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# Matrix representation of the q-Jacobsthal numbers

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**Abstract**

*In this paper, we consider a q-Jacobsthal sequence* $\{J_{q,n}\}$*, with initial conditions* $J_{q,0} = 0$ *and* $J_{q,1} = 1$*. Then give a generating matrix for the terms of sequence* $\{J_{q,kn}\}$ *for a positive integer* $k$*. With the aid of this matrix, we derive some new identities for the sequence* $\{J_{q,kn}\}$*.*

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## 1. Introduction

In [H1], Horadam introduce a sequence $\{W_n(a,b;p,q)\}$, or briefly $\{W_n\}$, defined by the recurrence relation

$$W_n = pW_{n-1} - qW_{n-2}, n \geq 2, \tag{1.1}$$

with $W_0 = a$, $W_1 = b$, where $a, b, p$ and $q$ are integers with $p > 0$, $q \neq 0$.

We are interested in the following two special cases of $\{W_n\}$, the $q$-Jacohsthal sequence $\{J_{q,n}\}$ defined by

$$J_{q,n} = J_{q,n-1} - qJ_{q,n-2},\ J_{q,0} = 0,\ J_{q,1} = 1,\ n \geq 2, \tag{1.2}$$

and the $q$-Jacobsthal-Lucas sequence $\{j_{q,n}\}$ defined by

$$j_{q,n} = j_{q,n-1} - qj_{q,n-2},\ j_{q,0} = 2,\ j_{q,1} = 1,\ n \geq 2. \tag{1.3}$$

The above recurrences involve the characteristic equation

$$x^2 - x + q = 0 \tag{1.4}$$

with roots $\alpha_q = \frac{1+\sqrt{1-4q}}{2}$ and $\beta_q = \frac{1+\sqrt{1-4q}}{2}$. Explicit closed form expressions for $J_{q,n}$ and $j_{q,n}$ are $(n \geq 1)$

$$J_{q,n} = \frac{\alpha_q^n - \beta_q^n}{\alpha_q - \beta_q}, \tag{1.5}$$

and

$$j_{q,n} = \alpha_q^n + \beta_q^n. \tag{1.6}$$

Particular cases of the previous definition are:

- If $q = -1$, the classic Fibonacci sequence appears by $F_0 = 0$, $F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$: $\{F_n\}_{n\in N} = \{0, 1, 1, 2, 3, 5, 8, ...\}$.
- If $q = -2$, the Jacobsthal sequence introduced in [H2], and defined by $J_{n+1} = J_n + 2J_{n-1}$ for $n \geq 1$, $J_0 = 0$, $J_1 = 1$: $\{J_n\}_{n\in N} = \{0, 1, 1, 3, 5, 11, ...\}$.

We define $J_q$ be the $2 \times 2$ matrix

$$J_q = \begin{bmatrix} 1 & -q \\ 1 & 0 \end{bmatrix}, \tag{1.7}$$

then for an integer $n$ with $n \geq 1$, $J_q^n$ has the form

$$J_q^n = \begin{bmatrix} J_{q,n+1} & -qJ_{q,n} \\ J_{q,n} & -qJ_{q,n-1} \end{bmatrix}. \tag{1.8}$$

The particular case $q = -2$, was introduced by *Köken* and Bozkurt in [KB2, KB3]. Moreover, they have obtained the Cassini formula for the Jacobsthal numbers. In this paper, we study new relations on $q$-Jacobsthal sequence, using the matrix $J_q$ defined in (1.7).

Initially, the $q$-Jacobsthal numbers are defined for $n \geq 0$ but their existence for $n < 0$ is readily extended, yielding

$$J_{q,-n} = -q^{-n}J_{q,n} \text{ and } j_{q,-n} = q^{-n}j_{q,n}.$$

For $n \geq 2$ and a fixed positive integer $k$, in [KS] the authors study the sequence $\{W_{q,kn}(x)\}$ and prove the following relation:

$$J_{q,kn} = j_{q,k}J_{q,k(n-1)} - q^k J_{q,k(n-2)}, \tag{1.9}$$

$$j_{q,kn} = j_{q,k}j_{q,k(n-1)} - q^k j_{q,k(n-2)}, \tag{1.10}$$

where the initial conditions of the sequences $\{J_{q,n}\}$ and $\{j_{q,n}\}$ are 0 and $\{J_{q,k}\}$, and 2 and $\{j_{q,k}\}$, respectively.

If $\alpha_{q,k}$ and $\beta_{q,k}$ are the roots of equation $\lambda^2 - j_{q,k}\lambda + q^k = 0$, then the Binet formulas of the sequences $\{J_{q,kn}\}$ and $\{j_{q,kn}\}$ are given by

$$J_{q,kn} = J_{q,k}\left(\frac{\alpha_{q,k}^n - \beta_{q,k}^n}{\alpha_{q,k} - \beta_{q,k}}\right) \text{ and } j_{q,kn} = \alpha_{q,k}^n + \beta_{q,k}^n,$$

respectively. It is clear that $\alpha_{q,1} = \alpha_q$ and $\beta_{q,1} = \beta_q$.

From the Binet formulas, one can see that $J_{q,2kn} = j_{q,kn}J_{q,kn}$.

## 2. Companion matrix for the sequence $\{J_{q,kn}\}$

In this section, we define a $2 \times 2$ matrix $A_q$ and then we give some new results for the $q$-Jacobsthal numbers $J_{q,kn}$ by matrix methods.

Define the $2 \times 2$ matrix $A_q$ as follows:

$$A_q = \begin{bmatrix} j_{q,k} & -q^k \\ 1 & 0 \end{bmatrix}. \tag{2.1}$$

By an inductive argument and using (1.9), we get

**Proposition 2.1.** *For any integer $n \geq 1$ holds:*

$$A_q^n = \frac{1}{J_{q,k}}\begin{bmatrix} J_{q,k(n+1)} & -q^k J_{q,kn} \\ J_{q,kn} & -q^k J_{q,k(n-1)} \end{bmatrix}. \tag{2.2}$$

**Proof.** (By induction). For $n=1$:

$$A_q^1 = \begin{bmatrix} j_{q,k} & -q^k \\ 1 & 0 \end{bmatrix} = \frac{1}{J_{q,k}} \begin{bmatrix} J_{q,2k} & -q^k J_{q,k} \\ J_{q,k} & -q^k J_{q,0} \end{bmatrix} \tag{2.3}$$

since $J_{q,0}=0$ and $J_{q,2k}=j_{q,k}J_{q,k}$. Let us suppose that the formula is true for $n-1$:

$$A_q^{n-1} = \frac{1}{J_{q,k}} \begin{bmatrix} J_{q,kn} & -q^k J_{q,k(n-1)} \\ J_{q,k(n-1)} & -q^k J_{q,k(n-2)} \end{bmatrix}. \tag{2.4}$$

Then, $\mathrm{A}_q^n = A_q^{n-1}A_q^1 = \frac{1}{J_{q,k}} \begin{bmatrix} J_{q,kn} & -q^k J_{q,k(n-1)} \\ J_{q,k(n-1)} & -q^k J_{q,k(n-2)} \end{bmatrix} \begin{bmatrix} j_{q,k} & -q^k \\ 1 & 0 \end{bmatrix}$

$= \frac{1}{J_{q,k}} \begin{bmatrix} j_{q,k}J_{q,kn} - q^k J_{q,k(n-1)} & -q^k J_{q,kn} \\ j_{q,k}J_{q,k(n-1)} - q^k J_{q,k(n-2)} & -q^k J_{q,k(n-1)} \end{bmatrix}$

$= \frac{1}{J_{q,k}} \begin{bmatrix} J_{q,k(n+1)} & -q^k J_{q,kn} \\ J_{q,kn} & -q^k J_{q,k(n-1)} \end{bmatrix}$ □

Clearly the matrix $A_q^n$ satisfies the recurrence relation, for $n \geq 1$

$$A_q^{n+1} = j_{q,k}A_q^n - q^k A_q^{n-1}, \tag{2.5}$$

where $A_q^0 = I_2$, $A_q^1 = A_q$ and $I_2$ is the $2\times 2$ unit matrix.

In this study, we define the $q$-Jacobsthal-Lucas $j_q$-matrix by

$$j_q = \begin{bmatrix} j_{q,k}^2 - 2q^k & -j_{q,k}q^k \\ j_{q,k} & -2q^k \end{bmatrix}. \tag{2.6}$$

It is easy to see that

$$\begin{bmatrix} j_{q,k(n+1)} \\ j_{q,kn} \end{bmatrix} = j_q \begin{bmatrix} J_{q,kn} \\ J_{q,k(n-1)} \end{bmatrix} \text{ and } \Delta \begin{bmatrix} J_{q,k(n+1)} \\ J_{q,kn} \end{bmatrix} = j_q \begin{bmatrix} j_{q,kn} \\ j_{q,k(n-1)} \end{bmatrix}$$

where $J_{q,kn}$, $j_{q,kn}$ are as above, and $\Delta = 1-4q$.

We obtain Cassini's formula and properties of these numbers by a similar matrix method to the Lucas numbers [KB1].

**Proposition 2.2.** *Let $j_q$ be a matrix as in (2.6). Then, for all integers $n \geq 1$, the following matrix power is held below*

$$j_q^n = \begin{cases} \Delta^{\frac{n}{2}} \begin{bmatrix} J_{q,k(n+1)} & -q^k J_{q,kn} \\ J_{q,kn} & -q^k J_{q,k(n-1)} \end{bmatrix} & if\ n\ even \\ \Delta^{\frac{n-1}{2}} \begin{bmatrix} j_{q,k(n+1)} & -q^k j_{q,kn} \\ j_{q,kn} & -q^k j_{q,k(n-1)} \end{bmatrix} & if\ n\ odd, \end{cases} \tag{2.7}$$

*where $J_{q,kn}$ and $j_{q,kn}$ are the $kn$-th $q$-Jacobsthal and $q$-Jacobsthal-Lucas numbers, respectively.*

**Proof.** We use mathematical induction on $n$. First, we consider odd $n$. For $n = 1$,

$$j_q^1 = \left[\begin{array}{cc} j_{q,2k} & -q^k j_{q,k} \\ j_{q,k} & -q^k j_{q,0} \end{array}\right],$$

since $j_{q,2k} = j_{q,k}^2 - 2q^k$ and $j_{q,0} = 2$. So, (2.7) is indeed true for $n = 1$. Now we suppose it is true for $n = t$, that is

$$j_q^t = \Delta^{\frac{t-1}{2}} \left[\begin{array}{cc} j_{q,k(t+1)} & -q^k j_{q,kt} \\ j_{q,kt} & -q^k j_{q,k(t-1)} \end{array}\right].$$

Using the induction hypothesis and $j_q^2$ by a direct computation. we can write

$$j_q^{t+2} = j_q^t j_q^2 = \Delta^{\frac{t+1}{2}} \left[\begin{array}{cc} j_{q,k(t+3)} & -q^k j_{q,k(t+2)} \\ j_{q,k(t+2)} & -q^k j_{q,k(t+1)} \end{array}\right],$$

as desired. Secondly, let us consider even n. For $n = 2$ we can write

$$j_q^2 = \Delta \left[\begin{array}{cc} J_{q,3k} & -q^k J_{q,2k} \\ J_{q,2k} & -q^k J_{q,k} \end{array}\right].$$

So, (2.7) is true for $n = 2$. Now, we suppose it is true for $n = t$, using properties of the $q$-Jacobsthal numbers and the induction hypothesis, we can write

$$j_q^{t+2} = \Delta^{\frac{t+2}{2}} \left[\begin{array}{cc} J_{q,k(t+3)} & -q^k J_{q,k(t+2)} \\ J_{q,k(t+2)} & -q^k J_{q,k(t+1)} \end{array}\right],$$

as desired. Hence, (2.7) holds for all $n$. $\square$

If we use the equation (2.5), we can write $\mathrm{j}_{q,k} A_q^{n+1} = j_{q,k}^2 A_q^n - q^k j_{q,k} A_q^{n-1}$
$= j_{q,k}^2 A_q^n - q^k (A_q^n + q^k A_q^{n-2})$
$= (j_{q,k}^2 - q^k) A_q^n - q^{2k} A_q^{n-2}$.

Comparing the entries in the first row and first column for the above matrix equation, we get

$$j_{q,k} = \frac{(j_{q,k}^2 - q^k) J_{q,k(n+1)} - q^{2k} J_{q,k(n-1)}}{J_{q,k(n+2)}}. \tag{2.8}$$

For $n \geq 0$, if we consider the fact that $\det(A_q^n) = (\det(A_q))^n$, then we obtain the generalized Cassini identity

$$J_{q,k(n+1)} J_{q,k(n-1)} - J_{q,kn}^2 = -q^{k(n-1)} J_{q,k}^2. \tag{2.9}$$

For example, for $k = 1$, we get $J_{q,n+1}J_{q,n-1} - J^2_{q,n} = -q^{n-1}$, the generalized Cassini identity with $q$-Jacobsthal numbers. In this case, if $q = -1$, we get Cassini identity on classic Fibonacci sequence.

Now we shall derive some results for $\{J_{q,kn}\}$ by matrix methods.

**Proposition 2.3.** *For all $n, m \in Z$*

$$J_{q,k}J_{q,k(n+m)} = J_{q,km}J_{q,k(n+1)} - q^k J_{q,k(m-1)}J_{q,kn}. \tag{2.10}$$

**Proof.** Since $A_q^{n+m} = A_q^n A_q^m$ and after some simplifications, we obtain $A_q^{n+m} = \frac{1}{J^2_{q,k}} \begin{bmatrix} J_{q,k(n+1)} & -q^k J_{q,kn} \\ J_{q,kn} & -q^k J_{q,k(n-1)} \end{bmatrix} \begin{bmatrix} J_{q,k(m+1)} & -q^k J_{q,km} \\ J_{q,km} & -q^k J_{q,k(m-1)} \end{bmatrix}$ $= \frac{J_{q,km}}{J_{q,k}} A_q^{n+1} - q^k \frac{J_{q,k(m-1)}}{J_{q,k}} A_q^n$. Thus we obtain

$$J_{q,k}A_q^{n+m} = J_{q,km}A_q^{n+1} - q^k J_{q,k(m-1)}A_q^n, \tag{2.11}$$

which, Comparing the entries in the second row and first column for the matrix equation (2.11), gives the conclusion. □

When $m = n$ in (2.10), we obtain

$$J_{q,k}J_{q,2kn} = J_{q,kn}(J_{q,k(n+1)} - q^k J_{q,k(n-1)}), \tag{2.12}$$

and the follow equality for $J_{q,kn}(x) \neq 0$

$$J_{q,k}j_{q,kn} = J_{q,k(n+1)} - q^k J_{q,k(n-1)}. \tag{2.13}$$

Comparing the entries in the first row and first column of the equality (2.11) and by taking $m = n$, we obtain

$$J_{q,k}J_{q,k(2n+1)} = J^2_{q,k(n+1)} - q^k J^2_{q,kn}, \tag{2.14}$$

a particular case of the next equality

$$J_{q,k}A_q^{2n} = J_{q,kn}A_q^{n+1} - q^k J_{q,k(n-1)}A_q^n. \tag{2.15}$$

**Corollary 2.4.** *For $k \geq 1$ and $n \in Z$,*

$$J_{q,k(2n+1)} + J_{q,k(2n-1)} = \frac{1}{J_{q,k}} \left( J^2_{q,k(n+1)} + (1 - q^k)J^2_{q,kn} - J^2_{q,k(n+1)} \right). \tag{2.16}$$

**Proof.** Considering the first row and first column of the matrix $A_q^{2n} = (A_q^n)^2$, we get $J_{q,k}J_{q,k(2n+1)} = J^2_{q,k(n+1)} - q^k J^2_{q,kn}$, and the second row and second column of the matrix $A_q^{2n}$, $J_{q,k}J_{q,k(2n-1)} = J^2_{q,kn} - q^k J^2_{q,k(n-1)}$. By adding side by side in the above equations, we have the conclusion. □

For any integer $m$, we have $A_q^{2n} = A_q^{n+m}A_q^{n-m}$. Here if we consider the entries in the second row and first column in the product $A_q^{n+m}A_q^{n-m}$ and the matrix $A_q^{2n}$, we get

$$J_{q,k}J_{q,2kn} = J_{q,k(n+m)}J_{q,k(n-m+1)} - q^k J_{q,k(n+m-1)}J_{q,k(n-m)}. \tag{2.17}$$

## 3. sums with $q$-Jacobsthal numbers

We define the $q$-Jacobsthal $G_q$-matrix by

$$G_q = \begin{bmatrix} 1 & 0 & 0 \\ 1 & j_{q,k} & -q^k \\ 0 & 1 & 0 \end{bmatrix}. \tag{3.1}$$

By an inductive argument and using (1.9), we get

**Proposition 3.1.** *For any integer $n \geq 1$ holds:*

$$G_q^n = \frac{1}{J_{q,k}} \begin{bmatrix} 1 & 0 & 0 \\ J^s_{q,kn} & J_{q,k(n+1)} & -q^k J_{q,kn} \\ J^s_{q,k(n-1)} & J_{q,kn} & -q^k J_{q,k(n-1)} \end{bmatrix}, \tag{3.2}$$

*where $J^s_{q,kn}$ is defined such that $J^s_{q,kn} = \sum_{i=1}^n J_{q,ki}$.*

**Proof.** (By induction). For $n = 1$:

$$G_q^1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & j_{q,k} & -q^k \\ 0 & 1 & 0 \end{bmatrix} = \frac{1}{J_{q,k}} \begin{bmatrix} 1 & 0 & 0 \\ J^s_{q,k} & J_{q,2k} & -q^k J_{q,k} \\ J^s_{q,0} & J_{q,k} & -q^k J_{q,0} \end{bmatrix} \tag{3.3}$$

since $J_{q,0} = 0$ and $J_{q,2k} = j_{q,k}J_{q,k}$. Let us suppose that the formula is true for $n - 1$:

$$G_q^{n-1} = \frac{1}{J_{q,k}} \begin{bmatrix} 1 & 0 & 0 \\ J^s_{q,k(n-1)} & J_{q,kn} & -q^k J_{q,k(n-1)} \\ J^s_{q,k(n-2)} & J_{q,k(n-1)} & -q^k J_{q,k(n-2)} \end{bmatrix} \tag{3.4}$$

Then,

$$\mathrm{G}_q^n = G_q^{n-1}G_q^1 = \frac{1}{J_{q,k}}\begin{bmatrix} 1 & 0 & 0 \\ J^s_{q,k(n-1)} & J_{q,kn} & -q^kJ_{q,k(n-1)} \\ J^s_{q,k(n-2)} & J_{q,k(n-1)} & -q^kJ_{q,k(n-2)} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & j_{q,k} & -q^k \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \frac{1}{J_{q,k}}\begin{bmatrix} 1 & 0 & 0 \\ J^s_{q,k(n-1)} + J_{q,kn} & j_{q,k}J_{q,kn} - q^kJ_{q,k(n-1)} & -q^kJ_{q,kn} \\ J^s_{q,k(n-2)} + J_{q,k(n-1)} & j_{q,k}J_{q,k(n-1)} - q^kJ_{q,k(n-2)} & -q^kJ_{q,k(n-1)} \end{bmatrix}$$

$$= \frac{1}{J_{q,k}}\begin{bmatrix} 1 & 0 & 0 \\ J^s_{q,kn} & J_{q,k(n+1)} & -q^kJ_{q,kn} \\ J^s_{q,k(n-1)} & J_{q,kn} & -q^kJ_{q,k(n-1)} \end{bmatrix}. \quad \square$$

**Corollary 3.2.** *If $J_{q,kn}$ is the $kn$-th $q$-Jacobsthal number, then*

$$J_{q,k}J^s_{q,k(n+m)} = J^s_{q,kn} + J^s_{q,k(m-1)}(J_{q,k(n+1)} - q^kJ_{q,kn}) + J_{q,k(n+1)}J_{q,km}.$$

(3.5)

**Proof.** The second row and first column in the matrix $G_q^{n+m}$ is equal to $J^s_{q,k(n+m)}$. Considering the (2,1)-entries of the matrix equation $G_q^nG_q^m$,
$\mathrm{J}_{q,k}J^s_{q,k(n+m)} = J^s_{q,kn} + J_{q,k(n+1)}J^s_{q,km} - q^kJ_{q,kn}J^s_{q,k(m-1)}$
$= J^s_{q,kn} + J_{q,k(n+1)}(J^s_{q,k(m-1)} + J_{q,km}) - q^kJ_{q,kn}J^s_{q,k(m-1)}$
$= J^s_{q,kn} + J^s_{q,k(m-1)}(J_{q,k(n+1)} - q^kJ_{q,kn}) + J_{q,k(n+1)}J_{q,km}$. Thus, the proof is completed. $\square$

If $q = -2$ and $k = 1$, the classic Jacobsthal sequence satisfy

$$J^s_{n+m} = J^s_n + J^s_{m-1}J_{n+2} + J_{n+1}J_{q,m}. \tag{3.6}$$

## 4. conclusion

In this paper, we consider the amazing relationships between the $q$-Jacobsthal numbers and matrices. Sum formula involving the terms of $q$-Jacobsthal numbers is one of the most important results obtained in this study.

Note that the $q$-Jacobsthal numbers represent a generalization of the classical Fibonacci sequence. In particular, we have obtained new results by matrix method in these numbers.

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