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EXPLICIT FORMULAS FOR THE DUMONT–FOATA POLYNOMIAL

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Dumont and Foata have defined a polynomial $F_n(x, y, z)$ recursively. They proved that $F_n(x, y, z)$ is symmetric in x, y, z and that $F_n(1, 1, 1) = G_{2n+2}$, the Genocchi number. Moreover, they gave an elegant combinatorial interpretation for the coefficients of $F_n(x, y, z)$. In the present paper explicit formulas and generating functions for $F_n(x, y, z)$ are obtained.

1. Introduction

Dumont and Foata [4] have defined a polynomial $F_n(x, y, z)$ in three variables recursively by means of

$$(1.1)\qquad F_n(x, y, z) = (x+z)(y+z)F_{n-1}(x, y, z+1) - z^2F_{n-1}(x, y, z) \quad (n \geq 2),$$

where $F_1(x, y, z) = 1$. They proved that $F_n(x, y, z)$ is symmetric in the three variables x, y, z and that

$$(1.2)\qquad F_n(1, 1, 1) = G_{2n+2} \quad (n \geq 1),$$

where G_{2n+2} is the Gennochi number defined by

$$(1.3)\qquad \frac{2u}{e^u+1} = u + \sum_{n=1}^{\infty} (-1)^n G_{2n} \frac{u^{2n}}{(2n)!}$$

or equivalently

$$(1.4)\qquad G_{2n} = 2(2^{2n}-1)|B_{2n}| \quad (n \geq 1),$$

where B_{2n} is the Bernoulli number in the even suffix notation [7, Ch. 2]

$$(1.5)\qquad \frac{u}{e^u-1} = \sum_{n=0}^{\infty} B_n \frac{u^n}{n!}.$$

Moreover, they obtain an elegant combinatorial interpretation for the coefficients $a_{n,i,j,k}$ defined by

$$(1.6)\qquad F_n(x, y, z) = \sum_{i,j,k=1}^{\infty} a_{n,i,j,k} x^{i-1} y^{j-1} z^{k-1}.$$

It is evident from (1.2) and (1.6) that

$$G_{2n+2} = \sum_{i,j,k=1}^{n} a_{n,i,j,k}. \tag{1.7}$$

Gandhi [5] had conjectured that if $\{P_n(z)\}_{n\geq 0}$ is a sequence of polynomials defined by $P_0(z) = 1$ and

$$P_n(z) = z^2 P_{n-1}(z+1) - (z-1)^2 P_{n-1}(z) \quad (n \geq 1), \tag{1.8}$$

then

$$G_{2n+2} = P_n(1). \tag{1.9}$$

This conjecture was proved independently by the present writer [1] and by Riordan and Stein [8].

If we put

$$Q_n(z) = P_{n-1}(z+1) \quad (n \geq 1),$$

it is clear that $Q_1(z) = 1$ and

$$Q_n(z) = (z+1)^2 Q_{n-1}(z+1) - z^2 Q_n(z) \quad (n \geq 2); \tag{1.10}$$

thus (1.9) becomes

$$G_{2n+2} = Q_{n-1}(1) \quad (n \geq 1). \tag{1.11}$$

Moreover comparison of (1.1) with (1.10) gives

$$Q_n(z) = F_n(1, 1, z) \quad (n \geq 1). \tag{1.12}$$

Explicit formulas for the polynomial have not previously been obtained. In the present paper we show first that

$$F_{n+1}(x, y, z) = \sum_{k=0}^{n} (-1)^{n-k} (x+z)_k (y+z)_k A_{n,k}(z) \tag{1.13}$$

and secondly that

$$F_n(x, y, z) = S_{n,1} + S_{n,2} + S_{n,3}, \tag{1.14}$$

where

$$(x)_k = x(x-1) \cdots (x+k-1), \quad (x)_0 = 1,$$

$$S_{n,j} = \sum_{r+s+t=n-j} \sum_{k=0}^{\min(r,s,t)} (-1)^{n-k} (y+z)_{2k} (z+x)_{2k} (x+y)_{2k} \cdot A_{r,k}(x) A_{s,k}(y) A_{t,k}(z) E^k C_j \quad (j = 1, 2, 3), \tag{1.15}$$

the C_j are certain explicit symmetric polynomials in x, y, z, E is the operator defined by

$$E\, f(x, y, z) = f(x+1, y+1, z+1)$$

(5.9), *where* C_j *is defined by* (5.7) *and*

$$A_{n,k}(x)=\frac{2}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\frac{(x+j)^{2n+1}}{(2x+j)_{k+1}}.$$

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