Available online at www.sciencedirect.com
ScienceDirect
Discrete Mathematics 307 (2007) 2359–2375
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Notes on degenerate numbers

M. Cenkci[a], F.T. Howard[b]

[a]Department of Mathematics, Akdeniz University, 07058, Antalya, Turkey
[b]Department of Mathematics, Wake Forest University, Winston-Salem, NC 27109, USA

Received 17 January 2004; received in revised form 20 March 2006; accepted 2 October 2006
Available online 2 January 2007

Abstract

In this paper, using a theorem relating the potential polynomial $F_k^{(z)}$ and the exponential Bell polynomial $B_{n,j}(0,\ldots,0,f_r,f_{r+1},\ldots)$, we obtain some explicit formulas for higher order degenerate Bernoulli numbers of the first and second kinds. We also prove new recurrence formulas for these numbers. Furthermore, we discuss other applications of the theorem, from which we deduce several formulas for degenerate Genocchi numbers, degenerate tangent numbers, and the coefficients of the higher order degenerate Euler polynomials. Finally, we examine the polynomials $V(k,j,z|\lambda)$ and $V_1(k,l,z|\lambda)$, and, in particular, we show how these polynomials are related to the degenerate Bernoulli, Genocchi, tangent, and van der Pol numbers, and the numbers generated by the reciprocal of $(1+\lambda x)^{1/\lambda}-x-1$.

MSC: 11B68; 11B73; 11B37

Keywords: Potential and Bell polynomials; Bernoulli and Stirling numbers

1. Introduction

For $r\geqslant 0$ and $f_r\neq 0$, let $F(x)=\sum_{k=r}^{\infty} f_k x^k/k!$ be a formal power series. For a variable z, we define the potential polynomial $F_k^{(z)}$ by means of

$$\left(\frac{f_r x^r/r!}{F(x)}\right)^z=\sum_{k=0}^{\infty}F_k^{(z)}\frac{x^k}{k!}, \tag{1.1}$$

and if $r\geqslant 1$, we define the exponential Bell polynomial $B_{n,j}(0,\ldots,0,f_r,f_{r+1},\ldots)$ by

$$(F(x))^j=j!\sum_{n=0}^{\infty}B_{n,j}(0,\ldots,0,f_r,f_{r+1},\ldots)\frac{x^n}{n!}. \tag{1.2}$$

Thus if j is a positive integer,

$$F_k^{(-j)}=\left(\frac{r!}{f_r}\right)^j\frac{k!j!}{(k+rj)!}B_{k+rj,j}(0,\ldots,0,f_r,f_{r+1},\ldots).$$

E-mail addresses: cenkci@akdeniz.edu.tr (M. Cenkci), howard@wfu.edu (F.T. Howard).

doi:10.1016/j.disc.2006.10.013

Howard [9] stated the following theorem:

Theorem 1.1. *If* $F_k^{(z)}$ *is defined by* (1.1) *and if* $B_{n,j}$ *is defined by* (1.2), *then*

$$\binom{k-z}{k} F_k^{(z)} = \sum_{j=0}^{k} \left(\frac{r!}{f_r}\right)^j \binom{k+z}{k-j} \binom{k-z}{k+j} \frac{(k+j)!}{(k+rj)!} B_{k+rj,j}(0, \ldots, 0, f_r, f_{r+1}, \ldots)$$

$$= \sum_{j=0}^{k} \left(\frac{r!}{f_r}\right)^j \binom{k-z}{k+j} \frac{(k+j)!}{(k+rj)!} B_{k+rj,j}(0, \ldots, 0, f_{r+1}, \ldots).$$

In [9], Howard gave a proof of Theorem 1.1 and pointed out some of its applications.

In this paper we discuss other applications of Theorem 1.1. To the authors' knowledge, these applications are new and provide a general approach to a large class of degenerate numbers. In particular, these applications generalize some well-known formulas relating Bernoulli and Stirling numbers to the degenerate forms of those numbers, which is the main motivation of the paper.

A summary by sections follows:

Section 2 is a preliminary section containing the basic definitions, theorems, notation, and terminology we need. In Section 3, we examine the higher order degenerate Bernoulli numbers of the first and second kinds. In Section 4, we give new formulas for the higher order degenerate Bernoulli numbers of both kinds. In particular, we obtain formulas and recurrence relations for the higher order degenerate Bernoulli numbers of the first kind. In Section 5, we define the higher order degenerate Genocchi numbers, higher order degenerate tangent numbers, and the coefficients of the higher order degenerate Euler polynomials, and we obtain explicit formulas for them. Finally, in Section 6, we define the polynomials $V(k, j, z|\lambda)$ and $V_1(k, l, z|\lambda)$ by means of

$$\left(1-\frac{x}{2}\right)^z \{x((1+\lambda x)^{1/\lambda}+1) - 2((1+\lambda x)^{1/\lambda}-1)\}^j$$
$$= j! \sum_{k=2j}^{\infty} V(k, j, z|\lambda) \frac{x^k}{k!}$$

and

$$\left(1-\frac{x}{2}\right)^z \{x((1+\lambda x)^{1/\lambda}+1) - 2((1+\lambda x)^{1/\lambda}-1) - \lambda x^2\}^l$$
$$= l! \sum_{k=3l}^{\infty} V_1(k, l, z|\lambda) \frac{x^k}{k!},$$

respectively, and, in particular, we show how these polynomials are related to the degenerate Bernoulli, Genocchi, tangent, and van der Pol numbers, and the numbers generated by the reciprocal of $(1+\lambda x)^{1/\lambda} - x - 1$.

2. Preliminaries

Throughout this paper, we use the theory of formal power series as outlined in [4, pp. 36–43]. Also, if z is a variable, we use the following notation:

$$(z)_k = z(z-1)\cdots(z-k+1),$$
$$\langle z\rangle_k = z(z+1)\cdots(z+k-1),$$
$$\binom{z}{k} = \frac{(z)_k}{k!}.$$

We also need the binomial theorem [4, p. 37]: for a variable z,

$$(1+x)^z = \sum_{k=0}^{\infty} \binom{z}{k} x^k.$$

We now define the Stirling numbers. These numbers have been extensively studied and their properties are well known. See, for example [4, Chapter 5] and [12, Chapter 4]. We include here their definitions in terms of generating functions and a few basic properties.

The Stirling number of the first kind, $s(n, j)$, is defined by

$$s(n, j) = B_{n,j}(0!, -1!, 2!, -3!, 4! \ldots),$$

that is,

$$(\log(1+x))^j = \left(\sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k}\right)^j = j! \sum_{n=j}^{\infty} s(n, j) \frac{x^n}{n!}.$$

The unsigned Stirling number of the first kind, $\bar{s}(n, j)$, is defined by

$$\bar{s}(n, j) = B_{n,j}(0!, 1!, 2!, \ldots), \tag{2.1}$$

that is,

$$(-\log(1-x))^j = \left(\sum_{k=1}^{\infty} \frac{x^k}{k}\right)^j = j! \sum_{n=j}^{\infty} \bar{s}(n, j) \frac{x^n}{n!}.$$

It is well known [4, p. 213] that

$$\langle z \rangle_n = \sum_{j=0}^{n} \bar{s}(n, j) z^j, \tag{2.2}$$

and, in fact, (2.2) is often used as the definition of $\bar{s}(n, j)$. We also have

$$\bar{s}(n+1, j) = n\bar{s}(n, j) + \bar{s}(n, j-1). \tag{2.3}$$

The Stirling number of the second kind, $S(n, j)$, is defined by

$$S(n, j) = B_{n,j}(1, 1, 1, \ldots), \tag{2.4}$$

that is,

$$(e^x - 1)^j = \left(\sum_{k=1}^{\infty} \frac{x^k}{k!}\right)^j = j! \sum_{n=j}^{\infty} S(n, j) \frac{x^n}{n!}.$$

The recurrence formula is

$$S(n+1, j) = jS(n, j) + S(n, j-1). \tag{2.5}$$

Carlitz [3] has defined the degenerate Stirling numbers of the first and second kinds, $\bar{s}(n, j|\lambda)$ and $S(n, j|\lambda)$, by means of

$$\left(\frac{1-(1-x)^{\lambda}}{\lambda}\right)^j = j! \sum_{n=j}^{\infty} \bar{s}(n, j|\lambda) \frac{x^n}{n!} \tag{2.6}$$

and

$$((1+\lambda x)^{1/\lambda}-1)^j = j!\sum_{n=j}^{\infty} S(n,j|\lambda)\frac{x^n}{n!}, \tag{2.7}$$

respectively. The limiting case $\lambda=0$ gives the ordinary Stirling numbers. Quite a few properties of degenerate Stirling numbers have been worked out in [3,7]. For example, we have the recurrence formulas

$$\bar{s}(n+1,k|\lambda) = (n-k\lambda)\bar{s}(n,k|\lambda) + \bar{s}(n,k-1|\lambda), \tag{2.8}$$

$$S(n+1,k|\lambda) = (k-n\lambda)S(n,k|\lambda) + S(n,k-1|\lambda). \tag{2.9}$$

A special case of (1.1) of interest is the higher order degenerate Bernoulli numbers of the first kind, $\beta_k^{(z)}(\lambda)$, defined by [3]

$$\left(\frac{x}{(1+\lambda x)^{1/\lambda}-1}\right)^z = \sum_{k=0}^{\infty} \beta_k^{(z)}(\lambda)\frac{x^k}{k!}. \tag{2.10}$$

The limiting case $\lambda=0$ is the Nörlund's polynomial $B_k^{(z)}$ [13, p. 146] and $\beta_k^{(1)}(\lambda)=\beta_k(\lambda)$ is the degenerate Bernoulli number. Degenerate Bernoulli numbers have been extensively studied in the past 10 years. In [11] Howard gave the explicit formula

$$\beta_k(\lambda) = k!\lambda^k b_k + \sum_{j=1}^{[k/2]} \frac{k}{2j} B_{2j}\lambda^{k-2j} s(k-1,2j-1), \tag{2.11}$$

where $[t]$ denotes the integer part of any real number t. Here B_{2j} is the ordinary Bernoulli number and b_k is the Bernoulli number of the second kind defined by [12, p. 279]

$$\frac{x}{\log(1+x)} = \sum_{k=0}^{\infty} b_k x^k.$$

Adelberg [1] developed a class of multivalued polynomials, which includes degenerate Bernoulli and Stirling polynomials and various generalizations. His approach is different from Carlitz's, which starts with generating functions, and is more general. Recently these numbers and polynomials have been covered from other arithmetical aspects. In [14] Young showed that degenerate Stirling and related numbers and similar sequences may in fact be expressed as p-adic integrals of generalized factorials. As an application of this identification he deduced systems of congruences which are analogues and generalizations of the Kummer congruences for the ordinary Bernoulli numbers. Furthermore, in [15] he proved as analogue of the Kummer congruences for expressions involving the degenerate Bernoulli polynomials by relating them to the general theory of "degenerate number sequences" developed in [14].

In [2], Carlitz defined the higher order Bernoulli numbers of the second kind, $b_k^{(z)}$, by means of

$$\left(\frac{x}{\log(1+x)}\right)^z = \sum_{k=0}^{\infty} b_k^{(z)} x^k. \tag{2.12}$$

Analogous to $\beta_k^{(z)}(\lambda)$, the polynomial $\alpha_k^{(z)}(\lambda)$ is defined by

$$\left(\frac{\lambda x}{1-(1-x)^{\lambda}}\right)^z = \sum_{k=0}^{\infty} \alpha_k^{(z)}(\lambda)\frac{x^k}{k!}. \tag{2.13}$$

For $\lambda=0$, we have the higher order Bernoulli numbers of the second kind, and for $\lambda=0$ and $z=1$, $\alpha_k^{(1)}(0)=(-1)^k k! b_k$.

If we differentiate both sides of (2.10) with respect to x and compare the coefficients of x, we have

$$\beta_k^{(z+1)}(\lambda) = \frac{z-k}{z}\beta_k^{(z)}(\lambda) + \frac{k}{z}(\lambda(z-k+1)-z)\beta_{k-1}^{(z)}(\lambda). \tag{2.14}$$

It follows from (2.8) and (2.14) that

$$\bar{s}(n, n-k|\lambda) = \binom{k-n}{k}\beta_k^{(n)}(\lambda), \quad 0 \leqslant k < n. \tag{2.15}$$

Similarly we have

$$\alpha_k^{(z+1)}(\lambda) = \frac{z-k}{z}\alpha_k^{(z)}(\lambda) - k\left(1-\lambda-\frac{k-1}{z}\right)\alpha_{k-1}^{(z)}(\lambda), \tag{2.16}$$

and (2.9), (2.16) yield

$$S(n, n-k|\lambda) = \binom{k-n}{k}\alpha_k^{(n)}(\lambda), \quad 0 \leqslant k < n. \tag{2.17}$$

Other special numbers and polynomials will be defined and used later in the paper as they are needed.

3. Higher order degenerate Bernoulli numbers

In this section we examine the higher order degenerate Bernoulli numbers of the first and second kinds, and we obtain explicit formulas for them. These formulas appear to be new; in particular, the formulas for the higher order degenerate Bernoulli numbers of the first kind are generalizations for the degenerate Bernoulli numbers, worked out by Howard [11] in detail.

Let $F(x) = f_0 + f_1 x + \cdots + f_n x^n/n! + \cdots$ be a formal exponential generating function. If $f_0 = 0$, $f_1 \neq 0$, we define

$$G(x) = \log(F(x)+1),$$

$$\left(\frac{x}{G(x)}\right)^k = \sum_{m=0}^{\infty} a_m^{(k)} x^m,$$

$$(G(x))^j = j! \sum_{m=j}^{\infty} T(m, j)\frac{x^m}{m!},$$

and if $f_0 = 0$, $f_1 = 1$,

$$\left(\frac{x}{F(x)}\right)^k = \sum_{m=0}^{\infty} F_m^{(k)}\frac{x^m}{m!}.$$

Theorem 3.1. *If $F_m^{(k)}$, $a_m^{(k)}$, and $T(m, j)$ are defined as above, then for positive integers k and m with $m \geqslant k$, we have*

$$\frac{F_m^{(k)}}{m!} = \sum_{j=0}^{k-1}\frac{B_j^{(k)}}{j!}a_{m-j}^{(k-j)} + \sum_{j=0}^{m-k}\frac{B_{j+k}^{(k)}}{(j+k)\cdots(j+1)}\frac{T(m-k, j)}{(m-k)!}. \tag{3.1}$$

Proof. Let $G(x)=t=\log(F(x)+1)$, so that $F(x)=e^t-1$. Then

$$\begin{aligned}
\sum_{m=0}^{\infty} F_m^{(k)}\frac{x^m}{m!} &= \frac{x^k}{t^k}\left(\frac{t}{e^t-1}\right)^k = \frac{x^k}{t^k}\sum_{j=0}^{\infty} B_j^{(k)}\frac{t^j}{j!} = x^k\sum_{j=0}^{\infty} B_j^{(k)}\frac{t^{j-k}}{j!}\\
&= x^k\sum_{j=0}^{k-1} B_j^{(k)}\frac{t^{j-k}}{j!} + x^k\sum_{j=k}^{\infty} B_j^{(k)}\frac{t^{j-k}}{j!}\\
&= \sum_{j=0}^{k-1} B_j^{(k)}\frac{x^j}{j!}\left(\frac{x}{t}\right)^{k-j} + x^k\sum_{j=0}^{\infty}\frac{B_{j+k}^{(k)}}{(j+k)\cdots(j+1)}\frac{t^j}{j!}\\
&= \sum_{j=0}^{k-1} B_j^{(k)}\frac{x^j}{j!}\sum_{m=0}^{\infty} a_m^{(k-j)}x^m + \sum_{j=0}^{\infty}\frac{B_{j+k}^{(k)}}{(j+k)\cdots(j+1)}\sum_{m=j}^{\infty} T(m,j)\frac{x^{m+k}}{m!}\\
&= \sum_{m=0}^{\infty}\left(\sum_{j=0}^{k-1}\frac{B_j^{(k)}}{j!}a_m^{(k-j)}\right)x^{m+j}\\
&\quad + \sum_{m=0}^{\infty}\left(\sum_{j=0}^{m}\frac{B_{j+k}^{(k)}}{(j+k)\cdots(j+1)}\frac{T(m,j)}{m!}\right)x^{m+k}.
\end{aligned}$$

Equating coefficients of x^m gives Theorem 3.1. □

Corollary 3.2. *For $F(x)=(1+\lambda x)^{1/\lambda}-1$, we have*

$$\begin{aligned}
\frac{\beta_m^{(k)}(\lambda)}{m!} &= \sum_{j=0}^{k-1}\frac{B_j^{(k)}}{j!}b_{m-j}^{(k-j)}\lambda^{m-j}\\
&\quad + \sum_{j=0}^{m-k}\frac{B_{j+k}^{(k)}}{(j+k)\cdots(j+1)}\frac{\lambda^{m-k-j}}{(m-k)!}s(m-k,j).
\end{aligned}\tag{3.2}$$

Proof. We have

$$\sum_{m=0}^{\infty} F_m^{(k)}\frac{x^m}{m!} = \left(\frac{x}{(1+\lambda x)^{1/\lambda}-1}\right)^k = \sum_{m=0}^{\infty}\beta_m^{(k)}(\lambda)\frac{x^m}{m!},$$

so that $F_m^{(k)}=\beta_m^{(k)}(\lambda)$,

$$\sum_{m=0}^{\infty} a_m^{(k)}x^m = \left(\frac{\lambda x}{\log(1+\lambda x)}\right)^k = \sum_{m=0}^{\infty} b_m^{(k)}\lambda^m x^m,$$

so that $a_m^{(k)}=\lambda^m b_m^{(k)}$, and

$$\sum_{m=j}^{\infty} T(m,j)\frac{x^m}{m!} = \frac{(\log(1+\lambda x)/\lambda)^j}{j!} = \sum_{m=j}^{\infty}\lambda^{m-j}s(m,j)\frac{x^m}{m!},$$

so that $T(m,j)=\lambda^{m-j}s(m,j)$. □

We note that for $k=1$, (3.2) obviously entails (2.11).

For positive integers k and m with $m \geqslant k$, we have (3.2) as an explicit formula for the higher order degenerate Bernoulli numbers. When $k > m$ or k is negative, we have the following formulas:

Theorem 3.3. *For $m = 0, 1, \ldots, k-1$, we have*

$$\beta_m^{(k)}(\lambda) = \frac{1}{\binom{m-k}{m}} \sum_{j=0}^{m} (-1)^{m-j} \bar{s}(k, k-j) S(k-j, k-m) \lambda^{m-j},$$

and for $k > 0$,

$$\beta_m^{(-k)}(\lambda) = \frac{1}{\binom{m+k}{m}} \sum_{j=0}^{m} (-1)^{j} \bar{s}(m+k, m+k-j) S(m+k-j, k) \lambda^{j}.$$

Proof. From (2.15), for $m = 0, 1, \ldots, k-1$, we have

$$\bar{s}(k, k-m|\lambda) = \binom{m-k}{m} \beta_m^{(k)}(\lambda).$$

Carlitz [3] proved that

$$\bar{s}(k, m|\lambda) = \sum_{j=m}^{k} (-1)^{j-m} \bar{s}(k, j) S(j, m) \lambda^{j-m}. \tag{3.3}$$

Thus, we have

$$\beta_m^{(k)}(\lambda) = \frac{1}{\binom{m-k}{m}} \sum_{j=0}^{m} (-1)^{m-j} \bar{s}(k, k-j) S(k-j, k-m) \lambda^{m-j}.$$

For the second equation, we make use of Carlitz's formulas [3]

$$\bar{s}(m, k|\lambda) = \binom{m}{k} \beta_{m-k}^{(-k)}(\lambda)$$

and

$$\bar{s}(m, k|-\lambda) = \sum_{j=k}^{m} \bar{s}(m, j) S(j, k) \lambda^{m-j}.$$

So we have

$$\bar{s}(m+k, k|\lambda) = \sum_{j=k}^{m+k} (-1)^{m+k-j} \bar{s}(m+k, j) S(j, k) \lambda^{m+k-j}$$

and

$$\begin{aligned}\beta_m^{(-k)}(\lambda) &= \frac{1}{\binom{m+k}{m}} \bar{s}(m+k, k|\lambda) \\ &= \frac{1}{\binom{m+k}{m}} \sum_{j=0}^{m} (-1)^{j} \bar{s}(m+k, m+k-j) S(m+k-j, k) \lambda^{j}. \qquad \square\end{aligned}$$

Remark 3.4. Theorem 3.3 is not new. It is in fact a special case of Corollary 4.3 of Adelberg's paper [1], taking into account $\beta_n^{(w)}(\lambda) = \lambda^n n! A_n(1/\lambda, -w)$, the definitions of Stirling polynomials, and the Stirling duality.

There are similar results for the higher order degenerate Bernoulli numbers of the second kind, $\alpha_m^{(k)}(\lambda)$. From the definitions of $\alpha_m^{(k)}(\lambda)$, $\bar{s}(m,k|\lambda)$, and $S(m,k|\lambda)$, it is clear that

$$\alpha_m^{(-k)}(\lambda) = \frac{1}{\binom{m+k}{k}} \bar{s}(m+k,k|\lambda), \quad k>0,$$

and from (2.17) we have

$$\alpha_m^{(k)}(\lambda) = \frac{1}{\binom{m-k}{k}} S(k,k-m|\lambda), \quad m=0,1,\ldots,k-1.$$

Theorem 3.5. *The following relations hold.*

For $k>0$,

$$\alpha_m^{(-k)}(\lambda) = (-\lambda)^m \beta_m^{(-k)}(\lambda^{-1}),$$

$$\alpha_m^{(-k)}(\lambda) = \frac{1}{\binom{m+k}{m}} \sum_{j=0}^{m} (-1)^{m-j} \bar{s}(m+k,m+k-j) S(m+k-j,k) \lambda^{m-j},$$

and for $m=0,1,\ldots,k-1$,

$$\alpha_m^{(k)}(\lambda) = (-\lambda)^m \beta_m^{(k)}(\lambda^{-1}),$$

$$\alpha_m^{(k)}(\lambda) = \frac{1}{\binom{m-k}{m}} \sum_{j=0}^{m} (-1)^j \bar{s}(k,k-j) S(k-j,k-m) \lambda^j.$$

Proof. In [3] Carlitz proved that

$$\bar{s}(m,k|\lambda) = (-1)^{m-k} \lambda^{m-k} S(m,k|\lambda^{-1}). \tag{3.4}$$

Therefore, we obtain

$$\alpha_m^{(-k)}(\lambda) = \frac{1}{\binom{m+k}{k}} \bar{s}(m+k,k|\lambda) = \frac{1}{\binom{m+k}{k}} (-\lambda)^m S(m+k,k|\lambda^{-1}) = (-\lambda)^m \beta_m^{(-k)}(\lambda^{-1})$$

and

$$\alpha_m^{(k)}(\lambda) = \frac{1}{\binom{m-k}{k}} S(k,k-m|\lambda) = \frac{1}{\binom{m-k}{k}} (-\lambda)^m \bar{s}\left(k,k-m|\lambda^{-1}\right) = (-\lambda)^m \beta_m^{(k)}(\lambda^{-1}),$$

which are the first and third equations. Other equations follow from (2.17), (3.3), and (3.4). □

Remark 3.6. One of the referees has suggested the following simple proof of Theorem 3.5:

Replacing λ by λ^{-1} and x by $-\lambda x$ in the generating function (2.10) for $\{\beta_m^{(z)}(\lambda)\}$, we get the generating function for $\{\alpha_m^{(z)}(\lambda)\}$, from which the whole proof follows.

4. New formulas and recurrence relations

Deeba and Rodriguez [5] proved the following recurrence formula: for any positive integer m and any integer $n > 1$, we have

$$B_m = \frac{1}{n(1-n^m)} \sum_{k=0}^{m-1} n^k \binom{m}{k} B_k \sum_{j=1}^{n-1} j^{m-k}. \tag{4.1}$$

In 1995, Howard [10] pointed out that (4.1) is a special case of the multiplication theorem for Bernoulli polynomials. Later, in [11], he proved an analogue of (4.1) for the degenerate Bernoulli numbers.

In this section we prove several recurrence formulas for the higher order degenerate Bernoulli numbers of the first and second kinds, some of which are useful to obtain new explicit formulas for these numbers. We also prove an analogue of (4.1) for the higher order degenerate Bernoulli numbers of the first kind, which generalizes the recurrence formula for the degenerate Bernoulli numbers presented by Howard in [11].

Theorem 4.1. *Let k be a positive integer. For any integer $m \geqslant 1$, we have*

$$\sum_{j=0}^{m} \frac{S(m+k-j,k|\lambda)}{(m+k-j)!} \frac{\beta_j^{(k)}(\lambda)}{j!} = 0. \tag{4.2}$$

Proof. From the definition of the higher order degenerate Bernoulli numbers of the first kind, we have

$$\begin{aligned}
x^k &= ((1+\lambda x)^{1/\lambda}-1)^k \sum_{m=0}^{\infty} \beta_m^{(k)}(\lambda)\frac{x^m}{m!} \\
&= k! \sum_{m=k}^{\infty} S(m,k|\lambda)\frac{x^m}{m!} \sum_{m=0}^{\infty} \beta_m^{(k)}(\lambda)\frac{x^m}{m!} \\
&= k! \sum_{m=0}^{\infty} \left(\sum_{j=0}^{m} \frac{S(m+k-j,k|\lambda)\beta_j^{(k)}(\lambda)}{(m+k-j)!j!} \right) x^{m+k} \\
&= S(k,k|\lambda)\beta_0^{(k)}(\lambda)x^k + \sum_{m=1}^{\infty} \left(\sum_{j=0}^{m} \frac{S(m+k-j,k|\lambda)\beta_j^{(k)}(\lambda)}{(m+k-j)!j!} \right) x^{m+k}.
\end{aligned}$$

Equating coefficients of x^k and using the fact that $S(k,k|\lambda)=1$, we have

$$\beta_0^{(k)}(\lambda) = 1,$$

and for all $m \geqslant 1$,

$$\sum_{j=0}^{m} \frac{S(m+k-j,k|\lambda)}{(m+k-j)!} \frac{\beta_j^{(k)}(\lambda)}{j!} = 0. \quad \square$$

Lemma 4.2. *With the notation of Section* 1, *we have*

$$F_k^{(z)} = \sum_{j=0}^{k} (-1)^j \binom{z+j-1}{j} \binom{z+k}{k-j} F_k^{(-j)}. \tag{4.3}$$

Proof. Using the notation of Section 1, we have

$$\sum_{k=0}^{\infty} F_k^{(z)} \frac{x^k}{k!} = \left(\frac{f_r \frac{x^r}{r!}}{F(x)} \right)^z = \left(1 + \sum_{i=1}^{\infty} c_i \frac{x^i}{i!} \right)^{-z},$$

where $c_i = f_{r+i}/f_r\binom{r+i}{r}$. For convenience we let $g = 1 + \sum_{i=1}^{\infty} c_i x^i/i!$. By binomial theorem,

$$(g)^{-z} = \{1 + (g-1)\}^{-z} = \sum_{j=0}^{\infty} \binom{-z}{j} (g-1)^j.$$

Since x^k will not appear in $(g-1)^j$ for $j > k$, we see that $F_k^{(z)}$ is the coefficient of $x^k/k!$ in

$$\begin{aligned}
\sum_{m=0}^{k} \binom{-z}{m} (g-1)^m &= \sum_{m=0}^{k} \binom{-z}{m} \sum_{j=0}^{m} (-1)^{j-m} \binom{m}{j} g^j \\
&= \sum_{m=0}^{k} (-1)^j g^j \sum_{m=j}^{k} \binom{z+m-1}{m} \binom{m}{j} \\
&= \sum_{j=0}^{k} (-1)^j g^j \binom{z+j-1}{j} \sum_{m=j}^{k} \binom{z+m-1}{m-j} \\
&= \sum_{j=0}^{k} (-1)^j \binom{z+j-1}{j} \binom{z+k}{k-j} g^j.
\end{aligned}$$

Thus we have

$$F_k^{(z)} = \sum_{j=0}^{k} (-1)^j \binom{z+j-1}{j} \binom{z+k}{k-j} F_k^{(-j)}. \qquad \square$$

We now state a formula for the higher order degenerate Bernoulli numbers of the first kind from which several interesting results can be deduced.

Theorem 4.3. *For positive integers n and k, we have*

$$\beta_n^{(kz)}(\lambda) = \sum_{j=1}^{n} (-1)^j \binom{z+j-1}{j} \frac{\binom{n+z}{n-j}}{\binom{n+jk}{n}} S(n+jk, jk|\lambda). \tag{4.4}$$

Proof. Using Theorem 1.1 with $F(x) = ((1+\lambda x)^{1/\lambda} - 1)^k$, we have

$$\left(\frac{x^k}{F(x)} \right)^z = \sum_{n=0}^{\infty} \beta_n^{(kz)}(\lambda) \frac{x^n}{n!},$$

so that $F_n^{(z)} = \beta_n^{(kz)}(\lambda)$, and

$$\left(\frac{F(x)}{x^k}\right)^j = \sum_{n=0}^{\infty} F_n^{(-j)} \frac{x^n}{n!},$$

so that $F_n^{(-j)} = (1/\binom{n+jk}{n})S(n+jk, jk|\lambda)$. By (4.3), we obtain

$$\beta_n^{(kz)}(\lambda) = \sum_{j=1}^{n} (-1)^j \binom{z+j-1}{j} \frac{\binom{n+z}{n-j}}{\binom{n+jk}{n}} S(n+jk, jk|\lambda). \qquad \square$$

Taking $z=1$ in Theorem 4.3, we have the following formula, which generalizes a well-known result for ordinary Bernoulli numbers [6, p. 48, formula (11)].

Corollary 4.4. *For positive integers n and k, we have*

$$\beta_n^{(k)}(\lambda) = \sum_{j=1}^{n} (-1)^j \frac{\binom{n+1}{j+1}}{\binom{n+jk}{n}} S(n+jk, jk|\lambda). \tag{4.5}$$

Corollary 4.5. *We have the following explicit formula for $\beta_n^{(k)}(\lambda)$:*

$$\beta_n^{(k)}(\lambda) = \sum_{j=1}^{n} (-1)^j \frac{\binom{n+1}{j+1}}{\binom{n+jk}{n}} \times \frac{1}{(jk)!} \sum_{t=1}^{jk} (-1)^{jk-t} \binom{jk}{t} t(t-\lambda)\cdots(t-(n+jk-1)\lambda).$$

Proof. Carlitz [3] proved that

$$S(n, k|\lambda) = \frac{1}{k!} \sum_{t=0}^{k} (-1)^{k-t} \binom{k}{t} t(t-\lambda)\cdots(t-(n-1)\lambda). \tag{4.6}$$

Substituting (4.6) in (4.5), we obtain the desired formula. $\square$

For the higher order degenerate Bernoulli numbers of the second kind, we have the following result analogous to (4.4).

Theorem 4.6. *For positive integers n and k, we have*

$$\alpha_n^{(kz)}(\lambda) = \sum_{j=1}^{n} (-1)^j \binom{z+j-1}{j} \frac{\binom{n+z}{n-j}}{\binom{n+jk}{n}} \bar{s}(n+jk, jk|\lambda). \tag{4.7}$$

Proof. Using Theorem 1.1 with $F(x) = \{(1-(1-x)^{\lambda})/\lambda\}^k$, we have

$$\left(\frac{x^k}{F(x)}\right)^z = \sum_{n=0}^{\infty} \alpha_n^{(kz)}(\lambda) \frac{x^n}{n!},$$

so that $F_n^{(z)} = \alpha_n^{(kz)}(\lambda)$, and

$$\left(\frac{F(x)}{x^k}\right)^j = \sum_{n=0}^{\infty} F_n^{(-j)} \frac{x^n}{n!},$$

so that $F_n^{(-j)} = (1/\binom{n+jk}{n})\bar{s}(n+jk, jk|\lambda)$. By (4.3), we obtain

$$\alpha_n^{(kz)}(\lambda) = \sum_{j=1}^{n} (-1)^j \binom{z+j-1}{j} \frac{\binom{n+z}{n-j}}{\binom{n+jk}{n}} \bar{s}(n+jk, jk|\lambda). \qquad \square$$

Corollary 4.7. *For positive integers n and k, we have*

$$\alpha_n^{(k)}(\lambda) = \sum_{j=1}^{n} (-1)^j \frac{\binom{n+1}{j+1}}{\binom{n+jk}{n}} \bar{s}(n+jk, jk|\lambda). \tag{4.8}$$

Corollary 4.8. *We have the following explicit formula for* $\alpha_n^{(k)}(\lambda)$:

$$\alpha_n^{(k)}(\lambda) = \sum_{j=1}^{n} (-1)^j \frac{\binom{n+1}{j+1}}{\binom{n+jk}{n}} \frac{1}{(jk)!} \sum_{t=1}^{jk} (-1)^{n+jk-t} \binom{jk}{t}$$
$$\times\, t\lambda(t\lambda-1)\cdots(t\lambda-(n+jk)+1)\lambda^{-jk}.$$

Proof. Carlitz [3] proved that

$$\bar{s}(n,k|\lambda) = \frac{1}{k!} \sum_{t=0}^{k} (-1)^{n-t} \binom{k}{t} t\lambda(t\lambda-1)\cdots(t\lambda-n+1)\lambda^{-k}. \tag{4.9}$$

Exploiting (4.9) in (4.8), the result follows. $\square$

We now prove an analogue of (4.1) for the higher order degenerate Bernoulli numbers of the first kind.

Theorem 4.9. *Let k be a non-negative integer. For any positive integer m and any integer* $n>1$, *we have*

$$n^k \beta_m^{(k)}(n\lambda) = \sum_{s=0}^{m} n^s \binom{m}{s} \beta_s^{(k)}(\lambda) \sum_{j=0}^{k(n-1)} c_j^{(k)} j(j-n\lambda)\cdots(j-(m-s-1)n\lambda),$$

where $c_j^{(k)}$ *is defined by*

$$(1+x+\cdots+x^{n-1})^k = \sum_{j=0}^{k(n-1)} c_j^{(k)} x^j.$$

Proof. In [3] Carlitz defined the higher order degenerate Bernoulli polynomials of the first kind, $\beta_m^{(k)}(\lambda, z)$, by means of the generating function

$$\left(\frac{x}{(1+\lambda x)^{1/\lambda}-1}\right)^k (1+\lambda x)^{z/\lambda} = \sum_{m=0}^{\infty} \beta_m^{(k)}(\lambda, z)\frac{x^m}{m!},$$

for each integer k. It follows from this definition that

$$\beta_m^{(k)}(\lambda, z) = \sum_{s=0}^{m} \binom{m}{s} z(z-\lambda)(z-2\lambda)\cdots(z-(m-s-1)\lambda)\beta_s^{(k)}(\lambda). \tag{4.10}$$

In [2] Carlitz gave the multiplication formula for the higher order Bernoulli polynomials

$$n^{k-m} B_m^{(k)}(nz) = \sum_{j=0}^{k(n-1)} c_j^{(k)} B_m^{(k)}\left(z + \frac{j}{n}\right),$$

where $c_j^{(k)}$ is defined by

$$(1 + x + \cdots + x^{n-1})^k = \sum_{j=0}^{k(n-1)} c_j^{(k)} x^j.$$

Carlitz [3] also proved the multiplication formula for the degenerate Bernoulli polynomials of the first kind

$$n^{1-m}\beta_m(n\lambda, nz) = \sum_{j=0}^{n-1} \beta_m\left(\lambda, z + \frac{j}{n}\right). \tag{4.11}$$

In the logical way, we can extend (4.11) by

$$n^{k-m}\beta_m^{(k)}(n\lambda, nz) = \sum_{j=0}^{k(n-1)} c_j^{(k)} \beta_m^{(k)}\left(\lambda, z + \frac{j}{n}\right). \tag{4.12}$$

If we let $z = 0$ in (4.12) and use (4.10), we have

$$\begin{aligned} n^{k-m}\beta_m^{(k)}(n\lambda) &= \sum_{j=0}^{k(n-1)} c_j^{(k)} \beta_m^{(k)}\left(\lambda, \frac{j}{n}\right) \\ &= \sum_{j=0}^{k(n-1)} c_j^{(k)} \sum_{s=0}^{m} n^{s-m} \binom{m}{s} j(j-n\lambda)\cdots(j-(m-s-1)n\lambda)\beta_s^{(k)}(\lambda) \\ &= \sum_{s=0}^{m} n^{s-m} \binom{m}{s} \beta_s^{(k)}(\lambda) \\ &\quad \times \sum_{j=0}^{k(n-1)} c_j^{(k)} j(j-n\lambda)\cdots(j-(m-s-1)n\lambda). \end{aligned} \tag{4.13}$$

We multiply both sides of (4.13) by n^m to complete the proof. $\square$

5. Other degenerate numbers

In this section we discuss other degenerate numbers. To be precise, we define the higher order degenerate Genocchi numbers, higher order degenerate tangent numbers, and the coefficients of the higher order degenerate Euler polynomials. Furthermore, we obtain explicit formulas for these numbers.

Let z be a variable. For positive integer n and any k, the higher order degenerate Euler polynomial, $E_n^{(k)}(\lambda, z)$, may be defined by means of

$$\left(\frac{2}{(1+\lambda x)^{1/\lambda} + 1}\right)^k (1+\lambda x)^{z/\lambda} = \sum_{n=0}^{\infty} E_n^{(k)}(\lambda, z)\frac{x^n}{n!}. \tag{5.1}$$

For $\lambda = 0$, we have the higher order Euler polynomials defined by Nörlund [13, p. 143]. Taking $z = 0$ in (5.1), we get the coefficients of the higher order degenerate Euler polynomials, $e_n^{(k)}(\lambda)$, that is,

$$\left(\frac{2}{(1+\lambda x)^{1/\lambda}+1}\right)^k = \sum_{n=0}^{\infty} e_n^{(k)}(\lambda)\frac{x^n}{n!}. \tag{5.2}$$

The higher order degenerate Genocchi numbers, $G_n^{(k)}(\lambda)$, may be defined by

$$\left(\frac{2x}{(1+\lambda x)^{1/\lambda}+1}\right)^k = \sum_{n=0}^{\infty} G_n^{(k)}(\lambda)\frac{x^n}{n!}. \tag{5.3}$$

For $\lambda = 0$, $G_n^{(k)}$ are the higher order Genocchi numbers, and $G_n^{(1)}(0) = G_n$ are the ordinary Genocchi numbers. It follows from (5.2) and (5.3) that

$$\frac{e_n^{(k)}(\lambda)}{n!} = \frac{G_{n+k}^{(k)}(\lambda)}{(n+k)!}. \tag{5.4}$$

The higher order tangent numbers $T_n^{(k)}$ may be defined by

$$\left(\frac{2}{e^x+1}\right)^k = \sum_{n=0}^{\infty} \frac{T_n^{(k)}}{2^n}\frac{x^n}{n!}.$$

The higher order degenerate tangent numbers, $T_n^{(k)}(\lambda)$, may be defined by means of

$$\left(\frac{2}{(1+\lambda x)^{1/\lambda}+1}\right)^k = \sum_{n=0}^{\infty} \frac{T_n^{(k)}(\lambda)}{2^n}\frac{x^n}{n!}, \tag{5.5}$$

so that

$$T_n^{(k)}(\lambda) = 2^n e_n^{(k)}(\lambda). \tag{5.6}$$

Let $F(x) = f_0 + f_1 x + \cdots + f_n x^n/n! + \cdots$ be a formal exponential generating function as in Section 3. If $f_0 \neq 0$, $f_1 \neq 0$, we define

$$G(x) = \log(F(x) - f_0 + 1),$$

$$\left(\frac{x}{G(x)}\right)^k = \sum_{m=0}^{\infty} a_m^{(k)} x^m,$$

$$(G(x))^j = j!\sum_{m=j}^{\infty} T(m,j)\frac{x^m}{m!},$$

and if $f_0 = 2$, $f_1 \neq 0$,

$$\left(\frac{2}{F(x)}\right)^k = \sum_{m=0}^{\infty} F_m^{(k)}\frac{x^m}{m!},$$

$$\left(\frac{2}{e^x + f_0 - 1}\right)^k = \sum_{m=0}^{\infty} C_m^{(k)}\frac{x^m}{m!}.$$

Theorem 5.1. *If $F_m^{(k)}$, $T(m,j)$, and $C_m^{(k)}$ are defined as above, then we have*

$$F_m^{(k)} = \sum_{j=0}^{m} C_j^{(k)} T(m,j). \tag{5.7}$$

Proof. Since $G(x)=t=\log(F(x)-f_0+1)$, we have $F(x)=e^t+f_0-1$. Then

$$\sum_{m=0}^{\infty} F_m^{(k)}\frac{x^m}{m!}=\left(\frac{2}{F(x)}\right)^k=\left(\frac{2}{e^t+f_0-1}\right)^k=\sum_{j=0}^{\infty} C_j^{(k)}\frac{t^j}{j!}$$

$$=\sum_{j=0}^{\infty} C_j^{(k)}\frac{(G(x))^j}{j!}=\sum_{j=0}^{\infty} C_j^{(k)}\sum_{m=j}^{\infty} T(m,j)\frac{x^m}{m!}$$

$$=\sum_{m=0}^{\infty}\left(\sum_{j=0}^{m} C_j^{(k)} T(m,j)\right)\frac{x^m}{m!}.$$

Comparing the coefficients of $x^m/m!$, Theorem 5.1 follows. □

Taking $F(x)=(1+\lambda x)^{1/\lambda}+1$ in Theorem 5.1, we get some interesting results:

Corollary 5.2. *The following relations hold*:

$$e_m^{(k)}(\lambda)=\sum_{j=0}^{m} e_j^{(k)}\lambda^{m-j}s(m,j),$$

$$G_{m+k}^{(k)}(\lambda)=\sum_{j=0}^{m}\frac{\binom{m+k}{j+k}}{\binom{m}{j}}G_{j+k}^{(k)}\lambda^{m-j}s(m,j),$$

$$T_m^{(k)}(\lambda)=\sum_{j=0}^{m}(2\lambda)^{m-j}T_j^{(k)}s(m,j).$$

We also have the following explicit formula for the higher order degenerate Genocchi numbers, which can be proved following exactly the same steps of the proof of Theorem 3.1.

Theorem 5.3. *For positive integers k and m with $m\geqslant k$, we have*

$$\frac{G_m^{(k)}(\lambda)}{m!}=\sum_{j=0}^{k-1}\frac{G_j^{(k)}}{j!}b_{m-j}^{(k-j)}\lambda^{m-j}+\sum_{j=0}^{m-k}\frac{G_{j+k}^{(k)}}{(j+k)\cdots(j+1)}\frac{\lambda^{m-k-j}}{(m-k)!}s(m-k,j).$$

6. The polynomials $V(k,j,z|\lambda)$ and $V_1(k,l,z|\lambda)$

Let z be a variable, j is a non-negative integer, and $k\geqslant 2j$. We define $V(k,j,z|\lambda)$ by means of

$$\left(1-\frac{x}{2}\right)^z\left\{x((1+\lambda x)^{1/\lambda}+1)-2((1+\lambda x)^{1/\lambda}-1)\right\}^j$$
$$=j!\sum_{k=2j}^{\infty} V(k,j,z|\lambda)\frac{x^k}{k!}. \tag{6.1}$$

For $z=0$ and $z=1$, we set $V(k,j,0|\lambda)=V(k,j|\lambda)$ and $V(k,j,1|\lambda)=A(k,j|\lambda)=V(k,j|\lambda)-(k/2)V(k-1,j|\lambda)$, respectively. The limiting case $\lambda=0$ was studied in [8,9]. It is clear from (6.1) that

$$V(k,j,z|\lambda)=\sum_{i=0}^{k-2j}(z)_i\binom{k}{i}(-2)^{-i}V(k-i,j|\lambda). \tag{6.2}$$

For a variable z, a non-negative integer l, and $k \geqslant 3l$, we define $V_1(k, l, z|\lambda)$ by

$$\left(1 - \frac{x}{2}\right)^z \left\{x((1+\lambda x)^{1/\lambda} + 1) - 2((1+\lambda x)^{1/\lambda} - 1) - \lambda x^2\right\}^l = l! \sum_{k=3l}^{\infty} V_1(k, l, z|\lambda) \frac{x^k}{k!}. \tag{6.3}$$

For $z = 0$, we adopt the notation $V_1(k, l, 0|\lambda) = V_1(k, l|\lambda)$. For the limiting case $\lambda = 0$, $V(k, j, z|\lambda)$ and $V_1(k, l, z|\lambda)$ coincide, since the left sides of (6.1) and (6.3) are then equal. Similar to (6.2), we have

$$V_1(k, l, z|\lambda) = \sum_{i=0}^{k-3l} (z)_i \binom{k}{i} (-2)^{-i} V_1(k-i, l|\lambda). \tag{6.4}$$

In this section we use Theorem 1.1 to obtain several formulas, which are generalizations of the results in [8,9]. We first need to define some special numbers. We define $A_k^{(z)}(\lambda)$ and $V_k^{(z)}(\lambda)$ by means of

$$\left(\frac{((1-\lambda)/2)x^2}{(1+\lambda x)^{1/\lambda} - x - 1}\right)^z = \sum_{k=0}^{\infty} A_k^{(z)}(\lambda) \frac{x^k}{k!} \tag{6.5}$$

and

$$\left(\frac{((1-\lambda)(1+4\lambda)/6)x^3}{x((1+\lambda x)^{1/\lambda} + 1) - 2((1+\lambda x)^{1/\lambda} - 1) - \lambda x^2}\right)^z = \sum_{k=0}^{\infty} V_k^{(z)}(\lambda) \frac{x^k}{k!}, \tag{6.6}$$

respectively. The limiting case $\lambda = 0$ gives $A_k^{(z)}$ and $V_k^{(z)}$, which were examined in [8]. The polynomial $T_n^{(z)}(\lambda)$ in the next theorem is defined by (5.5).

Theorem 6.1. *The following relations hold*:

$$\beta_n^{(z)}(\lambda) = \sum_{j=0}^{n} 2^{-j} \langle z \rangle_j \frac{n!}{(n+j)!} V(n+j, j, z|\lambda),$$

$$T_n^{(z)}(\lambda) = 2^n \sum_{j=0}^{[n/2]} 4^{-j} \langle z \rangle_j V(n, j, z|\lambda),$$

$$A_n^{(z)}(\lambda) = \sum_{l=0}^{n} (1-\lambda)^{-l} \langle z \rangle_l \frac{n!}{(n+2l)!} V_1(n+2l, l, z|\lambda) \quad (\lambda \neq 1),$$

$$\binom{n-z}{n} V_n^{(z)}(\lambda) = \sum_{l=0}^{n} \left(\frac{6}{(1-\lambda)(1+4\lambda)}\right)^l \binom{n+z}{n-l} \binom{n-z}{n+l} \times \frac{(n+l)!}{(n+3l)!} V_1(n+3l, l|\lambda).$$

Proof. Let $F_n^{(z)}(\lambda)$ be either $\beta_n^{(z)}(\lambda)$ or $T_n^{(z)}(\lambda)$, and define $K_n^{(z)}(\lambda)$ by

$$\left(\frac{2 f_r x^r / r!}{(2-x) F(\lambda, x)}\right)^z = \sum_{n=0}^{\infty} K_n^{(z)}(\lambda) \frac{x^n}{n!}, \tag{6.7}$$

where

$$F(\lambda, x) = \begin{cases} (1+\lambda x)^{1/\lambda} - 1, & r = 1, \quad f_r = 1 \text{ if } F_n^{(z)}(\lambda) = \beta_n^{(z)}(\lambda), \\ (1+\lambda x)^{1/\lambda} + 1, & r = 0, \quad f_r = 2 \text{ if } F_n^{(z)}(\lambda) = T_n^{(z)}(\lambda). \end{cases}$$

By Theorem 1.1,

$$K_i^{(z)}(\lambda) = \sum_{j=0}^{i} \left(\frac{r!}{2f_r}\right)^j \langle z\rangle_j \frac{i!}{(i+rj)!} V(i+rj, j|\lambda), \tag{6.8}$$

and by (6.7),

$$\beta_n^{(z)}(\lambda) = \sum_{i=0}^{n} (-2)^{i-n} \binom{z}{n-i} \frac{n!}{i!} K_i^{(z)}(\lambda), \tag{6.9}$$

$$T_n^{(z)}(\lambda) = 2^n \sum_{i=0}^{n} (-2)^{i-n} \binom{z}{n-i} \frac{n!}{i!} K_i^{(z)}(\lambda). \tag{6.10}$$

We now substitute (6.8) into (6.9) and (6.10), switch summations, and simplify; using (6.2) we obtain the first two formulas in Theorem 6.1. With an argument analogous to the one above, using the polynomial V_1 instead of V with $F(\lambda, x) = (1+\lambda x)^{1/\lambda} - x - 1$ and (6.4), we can prove the third formula. The last formula follows immediately from Theorem 1.1, with $r = 3$ and $f_r = (1-\lambda)(1+4\lambda)$. □

7. Final remarks

One of the referees has pointed out that in this paper we need only be in an integral domain of characteristic zero.

The authors wish to express their sincere gratitude to the referees for their valuable comments and suggestions.

The first author was supported by Akdeniz University Scientific Research Project Unit.

References

[1] A. Adelberg, A finite difference approach to degenerate Bernoulli and Stirling polynomials, Discrete Math. 140 (1995) 1–21.
[2] L. Carlitz, A note on Bernoulli and Euler polynomials of the second kind, Scripta Math. 25 (1961) 323–330.
[3] L. Carlitz, Degenerate Stirling, Bernoulli and Eulerian numbers, Utilitas Math. 15 (1979) 51–88.
[4] L. Comtet, Advanced Combinatorics, Riedel, Dordrecht, Boston, 1974.
[5] E. Deeba, D. Rodriguez, Stirling's series and Bernoulli numbers, Amer. Math. Monthly 98 (1991) 423–426.
[6] H.W. Gould, Explicit formulas for Bernoulli numbers, Amer. Math. Monthly 79 (1972) 44–51.
[7] F.T. Howard, Bell polynomials and degenerate Stirling numbers, Rend. Sem. Math. Univ. Padova 61 (1979) 203–219.
[8] F.T. Howard, A special class of Bell polynomials, Math. Comp. 35 (1980) 977–989.
[9] F.T. Howard, A theorem relating potential and Bell polynomials, Discrete Math. 39 (1982) 129–143.
[10] F.T. Howard, Applications of a recurrence formula for Bernoulli numbers, J. Number Theory 52 (1995) 157–172.
[11] F.T. Howard, Explicit formulas for degenerate Bernoulli numbers, Discrete Math. 162 (1996) 175–185.
[12] C. Jordan, Calculus of Finite Differences, Chelsea, New York, 1950.
[13] N. Nörlund, Vorlesungen über Differenzenrechnung, Chelsea, New York, 1954.
[14] P.T. Young, Congruences for degenerate number sequences, Discrete Math. 270 (2003) 279–289.
[15] P.T. Young, Degenerate and n-adic versions of Kummer's congruences for values of Bernoulli polynomials, Discrete Math. 285 (2004) 289–296.