Then

$$H(n,j) = \sum_{k=j}^{n} (-1)^{n-k}\binom{-k}{n-k} f_{n-k}(n)\binom{k}{k-j} f_{k-j}(-j)\,.$$

Since

$$\binom{-k}{n-k} = (-1)^{n-k}\binom{n-1}{k-1} = (-1)^{n-k}\frac{k}{n}\binom{n}{k}\,,$$

we get

$$H(n,j) = \frac{1}{n}\sum_{k=j}^{n}\binom{n}{k}\binom{k}{j} k f_{n-k}(n) f_{k-j}(-j) = \frac{1}{n}\binom{n}{j}\sum_{k=j}^{n}\binom{n-j}{k-j} k f_{n-k}(n) f_{k-j}(-j)$$

$$= \frac{j}{n}\binom{n}{j}\sum_{k=j}^{n}\binom{n-j}{k-j} f_{n-k}(n) f_{k-j}(-j) +$$

$$+ \frac{n-j}{n}\binom{n}{j}\sum_{k=j}^{n}\binom{n-j-1}{k-j-1} f_{n-k}(n) f_{k-j}(-j)$$

$$= \binom{n-1}{j-1} f_{n-j}(n-j) + \binom{n-1}{j}\sum_{k=j-1}^{n-1}\binom{n-j-1}{k-j} f_{n-k-1}(n) f_{k-j+1}(-j)$$

$$= \binom{n-1}{j-1} f_{n-j}(n-j) - \frac{j}{n-j}\binom{n-1}{j} f_{n-j}(n-j)\,.$$

Therefore, $H(n,j) = 0$ $(n > j)$. For $n = j$, it is obvious that $H(n,n) = 1$.

We may state

Theorem 6. *Let $\{f_k(n)\}_{k=0}^{\infty}$ denote a sequence of polynomials defined by* (5.3), *for some $\varphi(n)$ and define $F_1(n,k)$, $F(n,k)$ by* (5.10). *Then we have*

$$\sum_{k=j}^{n} (-1)^{n-k} F_1(n,k) F(k,j) = \sum_{k=j}^{n} (-1)^{k-j} F(n,k) F_1(n,k) = \delta_{nj}\,. \tag{5.11}$$

6. – Generating functions for the special $F_1(n,k)$ defined by (5.3) and (5.10) are implied by the Lagrange expansion ([**6**], p. 125).

Let $\varphi(x)$ and $f(x)$ be analytic about $x = 0$, $\varphi(0) = 1$. Put

$$u = x/\varphi(x)\,. \tag{6.1}$$

Then

$$f(x) = f(0) + \sum_{n=1}^{\infty}\frac{u^n}{n!}\left[\frac{d^{n-1}}{dz^{n-1}}\left(f'(z)\varphi^n(z)\right)\right]_{z=0} \tag{6.2}$$

and

$$\frac{f(x)}{1-u\varphi'(x)} = \sum_{n=0}^{\infty}\frac{u^n}{n!}\left[\frac{d^n}{dz^n}\left(f(z)\varphi^n(z)\right)\right]_{z=0}\,. \tag{6.3}$$

To begin with, we take $f(x) = x$ in (6.2). Since

$$\varphi^n(z) = \sum_{k=0}^{\infty} f_k(n)\frac{z^n}{n!}\,,$$

it follows that

$$\left[\frac{d^{n-1}}{dz^{n-1}}\left(\varphi^n(z)\right)\right]_{z=0} = f_{n-1}(n)\,,$$

so that (6.2) reduces to

$$x = \sum_{n=1}^{\infty} f_{n-1}(n) \frac{u^n}{n!}. \tag{6.4}$$

Taking $k=1$ in the first of (5.10) we get

$$F_1(n, 1) = \binom{-1}{n-1} f_{n-1}(n) = (-1)^{n-1} f_{n-1}(n).$$

Hence (6.4) becomes

$$x = \sum_{n=1}^{\infty} (-1)^{n-1} F_1(n, 1) \frac{u^n}{n!}. \tag{6.5}$$

More generally, if we take $f(x) = x^m$, $m \geqslant 1$, in (6.2), we get

$$x^m = m! \sum_{n=m}^{\infty} (-1)^{n-m} F_1(n, m) \frac{u^n}{n!} \qquad (m \geqslant 0). \tag{6.6}$$

Next, for $f(x) = \varphi^m(x)$, we find that

$$\varphi^m(x) = \sum_{n=0}^{\infty} \frac{m}{m+n} f_n(m+n) \frac{u^n}{n!} \qquad (m \geqslant 1). \tag{6.7}$$

This gives

$$\varphi^m(x) = \sum_{n=0}^{\infty} (-1)^n \frac{m!}{(m+n)!} F_1(m+n, m) u^n \qquad (m \geqslant 0). \tag{6.8}$$

This result is equivalent to (6.6).

The method also applies to the case of negative m. For convenience we replace m by $-m$. In place of (6.7) we now get

$$\varphi^{-m}(x) = \sum_{\substack{n=0 \\ n \neq m}}^{\infty} \frac{m}{m-n} f_n(n-m) \frac{u^n}{n!},$$

$$\varphi^{-m}(x) = \sum_{\substack{n=0 \\ n \neq m}}^{\infty} \frac{-m}{n-m} f_n(n-m) \frac{u^n}{n!} - mR_m \frac{u^m}{m!}, \tag{6.9}$$

where

$$R_m = \left[\frac{d^{m-1}}{dz^{m-1}} \left\{\frac{\varphi'(z)}{\varphi(z)}\right\}\right]_{z=0} = \left[\frac{d^m}{dz^m} \{\log \varphi(z)\}\right]_{z=0}.$$

Thus

$$\sum_{m=1}^{\infty} R_m \frac{z^m}{m!} = \log \varphi(z). \tag{6.10}$$

Those terms in the right member of (6.9) with $n < m$ are expressible in terms of $F(n, k)$; however those with $n > m$ are apparently not expressible in terms of either $F(n, k)$ or $F_1(n, k)$. Thus

$$\varphi^{-m}(x) = \sum_{n=0}^{m-1} (-1)^n \frac{(m-n-1)!}{(m-1)!} F(m, m-n) u^n - \tag{6.11}$$

$$- mR_m \frac{u^m}{m!} + \sum_{n=m+1}^{\infty} \frac{-m}{n-m} f_n(n-m) \frac{u^n}{n!} \qquad (m > 0).$$

As a partial check of (6.6) we take $u = e^x - 1$ and $F_1(n, k) = S_1(n, k)$. Then, denoting the right hand side of (6.6) by U_m, so that

$$U_m = \sum_{n=0}^{\infty} (-1)^n \frac{m!}{(n+m)!} S_1(m+n, m) u^n,$$

we have

$$\sum_{m=0}^{\infty} (-1)^m \frac{z^m}{m!} U_m = \sum_{n=0}^{\infty} (-1)^n \frac{u^n}{n!} \sum_{m=0}^{n} F_1(n, m) z^m$$

$$= \sum_{n=0}^{\infty} (-1)^n \binom{z+n-1}{n} u^n = (1+u)^{-z} = e^{-xz},$$

which is correct.

As an application of (6.3), we take $f(x) = x^m$, $m > 0$. The result, using (5.10), is

$$\frac{\varphi^m(x)}{1 - u\varphi'(x)} = \sum_{n=0}^{\infty} (-1)^n \frac{(m-1)!}{(m+n-1)!} F_1(m+n, m) u^n \qquad (m > 0). \tag{6.12}$$

Let $x = \lambda(u)$ denote the inverse of $u = u(x)$, $\lambda(u)$ analytic about $u = 0$, $\lambda(0) = 0$. Since $u = x/\varphi(x)$, we get

$$u = u(\lambda(u)) = \frac{\lambda(u)}{\varphi(\lambda(u))},$$

so that

$$\varphi(\lambda(u)) = \frac{\lambda(u)}{u}. \tag{6.13}$$

Substituting from (6.13) in (6.8), we get

$$\frac{1}{m!}\lambda^m(u) = \sum_{n=m}^{\infty}(-1)^{n-m}F_1(n, m)\frac{u^n}{n!} \qquad (m \geqslant 0). \tag{6.14}$$

Since

$$\varphi'(\lambda(u))\,\lambda'(u) = \frac{\lambda'(u)}{u} - \frac{\lambda(u)}{u^2},$$

$$1 - u\varphi'(\lambda(u)) = 1 - u\left\{\frac{1}{u} - \frac{\lambda(u)}{u^2\lambda'(u)}\right\} = \frac{\lambda(u)}{u\lambda'(u)},$$

substitution from (6.13) in (6.12) gives

$$\varphi^{m-1}(u)\lambda'(u) = \sum_{n=0}^{\infty}(-1)^n\frac{(m-1)!}{(m+n-1)}F_1(m+n, m)u^{m+n-1}$$

$$= \sum_{n=m}^{\infty}(-1)^{n-m}\frac{(m-1)!}{(n-1)!}F_1(n, m)u^{n-1}.$$

This is evidently implied by differentiation of (6.14).

For $u = e^x - 1$, $\lambda(u) = \log(1+u)$, (6.14) reduces to

$$\frac{1}{m!}(\log(1+u))^m = \sum_{n=m}^{\infty}(-1)^{n-m}F_1(n, m)\frac{u^n}{n!} \qquad (m \geqslant 0).$$

Finally we note that the generalized version of the familiar formula

$$\sum_{n=m}^{\infty}S(n, m)\frac{x^n}{n!} = \frac{1}{m!}(e^x - 1)^m \qquad (m \geqslant 0)$$

is given by

$$\sum_{n=m}^{\infty}F(n, m)\frac{x^n}{n!} = \frac{1}{m!}\left(\frac{x}{\varphi(x)}\right)^m \qquad (m \geqslant 0). \tag{6.15}$$

The results of this section may be compared with the similar ones in [4] (Ch. 6) and [5] (Ch. 6).

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Summary

Let $\{f_k(z)\}$ denote an arbitrary sequence of polynomials, $\deg f_k(z) = k$, $f_k(0) = 0$ $(k > 0)$. *Generalized Stirling numbers of the first and second kind are defined by*

$$F_1(n, n-k) = \binom{k-n}{k}f_k(n), \qquad F(n, n-k) = \binom{n}{k}f_k(-n+k), \tag{*}$$

respectively. For the ordinary Stirling numbers, $f_k(z)$ is the Nörlund polynomial $B_k^{(z)}$ defined by $(x/(e^x-1))^z = \sum_{k=0}^{\infty}B_k^{(z)}x^k/k!$.

By means of (*) many of the properties of the ordinary Stirling numbers are shown to hold for the generalized numbers. However, in order to obtain orthogonality relations, additional restrictions are introduced.

* * *