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WEIGHTED STIRLING NUMBERS OF THE FIRST AND SECOND KIND—I

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1. INTRODUCTION

The Stirling numbers of the first and second kind can be defined by

$$(1.1) \quad (x)_n \equiv x(x+1) \cdots (x+n-1) = \sum_{k=0}^n S_1(n, k)x^k$$

and

$$(1.2) \quad x^n = \sum_{k=0}^n S(n, k)x(x-1) \cdots (x-k+1),$$

respectively.

It is well known that $S_1(n, k)$ is the number of permutations of

$$Z_n = \{1, 2, \dots, n\}$$

with k cycles and that $S(n, k)$ is the number of partitions of the set Z_n into k blocks [1, Ch. 5], [2, Ch. 4]. These combinatorial interpretations suggest the following extensions.

Let n, k be positive integers, $n \geq k$, and let k_1, k_2, \dots, k_n be non-negative integers such that

$$(1.3) \quad \begin{cases} k = k_1 + k_2 + \cdots + k_n \\ n = k_1 + 2k_2 + \cdots + nk_n. \end{cases}$$

We define $\bar{S}(n, k, \lambda)$, $\bar{S}_1(n, k, \lambda)$, where λ is a parameter, in the following way.

$$(1.4) \quad \bar{S}(n, k, \lambda) = \sum \sum (k_1 \lambda + k_2 \lambda^2 + \cdots + k_n \lambda^n),$$

where the inner summation is over all partitions of Z_n into k_1 blocks of cardinality 1, k_2 blocks of cardinality 2, ..., k_n blocks of cardinality n ; the outer summation is over all k_1, k_2, \dots, k_n satisfying (1.3).

$$(1.5) \quad \bar{S}_1(n, k, \lambda) = \sum \sum \left\{ k_1 (\lambda)_1 + k_2 \frac{(\lambda)_2}{1!} + \cdots + k_n \frac{(\lambda)}{(n-1)!} \right\},$$

where the inner summation is over all permutations of Z_n with k_1 cycles of length 1, k_2 cycles of length 2, ..., k_n cycles of length n ; the outer summation is over all k_1, k_2, \dots, k_n satisfying (1.3).

We now put

$$(1.6) \quad \begin{cases} S(n, k, \lambda) = \frac{1}{k} \bar{S}(n, k, \lambda) \\ S_1(n, k, \lambda) = \frac{1}{n} \bar{S}_1(n, k, \lambda). \end{cases}$$

It is evident from (1.4) and (1.5) that

$$(1.7) \quad S(n, k, 1) = S(n, k), \quad S_1(n, k, 1) = S_1(n, k).$$

Indeed we shall show that if λ is an integer, then $S(n, k, \lambda)$ and $S_1(n, k, \lambda)$ are also integers. More precisely, we show that, for arbitrary λ ,

$$(1.8) \quad \bar{S}(n, k, \lambda) = \sum_{j=1}^{n-k+1} (k)_j S(n, j+k-1) \binom{\lambda}{j},$$

$$(1.9) \quad \bar{S}_1(n, k, \lambda) = \sum_{j=1}^{n-k+1} \binom{n}{j} (\lambda)_j S_1(n-j, k-1).$$

We obtain recurrences and generating functions for both $S(n, k, \lambda)$ and $S_1(n, k, \lambda)$. Simpler results hold for the functions

$$(1.10) \quad \begin{cases} R(n, k, \lambda) = \bar{S}(n, k+1, \lambda) + S(n, k) \\ R_1(n, k, \lambda) = \bar{S}_1(n, k+1, \lambda) + S_1(n, k). \end{cases}$$

For example, we have the recurrences

$$(1.11) \quad \begin{cases} R(n+1, k, \lambda) = R(n, k-1, \lambda) + (k+\lambda)R(n, k, \lambda) \\ R_1(n+1, k, \lambda) = R_1(n, k-1, \lambda) + (n+\lambda)R_1(n, k, \lambda) \end{cases}$$

and the orthogonality relations

$$(1.12) \quad \sum_{j=0}^n R(n, j, \lambda) \cdot (-1)^{j-k} R_1(j, k, \lambda) \\ = \sum_{j=0}^n (-1)^{n-j} R_1(n, j, \lambda) R(j, k, \lambda) = \begin{cases} 1 & (n = k) \\ 0 & (n \neq k). \end{cases}$$

For $\lambda = 0$ and $\lambda = 1$, (1.11) and (1.12) reduce to familiar formulas for $S(n, k)$ and $S_1(n, k)$.

The definitions (1.4) and (1.5) furnish combinatorial interpretations of $\bar{S}(n, k, \lambda)$ and $\bar{S}_1(n, k, \lambda)$ when λ is arbitrary. For λ a nonnegative integer, the recurrences (1.11) suggest combinatorial interpretations for $R(n, k, \lambda)$ and $R_1(n, k, \lambda)$ that generalize the interpretation of $S(n, k)$ and $S_1(n, k)$ described above. For the statement of the generalized interpretations, see Section 7 below.

2. THE FUNCTION $\bar{S}(n, k, \lambda)$

Let n, k be positive integers, $n \geq k$, and k_1, k_2, \dots, k_n nonnegative such that

$$(2.1) \quad \begin{cases} k = k_1 + k_2 + \dots + k_n \\ n = k_1 + 2k_2 + \dots + nk_n. \end{cases}$$

Put

$$(2.2) \quad S(n; k_1, k_2, \dots, k_n; \lambda) = \sum (k_1\lambda + k_2\lambda^2 + \dots + k_n\lambda^n),$$

where the summation is over all partitions of $Z_n = 1, 2, \dots, n$ into k_1 blocks of cardinality 1, k_2 blocks of cardinality 2, \dots , k_n blocks of cardinality n . Then we have (compare [2, p. 75]):

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{x^n}{n!} \sum_{k_1, k_2, \dots} S(n; k_1, k_2, \dots; \lambda) \frac{y^{k_1} y^{k_2} \dots}{k_1! k_2! \dots} \\ &= \sum_{n=1}^{\infty} \frac{x^n}{n!} \sum_{k_1, k_2, \dots} (k_1\lambda + k_2\lambda^2 + \dots) \frac{n!}{1!^{k_1} 2!^{k_2} \dots} \frac{y^{k_1} y^{k_2} \dots}{k_1! k_2! \dots} \\ &= \left(\frac{y_1 \lambda x}{1!} + \frac{y_2 \lambda^2 x^2}{2!} + \dots \right) \exp \left\{ \frac{y_1 x}{1!} + \frac{y_2 x^2}{2!} + \dots \right\}. \end{aligned}$$

For $y_1 = y_2 = \dots = y$, the extreme right member becomes

$$y(e^{\lambda x} - 1) \exp \{y(e^x - 1)\}.$$

Hence, we get the generating function

$$(2.3) \quad \sum_{n, k} \bar{S}(n, k, \lambda) \frac{x^n}{n!} y^k = y(e^{\lambda x} - 1) \exp \{y(e^x - 1)\}.$$

Recall that

$$(2.4) \quad \sum_{n, k} S(n, k) \frac{x^n}{n!} y^k = \exp \{y(e^x - 1)\}.$$

Thus, the right-hand side of (2.3) is equal to

$$y \sum_{m=1}^{\infty} \frac{\lambda^m x^m}{m!} \sum_{n, k} S(n, k) \frac{x^n}{n!} y^k$$

and therefore,

$$(2.5) \quad \bar{S}(n, k, \lambda) = \sum_{m=1}^{n-k+1} \binom{n}{m} \lambda^m S(n-m, k-1).$$

Note that, for $\lambda = 1$, (2.3) reduces to

$$\begin{aligned} \sum_{n, k} \bar{S}(n, k, 1) \frac{x^n}{n!} y^k &= y(e^x - 1) \exp \{y(e^x - 1)\} = y \frac{\partial}{\partial y} \exp \{y(e^x - 1)\} \\ &= \sum_{n, k} k S(n, k) \frac{x^n}{n!} y^k, \quad \text{by (2.4)}. \end{aligned}$$

Thus, we again get

$$\bar{S}(n, k, 1) = kS(n, k).$$

By (1.2),

$$\lambda^m = \sum_{j=0}^m S(m, j)j! \binom{\lambda}{j}.$$

Thus, (2.5) becomes

$$\begin{aligned} \bar{S}(n, k, \lambda) &= \sum_{m=1}^{n-k+1} \binom{n}{m} S(n-m, k-1) \sum_{j=1}^m S(m, j)j! \binom{\lambda}{j} \\ &= \sum_{j=1}^{n-k+1} j! \binom{\lambda}{j} \sum_{m=j}^n \binom{n}{m} S(m, j)S(n-m, k-1). \end{aligned}$$

The inner sum is equal to

$$\binom{j+k-1}{j} S(n, j+k-1),$$

so that

$$\begin{aligned} \bar{S}(n, k, \lambda) &= \sum_{j=1}^{n-k+1} j! \binom{\lambda}{j} \binom{j+k-1}{j} S(n, j+k-1) \\ (2.6) \quad &= \sum_{j=1}^{n-k+1} (k)_j S(n, j+k-1) \binom{\lambda}{j}. \end{aligned}$$

Hence,

$$(2.7) \quad S(n, k, \lambda) = \frac{1}{k} \bar{S}(n, k, \lambda) = \sum_{j=1}^{n-k+1} (k+1)_{j-1} S(n, j+k-1) \binom{\lambda}{j}.$$

Thus, for λ an integer, $S(n, k, \lambda)$ is an integer. For example, we have

$$S(n, k, 1) = S(n, k)$$

$$S(n, k, 2) = 2S(n, k) + (k+1)S(n, k+1)$$

$$S(n, k, 3) = 3S(n, k) + 3(k+1)S(n, k+2).$$

It follows readily from (2.7) that

$$\begin{aligned} (2.8) \quad &\sum_{t=0}^m (-1)^t \binom{m}{t} S(n, k, \lambda-t) \\ &= \sum_{j=m}^{n-k+1} (k+1)_{j-1} S(n, j+k-1) \binom{\lambda-m}{j-m}, \quad (m \geq 1). \end{aligned}$$

This result holds for all λ . However, if λ is a positive integer, then

$$(2.9) \quad \sum_{t=0}^{\lambda} (-1)^t \binom{\lambda}{t} S(n, k, \lambda-t) = (k+1)_{\lambda-1} S(n, \lambda+k-1),$$

and

$$(2.10) \quad \begin{aligned} & \sum_{t=0}^{\lambda+1} (-1)^t \binom{\lambda+1}{t} S(n, k, \lambda-t) \\ &= \sum_{j=\lambda+1}^{n-k+1} (-1)^{j-\lambda-1} (k+1)_{j-1} S(n, j+k-1). \end{aligned}$$

3. THE FUNCTION $R(n, k, \lambda)$

It is convenient to define

$$(3.1) \quad R(n, k, \lambda) = \bar{S}(n, k+1, \lambda) + S(n, k).$$

Thus, (2.5) implies

$$(3.2) \quad R(n, k, \lambda) = \sum_{m=0}^{n-k} \binom{n}{m} \lambda^m S(n-m, k),$$

while (2.7) gives

$$(3.3) \quad R(n, k, \lambda) = \sum_{j=0}^{n-k} (k+1)_j S(n, j+k) \binom{\lambda}{j}.$$

Multiplying (3.2) by $k! \binom{y}{k}$ and summing over k , we get

$$\begin{aligned} \sum_{k=0}^n k! \binom{y}{k} R(n, k, \lambda) &= \sum_{m=0}^n \binom{n}{m} \lambda^m \sum_{k=0}^{n-m} S(n-m, k) y(y-1) \cdots (y-k+1) \\ &= \sum_{m=0}^n \binom{n}{m} \lambda^m y^{n-m}. \end{aligned}$$

Hence,

$$(3.4) \quad \sum_{k=0}^n k! \binom{y}{k} R(n, k, \lambda) = (y+\lambda)^n.$$

It follows from (3.4) that

$$(3.5) \quad \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{k=0}^n k! \binom{y}{k} R(n, k, \lambda) = e^{x(y+\lambda)}.$$

To obtain a recurrence for $R(n, k, \lambda)$, take

$$\begin{aligned} \sum_{k=0}^n k! \binom{y}{k} (R(n+1, k, \lambda) - \lambda R(n, k, \lambda)) &= (y+\lambda)^{n+1} - \lambda(y+\lambda)^n \\ &= y(y+\lambda)^n. \end{aligned}$$

Since

$$k! \binom{y}{k} y = (k+1)! \binom{y}{k+1} + k \cdot k! \binom{y}{k},$$

it is clear that (3.4) gives

$$R(n+1, k, \lambda) - \lambda R(n, k, \lambda) = kR(n, k, \lambda) + R(n, k-1, \lambda),$$

that is

$$(3.6) \quad R(n+1, k, \lambda) = (\lambda + k)R(n, k, \lambda) + R(n, k-1, \lambda).$$

An equivalent result is

$$(3.7) \quad \bar{S}(n+1, k+1, \lambda) = (\lambda + k)\bar{S}(n, k+1, \lambda) + \bar{S}(n, k, \lambda) + S(n, k).$$

To get an explicit formula for $R(n, k, \lambda)$ we recall that

$$S(n, k) = \frac{1}{k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} j^n.$$

Thus, by (3.2),

$$R(n, k, \lambda) = \frac{1}{k!} \sum_{m=0}^{n-k} \binom{n}{m} \lambda^m \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} j^{n-m}.$$

For $n-k < m \leq n$, the inner sum vanishes, so that

$$\begin{aligned} R(n, k, \lambda) &= \frac{1}{k!} \sum_{m=0}^n \binom{n}{m} \lambda^m \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} j^{n-m} \\ &= \frac{1}{k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} \sum_{m=0}^n \binom{n}{m} \lambda^m j^{n-m}. \end{aligned}$$

Thus,

$$(3.8) \quad R(n, k, \lambda) = \frac{1}{k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} (\lambda + j)^n = \frac{1}{k!} \Delta^k \lambda^n.$$

It follows from (3.8) that

$$(3.9) \quad \sum_{n=k}^{\infty} R(n, k, \lambda) \frac{z^n}{n!} = \frac{1}{k!} e^{\lambda z} (e^z - 1)^k$$

in agreement with previous results. Also, since

$$\begin{aligned} \frac{1}{k!} \sum_{n=0}^{\infty} z^n \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} (\lambda + j)^n &= \frac{1}{k!} \sum_{j=0}^k \frac{(-1)^{k-j} \binom{k}{j}}{1 - (\lambda + j)z} \\ &= \frac{z^k}{(1 - \lambda z)(1 - (\lambda + 1)z) \dots (1 - (\lambda + k)z)}, \end{aligned}$$

we have

$$(3.10) \quad \sum_{n=0}^{\infty} R(n, k, \lambda) z^n = \frac{z^k}{(1 - \lambda z)(1 - (\lambda + 1)z) \dots (1 - (\lambda + k)z)}.$$

We also note that (3.9) implies the "addition theorem":

$$(3.11) \quad R(n, j+k, \lambda + \mu) = \binom{j+k}{j}^{-1} \sum_{m=0}^n \binom{n}{m} R(m, j, \lambda) R(n-m, k, \mu).$$

By the recurrence (3.6) together with $R(0, 0, \lambda) = 1$, or by means of (3.8), we have

$$(3.12) \quad R(n, 0, \lambda) = \lambda^n, \quad R(n, n, \lambda) = 1.$$

Moreover, if we put

$$x^n = \sum_{k=0}^n \bar{R}(n, k, \lambda) (x - \lambda)(x - \lambda - 1) \cdots (x - \lambda - k + 1),$$

then

$$\bar{R}(n+1, k, \lambda) = (\lambda + k)\bar{R}(n, k, \lambda) + \bar{R}(n, k-1, \lambda),$$

so that $\bar{R}(n, k, \lambda) = R(n, k, \lambda)$. Thus, we have

$$(3.13) \quad y^n = \sum_{k=0}^n R(n, k, \lambda) (y - \lambda)(y - \lambda - 1) \cdots (y - \lambda - k + 1),$$

or, replacing y by $-y$,

$$(3.14) \quad y^n = \sum_{k=0}^n (-1)^{n-k} R(n, k, \lambda) (y + \lambda)_k.$$

This, of course, is equivalent to (3.4).

It is clear from (3.8) or (3.13) that

$$(3.15) \quad R(n, k, 0) = S(n, k).$$

For $\lambda = 1$, since $\bar{S}(n, k, 1) = kS(n, k)$, then by (3.1)

$$R(n, k, 1) = (k+1)S(n, k+1) + S(n, k),$$

so that

$$(3.16) \quad R(n, k, 1) = S(n+1, k+1).$$

The function

$$(3.17) \quad B(n, \lambda) = \sum_{k=0}^n R(n, k, \lambda)$$

evidently reduces, for $\lambda = 0$, to the Bell number [1, p. 210]

$$B(n) = \sum_{k=0}^n S(n, k).$$

A few formulas may be noted. It follows from (3.2) that

$$(3.18) \quad B(n, \lambda) = \sum_{m=0}^n \binom{n}{m} \lambda^m B(n-m).$$

Also, by (3.9), we have

$$(3.19) \quad \sum_{n=0}^{\infty} B(n, \lambda) \frac{z^n}{n!} = e^{\lambda z} \exp(e^z - 1),$$

which, indeed, is implied by (3.18).

Differentiation of (3.19) gives

$$\sum_{n=0}^{\infty} B(n+1, \lambda) \frac{z^n}{n!} = \lambda e^{\lambda z} \exp(e^z - 1) + e^{(\lambda+1)z} \exp(e^z - 1).$$

Hence,

$$(3.20) \quad \begin{aligned} B(n+1, \lambda) &= \lambda B(n, \lambda) + B(n, \lambda+1) \\ &= B(n, \lambda) + \sum_{m=0}^n \binom{n}{m} B(m, \lambda). \end{aligned}$$

Iteration of the first half of (3.20) gives

$$(3.21) \quad B(n+m, \lambda) = \sum_{j=0}^m \frac{1}{j!} \Delta^j \lambda^m \cdot B(n, \lambda+j),$$

as can be proved by induction on m . Incidentally, by (3.8), (3.21) can be written in the form

$$(3.22) \quad B(n+m, \lambda) = \sum_{j=0}^m R(m, j, \lambda) B(n, \lambda+j).$$

To anticipate the first result in Section 6, the inverse of (3.22) is

$$(3.23) \quad B(n, \lambda+m) = \sum_{j=0}^m (-1)^{m-j} R_1(m, j, \lambda) B(n+j, \lambda),$$

where $R_1(m, j, \lambda)$ is defined by (5.1).

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Returning to (3.9), note that

$$\begin{aligned} \sum_{n=k}^{\infty} R(n, k, \lambda+1) \frac{z^n}{n!} &= \frac{1}{k!} e^{(\lambda+1)z} (e^z - 1)^k \\ &= \frac{1}{k!} e^{\lambda z} (e^z - 1)^{k+1} + \frac{1}{k!} e^{\lambda z} (e^z - 1)^k, \end{aligned}$$

which implies

$$(3.24) \quad R(n, k, \lambda+1) = (k+1)R(n, k+1, \lambda) + R(n, k, \lambda).$$

More generally, since

$$e^{mz} = ((e^z - 1) + 1)^m = \sum_{j=0}^m \binom{m}{j} (e^z - 1)^j,$$

we get

$$(3.25) \quad R(n, k, \lambda+m) = \sum_{j=0}^m \binom{m}{j} (k+1)_j R(n, k+j, \lambda).$$

We may also write (3.24) in the form

$$(3.26) \quad \Delta_\lambda R(n, k, \lambda) = (k+1)R(n, k+1, \lambda),$$

where Δ_λ is the finite difference operator. Iteration of (3.26) gives

$$(3.27) \quad \Delta_\lambda^m R(n, k, \lambda) = (k+1)_m R(n, k+m, \lambda).$$

4. THE FUNCTION $\bar{S}_1(n, k, \lambda)$

Corresponding to (2.2), we define

$$(4.1) \quad S_1(n; k_1, k_2, \dots, k_n; \lambda) = k_1(\lambda)_1 + k_2 \frac{(\lambda)_2}{1!} + \dots + k_n \frac{(\lambda)_n}{(n-1)!},$$

where the inner summation is over all permutations of Z_n ,

$$n = k_1 + 2k_2 + \dots + nk_n,$$

with k_1 cycles of length 1, k_2 cycles of length 2, ..., k_n cycles of length n . Then (compare [2, p. 68]), we have

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{x^n}{n!} \sum_{k_1, k_2, \dots} S_1(n; k_1, k_2, \dots, k_n; \lambda) \frac{y_1^{k_1} y_2^{k_2} \dots}{k_1! k_2! \dots} \\ &= \sum_{n=1}^{\infty} \frac{x^n}{n!} \sum_{k_1, k_2, \dots} \left\{ k_1(\lambda)_1 + k_2 \frac{(\lambda)_2}{1!} + \dots + k_n \frac{(\lambda)_n}{(n-1)!} \right\} \frac{n!}{1^{k_1} 2^{k_2} \dots n^{k_n}} \frac{y_1^{k_1} y_2^{k_2} \dots}{k_1! k_2! \dots} \\ &= \left\{ \frac{(\lambda)_1}{1!} y_1 x + \frac{(\lambda)_2}{2!} y_2 x^2 + \frac{(\lambda)_3}{3!} y_3 x^3 + \dots \right\} \exp \left\{ y_1 x + \frac{1}{2} y_2 x^2 + \frac{1}{3} y_3 x^3 + \dots \right\}. \end{aligned}$$

For $y_1 = y_2 = \dots = y$, the extreme right member becomes

$$y((1-x)^{-\lambda} - 1)(1-x)^{-y}.$$

Hence, we get

$$(4.2) \quad \sum_{n,k} \bar{S}_1(n, k, \lambda) \frac{x^n}{n!} y^k = y((1-x)^{-\lambda} - 1)(1-x)^{-y},$$

where

$$(4.3) \quad \bar{S}_1(n, k, \lambda) = \sum S_1(n; k_1, k_2, \dots, k_n; \lambda),$$

and the summation on the right is over all nonnegative k_1, k_2, \dots, k_n satisfying $n = k_1 + 2k_2 + \dots + nk_n$.

Since (see [2, p. 71]),

$$(4.4) \quad \sum_{n,k} S_1(n, k) \frac{x^n}{n!} y^k = (1-x)^{-y},$$

it follows from (4.2) that

$$\begin{aligned} & \sum_{n,k} \bar{S}_1(n, k+1, \lambda) \frac{x^n}{n!} y^k = \sum_{n,k} S_1(n, m) \frac{x^n}{n!} ((\lambda+y)^m - y^m) \\ &= \sum_{n,m} S_1(n, m) \frac{x^n}{n!} \sum_{k=0}^{m-1} \binom{m}{k} \lambda^{m-k} y^k = \sum_{n,k} \frac{x^n}{n!} y^k \sum_{m=k+1}^n \binom{m}{k} \lambda^{m-k} S_1(n, m). \end{aligned}$$

Therefore,

$$(4.5) \quad \bar{S}_1(n, k+1, \lambda) = \sum_{j=1}^{n-k} \binom{j+k}{j} \lambda^j S_1(n, j+k).$$

In the next place, it also follows from (4.2) that

$$\begin{aligned} \sum_{n,k} \bar{S}_1(n, k+1, \lambda) \frac{x^n}{n!} y^k &= ((1-x)^{-\lambda} - 1)(1-x)^{-y} \\ &= \sum_{m=1}^{\infty} (\lambda)_m \frac{x^m}{m!} \sum_{n,k} S_1(n, k) \frac{x^n}{n!} y^k. \end{aligned}$$

Equating coefficients, we get

$$\begin{aligned} \bar{S}_1(n, k+1, \lambda) &= \sum_{m=1}^{n-k} \binom{n}{m} (\lambda)_m S_1(n-m, k) \\ &= \sum_{m=1}^{n-k} \frac{(\lambda)_m}{m!} n(n-1) \cdots (n-m+1) S_1(n-m, k). \end{aligned}$$

Thus,

$$(4.7) \quad \begin{aligned} S_1(n, k+1, \lambda) &= \frac{1}{n} \bar{S}_1(n, k+1, \lambda) \\ &= \sum_{m=1}^{n-k} \frac{(\lambda)_m}{m!} (n-1) \cdots (n-m+1) S_1(n-m, k). \end{aligned}$$

It follows at once from (4.7) that, for λ integral, $S_1(n, k+1, \lambda)$ is also integral.

It is evident from (4.1) and (4.3) that

$$(4.8) \quad \bar{S}_1(n, k, 1) = n S_1(n, k).$$

Thus, for example, (4.5) and (4.6) yield

$$(4.9) \quad \sum_{j=1}^{n-k} \binom{j+k}{j} S_1(n, j+k) = n S_1(n, k+1),$$

and

$$(4.10) \quad \sum_{m=1}^{n-k} n(n-1) \cdots (n-m+1) S_1(n-m, k) = n S_1(n, k+1),$$

respectively.

5. THE FUNCTION $R_1(n, k, \lambda)$

We define the function $R_1(n, k, \lambda)$ by means of

$$(5.1) \quad R_1(n, k, \lambda) = \bar{S}_1(n, k+1, \lambda) + S_1(n, k).$$

Then, by (4.5),

$$(5.2) \quad R_1(n, k, \lambda) = \sum_{j=0}^{n-k} \binom{j+k}{j} \lambda^j S_1(n, j+k),$$

and by (4.6),

$$(5.3) \quad R_1(n, k, \lambda) = \sum_{m=0}^{n-k} \binom{n}{m} (\lambda)_m S_1(n-m, k) \\ = \sum_{m=0}^{n-k} \frac{(\lambda)_m}{m!} n(n-1) \cdots (n-m+1) S_1(n-m, k).$$

It is also evident from (4.2) and (4.4) that

$$(5.4) \quad \sum_{n,k} R_1(n, k, \lambda) \frac{x^n}{n!} y^k = (1-x)^{-\lambda-y}.$$

Differentiation of (5.4) with respect to x gives

$$\sum_{n,k} R_1(n+1, k, \lambda) \frac{x^n}{n!} y^k = (\lambda+y)(1-x)^{-\lambda-y-1},$$

so that

$$(1-x) \sum_{n,k} R_1(n+1, k, \lambda) \frac{x^n}{n!} y^k = (\lambda+y) \sum_{n,k} R_1(n, k, \lambda) \frac{x^n}{n!} y^k.$$

Equating coefficients, we get

$$R_1(n+1, k, \lambda) = nR_1(n, k, \lambda) = \lambda R_1(n, k, \lambda) + R_1(n, k-1, \lambda),$$

that is,

$$(5.5) \quad R_1(n+1, k, \lambda) = (\lambda+n)R_1(n, k, \lambda) + R_1(n, k-1, \lambda).$$

It follows at once from (5.5) and $R_1(0, 0, \lambda) = 1$ that

$$(5.6) \quad R_1(n, 0, \lambda) = (\lambda)_n, \quad R_1(n, n\lambda) = 1.$$

Also, taking $y = 1$ in (5.4), we get

$$(5.7) \quad \sum_{k=0}^n R_1(n, k, \lambda) = (\lambda+1)_n.$$

More generally, we have

$$(5.8) \quad \sum_{k=0}^n R_1(n, k, \lambda) y^k = (\lambda+y)_n.$$

Clearly, (5.5) is implied by (5.8).

It is clear from (5.4) that

$$(5.9) \quad R_1(n, k, 0) = S_1(n, k).$$

For $\lambda = 1$, we have, by (4.8) and (5.1),

$$(5.10) \quad R_1(n, k, 1) = S_1(n+1, k+1).$$

These formulas may be compared with (3.15) and (3.16).

In view of (5.10), (5.2) and (5.3) reduce to

$$(5.11) \quad S_1(n+1, k+1) = \sum_{j=0}^{n-k} \binom{j+k}{j} S_1(n, j+k),$$

and

$$(5.12) \quad S_1(n+1, k+1) = \sum_{m=0}^{n-k} n(n-1) \cdots (n-m+1) S_1(n-m, k).$$

It is not difficult to give direct proofs of (5.11) and (5.12).

Returning to (5.4), note that

$$(1-x) \sum_{n,k} R_1(n, k, \lambda+1) \frac{x^n}{n!} y^k = (1-x)^{-\lambda-y}.$$

This gives

$$(5.13) \quad R_1(n, k, \lambda) = R_1(n, k, \lambda+1) - nR_1(n-1, k, \lambda+1),$$

and generally,

$$(5.14) \quad R_1(n, k, \lambda) = \sum_{j=0}^m (-1)^j \binom{m}{j} n(n-1) \cdots (n-j+1) R_1(n-j, k, \lambda+m).$$

The inverse of (5.14) is

$$(5.15) \quad R_1(n, k, \lambda+m) = \sum_{j=0}^n \binom{n}{j} (m)_j R_1(n-j, k, \lambda).$$

We may write (5.13) in the form

$$(5.16) \quad \Delta_\lambda R_1(n, k, \lambda) = nR_1(n-1, k, \lambda+1).$$

Iteration gives

$$(5.17) \quad \Delta_\lambda^m R_1(n, k, \lambda) = n(n-1) \cdots (n-m+1) R_1(n-m, k, \lambda+m).$$

6. ORTHOGONALITY RELATIONS

Comparing (5.8) with (3.14), we have immediately the orthogonality relations

$$(6.1) \quad \begin{aligned} & \sum_{k=0}^n (-1)^{n-k} R(n, k, \lambda) R_1(k, j, \lambda) \\ &= \sum_{k=0}^n R_1(n, k, \lambda) \cdot (-1)^{k-j} R(k, j, \lambda) = \delta_{n,j}, \end{aligned}$$

the Kronecker delta.

It is of some interest to give a proof of (6.1) making use of (3.2) and (5.2). We have

$$\begin{aligned} & \sum_{k=0}^n (-1)^{n-k} R(n, k, \lambda) R_1(k, j, \lambda) \\ &= \sum_{k=0}^n (-1)^{n-k} \sum_{m=0}^{n-k} \binom{n-k}{m} \lambda^m S(n-m, k) \sum_{t=0}^{k-j} \binom{j+t}{t} \lambda^t S_1(k, k+t) \\ &= \sum_{m=0}^n \sum_{t=0}^{n-j} (-1)^m \binom{n}{m} \binom{j+t}{t} \lambda^{m+t} \sum_{k=0}^{n-m} (-1)^{n-m-k} S(n-m, k) S_1(k, j+t). \end{aligned}$$

The inner sum is equal to 1 if $n - m = j + t$, and vanishes otherwise. Thus, we have

$$\lambda^{n-j} \sum_{m=0}^n (-1)^m \binom{n}{m} \binom{n-m}{j} = \lambda^{n-j} \sum_{m=0}^n (-1)^{n-m} \binom{n}{m} \binom{m}{j} = \delta_{n,j},$$

so that

$$(6.2) \quad \sum_{k=0}^n (-1)^{n-k} R(n, k, \lambda) R_1(k, j, \lambda) = \delta_{n,j}.$$

As for the second half of (6.1), we have

$$\begin{aligned} & \sum_{k=0}^n R_1(n, k, \lambda) \cdot (-1)^{k-j} R(k, j, \lambda) \\ &= \sum_{k=0}^n \sum_{t=0}^{n-k} \binom{t+k}{t} \lambda^t S_1(n, t+k) \cdot (-1)^{k-j} \sum_{m=0}^{k-j} \binom{k}{m} \lambda^m S(k-m, j) \\ &= \sum_{k=0}^n \sum_{t=k}^n \binom{t}{k} \lambda^{t-k} S_1(n, t) \cdot (-1)^{k-j} \sum_{m=j}^k \binom{k}{m} \lambda^{k-m} S(m, j) \\ &= \sum_{t=0}^n \sum_{m=j}^n (-1)^{t-j} \lambda^{t-m} S_1(n, t) S(m, j) \sum_{k=0}^t (-1)^{t-k} \binom{t}{k} \binom{k}{m} \\ &= \sum_{t=0}^n \sum_{m=j}^n (-1)^{t-j} \lambda^{t-m} S_1(n, t) S(m, j) \delta_{t,m} \\ &= \sum_{t=j}^n (-1)^{t-j} S_1(n, t) S(t, j) = \delta_{n,j}. \end{aligned}$$

This, together with (6.2), completes the proof of (6.1).

The proof of (6.2) above suggests a more general result. As in the above proof, we have

$$\begin{aligned} \sum_{k=0}^n (-1)^{n-k} R(n, k, \lambda) R_1(k, j, \mu) &= \sum_{m=0}^n \sum_{t=0}^{n-j} (-1)^m \binom{n}{m} \binom{j+t}{j} \lambda^m \mu^t \delta_{n-m, j+t} \\ &= \sum_{m=0}^n (-1)^m \binom{n}{m} \binom{n-m}{j} \lambda^m \mu^{n-m-j} \\ &= \sum_{m=j}^n (-1)^{n-m} \binom{n}{m} \binom{m}{j} \lambda^{n-m} \mu^{m-j} \\ &= \binom{n}{j} \sum_{m=1}^n (-1)^{n-m} \binom{n-j}{m-j} \lambda^{n-m} \mu^{m-j} \\ &= (-1)^{n-j} \binom{n}{j} \sum_{m=0}^{n-j} (-1)^m \binom{n-j}{m} \lambda^{n-j-m} \mu^m, \end{aligned}$$

and therefore,

$$(6.3) \quad \sum_{k=0}^n (-1)^{n-k} R(n, k, \lambda) R_1(k, j, \mu) = \binom{n}{j} (\mu - \lambda)^{n-j}.$$

For $\mu = \lambda$, (6.3) reduces to (6.2).

In the next place

$$\begin{aligned} & \sum_{k=0}^n R_1(n, k, \mu) \cdot (-1)^{k-j} R(k, j, \lambda) \\ &= \sum_{k=0}^n \sum_{t=k}^n \binom{t}{k} \mu^{t-k} S_1(n, t) \cdot (-1)^{k-j} \sum_{m=j}^k \binom{k}{m} \lambda^{k-m} S(m, j) \\ &= \sum_{t=0}^n \sum_{m=j}^n (-1)^{t-j} \binom{t}{m} S_1(n, t) S(m, j) \sum_{k=m}^t (-1)^{t-k} \binom{t-m}{k-m} \mu^{t-k} \lambda^{k-m} \\ &= \sum_{t=0}^n \sum_{m=j}^t (-1)^{t-j} \binom{t}{m} S_1(n, t) S(m, j) (\lambda - \mu)^{t-m}. \end{aligned}$$

Let $U(n, j)$ denote this sum. Then,

$$\begin{aligned} \sum_{j=0}^n (-1)^j U(n, j) j! \binom{x}{j} &= \sum_{t=0}^n \sum_{m=0}^t (-1)^t \binom{t}{m} S_1(n, t) (\lambda - \mu)^{t-m} \sum_{j=0}^m S(m, j) j! \binom{x}{j} \\ &= \sum_{t=0}^n \sum_{m=0}^t (-1)^t \binom{t}{m} S_1(n, t) (\lambda - \mu)^{t-m} x^m \\ &= \sum_{t=0}^n (-1)^t S_1(n, t) (x + \lambda - \mu)^t \\ &= (-1)^n (x + \lambda - \mu) (x + \lambda - \mu - 1) \cdots (x + \lambda - \mu - n + 1). \end{aligned}$$

Replacing x by $-x$, this becomes

$$(6.4) \quad \sum_{j=0}^n U(n, j) (x)_j = (x - \lambda + \mu)_n.$$

Since

$$(x + y)_n = \sum_{j=0}^n \binom{n}{j} (x)_j (y)_{n-j},$$

it follows from (6.4) that

$$U(n, j) = \binom{n}{j} (\mu - \lambda)_{n-j}.$$

Therefore, we have

$$(6.5) \quad \sum_{k=0}^n R_1(n, k, \mu) \cdot (-1)^{k-j} R(k, j, \lambda) = \binom{n}{j} (\mu - \lambda)_{n-j}.$$

This result may be compared with (6.3). If we define matrices

$$M = [(-1)^{n-k} R(n, k, \lambda)] \quad (n, k = 0, 1, 2, \dots),$$

and

$$M_1 = [R_1(n, k, \mu)] \quad (n, k = 0, 1, 2, \dots),$$

then (6.3) and (6.5) become

$$(6.3)' \quad MM_1 = \left[\binom{n}{k} (\lambda - \mu)^{n-k} \right],$$

and

$$(6.5)' \quad M_1 M = \left[\binom{n}{k} (\mu - \lambda)_{n-k} \right],$$

respectively.

7. COMBINATORIAL INTERPRETATION OF $R(n, k, \lambda)$ AND $R_1(n, k, \lambda)$

Let λ be a nonnegative integer and let $B_1, B_2, \dots, B_\lambda$ denote λ open boxes. Let $P(n, k, \lambda)$ denote the number of partitions of $Z_n = \{1, 2, \dots, n\}$ into k blocks with the understanding that an arbitrary number of the elements of Z_n may be placed in any number (possibly none) of the boxes. For brevity, we shall call these " λ -partitions." Clearly,

$$(7.1) \quad P(n, k, 0) = S(n, k).$$

To evaluate $P(n, 0, \lambda)$, we place x_1 elements of Z_n in B_1, x_2 in B_2, \dots, x_λ in B_λ . Thus,

$$P(n, 0, \lambda) = \sum_{x_1 + x_2 + \dots + x_\lambda = n} \frac{n!}{x_1! x_2! \dots x_\lambda!}.$$

Hence,

$$(7.2) \quad P(n, 0, \lambda) = \lambda^n.$$

Also, clearly,

$$(7.3) \quad P(0, k, \lambda) = \delta_{0,k}.$$

To get a recurrence for $P(n, k, \lambda)$, we consider the effect of adding the element $n+1$ to a λ -partition of Z_n into k blocks. The added element may be placed in any of the blocks or any of the boxes without changing the value of k . On the other hand, if it constitutes an additional block, then of course the number of blocks becomes $k+1$. Thus, we have

$$(7.4) \quad P(n+1, k, \lambda) = (\lambda + k)P(n, k, \lambda) + P(n, k-1, \lambda).$$

Since

$$P(0, k, \lambda) = R(0, k, \lambda) = \delta_{0,k},$$

comparison of (7.4) with (3.6) gives

$$(7.5) \quad P(n, k, \lambda) = R(n, k, \lambda).$$

Hence, $R(n, k, \lambda)$ is equal to the number of λ -partitions of Z_n into k blocks.

Turning next to $R_1(n, k, \lambda)$, again let $B_1, B_2, \dots, B_\lambda$ denote λ open boxes. Let $P_1(n, k, \lambda)$ denote the number of permutations of Z_n with k cycles with the understanding that an arbitrary number of the elements of Z_n may be placed in any number (possibly none) of the boxes and then permuted in all possible ways in each box. For brevity, we call these " λ -permutations."

Clearly,

$$(7.6) \quad P_1(n, k, 0) = S_1(n, k).$$

To evaluate $P(n, 0, \lambda)$, note that $P(1, 0, \lambda) = \lambda$ and

$$P(n+1, 0, \lambda) = (\lambda + n)P(n, 0, \lambda),$$

since the element $n+1$ may occupy any one of the $n+\lambda$ positions. Thus,

$$(7.7) \quad P_1(n, 0, \lambda) = (\lambda)_n.$$

Also clearly,

$$(7.8) \quad P_1(0, k, \lambda) = \delta_{0,k}.$$

A recurrence for $P_1(n, k, \lambda)$ is obtained using the method of proof of (7.4); however, there are now $\lambda+n$ possible positions for the element $n+1$. Thus, we get

$$(7.9) \quad P_1(n+1, k, \lambda) = (\lambda+n)P_1(n, k, \lambda) + P_1(n, k-1, \lambda).$$

Comparison of (7.9) with (5.5) gives

$$(7.10) \quad P_1(n, k, \lambda) = R_1(n, k, \lambda).$$

Hence, $R_1(n, k, \lambda)$ is equal to the number of λ -permutations of Z_n with k cycles.

We remark that (7.5) can also be proved using (3.2) and that (7.10) can be proved using (5.3).

Finally, we note that the generalized Bell number defined by (3.17),

$$B(n, \lambda) = \sum_{k=0}^n R(n, k, \lambda),$$

is equal to the total number of λ -partitions of Z_n .

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