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ON GENERALIZED FIBONACCI AND LUCAS NUMBERS BY MATRIX METHODS

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Abstract

In this study we define the generalized Lucas $V(p,q)$-matrix similar to the generalized Fibonacci $U(1,-1)$-matrix. The $V(p,q)$-matrix is different from the Fibonacci $U(p,q)$-matrix, but is related to it. Using this matrix representation, we have found some well-known equalities and a Binet-like formula for the generalized Fibonacci and Lucas numbers.

Keywords: Matrix methods, generalized Lucas numbers, Binet's formula.

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1. Introduction

Consider a sequence $\{W_n\} = \{W_n(a,b,p,q)\}$ defined by the recurrence relation

(1.1) $$W_n = pW_{n-1} - qW_{n-2}, n \geq 2,$$

with $W_0 = a$, $W_1 = b$, where a, b, p and q are integers with $p > 0$, $q \neq 0$.

We are interested in the following two special cases of $\{W_n\}$: $\{U_n\}$ is defined by $U_0 = 0$, $U_1 = 1$, and $\{V_n\}$ is defined by $V_0 = 2$, $V_1 = p$. It is well known that $\{U_n\}$ and $\{V_n\}$ can be expressed in the form

(1.2) $$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, V_n = \alpha^n + \beta^n,$$

where $\alpha = \frac{p+\sqrt{\Delta}}{2}$, $\beta = \frac{p-\sqrt{\Delta}}{2}$ and the discriminant is $\Delta = p^2 - 4q$.

Especially, if $p = -q = 1$ and $2p = -q = 2$, $\{U_n\}$ is the usual Fibonacci and Jacobsthal sequence, respectively.

We define $U(p,q)$ be the 2×2 matrix

(1.3) $$U(p,q) = \begin{bmatrix} p & -q \\ 1 & 0 \end{bmatrix},$$

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then for an integer n with $n \geq 1$, $U^n(p,q)$ has the form

$$U^n(p,q) = \begin{bmatrix} U_{n+1} & -qU_n \\ U_n & -qU_{n-1} \end{bmatrix}. \tag{1.4}$$

This property provides an alternate proof of Cassini Fibonacci formula:

$$U_{n+1}U_{n-1} - U_n^2 = -q^{n-1}.$$

Also, let n and m be two integers such that $m, n \geq 1$. The following results are obtained from the identity $U^{n+m}(p,q) = U^n(p,q)U^m(p,q)$ for the matrix (1.4):

$$U_{n+m+1} = U_{n+1}U_{m+1} - qU_nU_m, \tag{1.5}$$

$$U_{n+m} = U_nU_{m+1} - qU_{n-1}U_m. \tag{1.6}$$

In this study, we define the Lucas $V(p,q)$-matrix by

$$V(p,q) = \begin{bmatrix} p^2 - 2q & -pq \\ p & -2q \end{bmatrix}. \tag{1.7}$$

It is easy to see that

$$\begin{bmatrix} V_{n+1} \\ V_n \end{bmatrix} = V(p,q) \begin{bmatrix} U_n \\ U_{n-1} \end{bmatrix} \text{ and } \Delta \begin{bmatrix} U_{n+1} \\ U_n \end{bmatrix} = V(p,q) \begin{bmatrix} V_n \\ V_{n-1} \end{bmatrix}$$

where U_n and V_n are as above. Our aim, is not to compute powers of matrices. Our aim is to find different relations between matrices containing generalized Fibonacci and Lucas numbers. That is, we obtain relations between the generalized Fibonacci $U(p,q)$-matrix and the Lucas $V(p,q)$ in Theorem 2.1.

2. $V(p,q)$-matrix representation of the generalized Lucas numbers

In this section, we will present a new matrix representation of the generalized Fibonacci and Lucas numbers. We obtain Cassini's formula and properties of these numbers by a similar matrix method to the Fibonacci $U(1,-1)$-matrix.

2.1. Theorem. *Let $V(p,q)$ be a matrix as in (1.7). Then, for all integers $n \geq 1$, the following matrix power is held below*

$$V^n(p,q) = \begin{cases} \Delta^{\frac{n}{2}} \begin{bmatrix} U_{n+1} & -qU_n \\ U_n & -qU_{n-1} \end{bmatrix} & if\ n\ even \\ \Delta^{\frac{n-1}{2}} \begin{bmatrix} V_{n+1} & -qV_n \\ V_n & -qV_{n-1} \end{bmatrix} & if\ n\ odd, \end{cases} \tag{2.1}$$

with $\Delta = p^2 - 4q$ and where U_n and V_n are the nth generalized Fibonacci and Lucas numbers, respectively.

Proof. We use mathematical induction on n. First, we consider odd n. For $n = 1$,

$$V^1(p,q) = \begin{bmatrix} V_2 & -qV_1 \\ V_1 & -qV_0 \end{bmatrix},$$

since $V_2 = p^2 - 2q$, $V_1 = p$ and $V_0 = 2$. So, (2.1) is indeed true for $n = 1$. Now we suppose it is true for $n = k$, that is

$$V^k(p,q) = \Delta^{\frac{k-1}{2}} \begin{bmatrix} V_{k+1} & -qV_k \\ V_k & -qV_{k-1} \end{bmatrix}.$$

Using the induction hypothesis and $V^2(p,q)$ by a direct computation. we can write

$$V^{k+2}(p,q) = V^k(p,q)V^2(p,q) = \Delta^{\frac{k+1}{2}} \begin{bmatrix} V_{k+3} & -qV_{k+2} \\ V_{k+2} & -qV_{k+1} \end{bmatrix},$$

as desired. Secondly, let us consider even n. For $n = 2$ we can write

$$V^2(p,q) = \Delta \begin{bmatrix} U_3 & -qU_2 \\ U_2 & -qU_1 \end{bmatrix}.$$

So, (2.1) is true for $n = 2$. Now, we suppose it is true for $n = k$, using properties of the generalized Fibonacci numbers and the induction hypothesis, we can write

$$V^{k+2}(p,q) = V^k(p,q)V^2(p,q) = \Delta^{\frac{k+2}{2}} \begin{bmatrix} U_{k+3} & -qU_{k+2} \\ U_{k+2} & -qU_{k+1} \end{bmatrix},$$

as desired. Hence, (2.1) holds for all n. □

2.2. Theorem. *Let $V(p,q)$ be a matrix as in (1.7). Then the following equalities are valid for all integers $n \geq 1$:*

(i) $\det(V^n(p,q)) = (-q\Delta)^n$,
(ii) $U_{n+1}U_{n-1} - U_n^2 = -q^{n-1}$,
(iii) $V_{n+1}V_{n-1} - V_n^2 = \Delta q^{n-1}$.

Proof. To establish (i) we use induction on n. Clearly $\det(V(p,q)) = -q\Delta$. If we make the induction hypothesis $\det(V^k(p,q)) = (-q\Delta)^k$, then from the multiplicative property of the determinant we have

$$\det(V^{k+1}(p,q)) = \det(V^k(p,q))\det(V^1(p,q)) = (-q\Delta)^{k+1},$$

which shows (i) for all $n \geq 1$. The identities (ii) and (iii) easily seen by using (2.1) and (i) for even and odd values of n, respectively. □

2.3. Theorem. *Let n be any integer. The well-known Binet formulas for the generalized Fibonacci and Lucas numbers are*

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } V_n = \alpha^n + \beta^n,$$

where $\alpha = \frac{p+\sqrt{\Delta}}{2}$ and $\beta = \frac{p-\sqrt{\Delta}}{2}$.

Proof. Let the matrix $V(p,q)$ be as in (1.7). We can write the characteristic equation of $V(p,q)$ as $x^2 - \Delta x - q\Delta = 0$. If we calculate the eigenvalues and eigenvectors of the matrix $V(p,q)$ we obtain $\lambda_1 = -\Delta^{\frac{1}{2}}\beta$, $\lambda_2 = \Delta^{\frac{1}{2}}\alpha$, $v_1 = (\beta, 1)$, $v_2 = (\alpha, 1)$, where $\alpha = \frac{p+\sqrt{\Delta}}{2}$ and $\beta = \frac{p-\sqrt{\Delta}}{2}$. Then we can diagonalize of the matrix $V(p,q)$ by $D = P^{-1}V(p,q)P$, where

$$P = (v_1^T, v_2^T) = \begin{bmatrix} \beta & \alpha \\ 1 & 1 \end{bmatrix}, \tag{2.2}$$

and

$$D = \begin{bmatrix} -\Delta^{\frac{1}{2}}\beta & 0 \\ 0 & \Delta^{\frac{1}{2}}\alpha \end{bmatrix}. \tag{2.3}$$

From properties of similar matrices, we can write $D^n = P^{-1}V^n(p,q)P$, where n is any integer. Furthermore, we can obtain $V^n(p,q) = PD^nP^{-1}$. By (2.1) and taking the nth power of the diagonal matrix, we get

$$V^n(p,q) = \Delta^{\frac{n-1}{2}} \begin{bmatrix} \alpha^{n+1} + (-\beta)^{n+1} & -q(\alpha^n - (-\beta)^n) \\ \alpha^n - (-\beta)^n & -q(\alpha^{n-1} + (-\beta)^{n-1}) \end{bmatrix}. \tag{2.4}$$

Thus, the proof follows from theorem (2.1). □

3. Generalized Fibonacci Numbers and main results

3.1. Theorem. *For all integers m and n, the following equalities are valid:*

(i) $\Delta U_{m+n} = V_{m+1}V_n - qV_mV_{n-1}$,
(ii) $U_{m+n} = U_{m+1}U_n - qU_mU_{n-1}$,
(iii) $V_{m+n} = U_{m+1}V_n - qU_mV_{n-1}$,
(iv) $\Delta U_{m-n} = -q^{-n}(V_mV_{n+1} - V_{m+1}V_n)$.

Proof. $V_{m+n}(p,q)$ can be written, using (2.1), as

$$V^{m+n}(p,q) = \begin{cases} \Delta^{\frac{m+n}{2}} \begin{bmatrix} U_{m+n+1} & -qU_{m+n} \\ U_{m+n} & -qU_{m+n-1} \end{bmatrix} & \text{if } m+n \text{ even,} \\ \Delta^{\frac{m+n-1}{2}} \begin{bmatrix} V_{m+n+1} & -qV_{m+n} \\ V_{m+n} & -qV_{m+n-1} \end{bmatrix} & \text{if } m+n \text{ odd.} \end{cases} \tag{3.1}$$

For the case of odd m and n, $V^m(p,q)V^n(p,q)$ is:

$$\Delta^{\frac{m+n}{2}-1} \begin{bmatrix} V_{m+1}V_{n+1} - qV_mV_n & -q(V_{m+1}V_n - qV_mV_{n-1}) \\ V_mV_{n+1} - qV_{m-1}V_n & -q(V_mV_n - qV_{m-1}V_{n-1}) \end{bmatrix} \tag{3.2}$$

Comparing the entries in the first row and second column of the matrices (3.1) and (3.2), we obtain

$$\Delta U_{m+n} = V_{m+1}V_n - qV_mV_{n-1},$$

while comparing the entries in the second row and first column gives

$$\Delta U_{m+n} = V_mV_{n+1} - qV_{m-1}V_n.$$

For the case of even m and n, $V^m(p,q)V^n(p,q)$ is:

$$\Delta^{\frac{m+n}{2}} \begin{bmatrix} U_{m+1}U_{n+1} - qU_mU_n & -q(U_{m+1}U_n - qU_mU_{n-1}) \\ U_mU_{n+1} - qU_{m-1}U_n & -q(U_mU_n - qU_{m-1}U_{n-1}) \end{bmatrix} \tag{3.3}$$

Comparing the entries in the first row and second column for the matrices (3.1) and (3.3), we find that

$$U_{m+n} = U_{m+1}U_n - qU_mU_{n-1}$$

and the entries in the second row and first column

$$U_{m+n} = U_mU_{n+1} - qU_{m-1}U_n.$$

For cases of odd m and even n, or odd n and even m, $V^m(p,q)V^n(p,q)$ is:

$$\Delta^{\frac{m+n-1}{2}} \begin{bmatrix} U_{m+1}V_{n+1} - qU_mV_n & -q(U_{m+1}V_n - qU_mV_{n-1}) \\ U_mV_{n+1} - qU_{m-1}V_n & -q(U_mV_n - qU_{m-1}V_{n-1}) \end{bmatrix}. \tag{3.4}$$

Comparing the entries in the first row and second column for the matrices (3.1) and (3.4), we obtain the equations

$$V_{m+n} = U_{m+1}V_n - qU_mV_{n-1},$$

and the entries in the second row and first column

$$V_{m+n} = U_mV_{n+1} - qU_{m-1}V_n.$$

The inverse of the matrix $V^n(p,q)$ in (2.1) is given by

$$V^{-n}(p,q) = \begin{cases} \frac{1}{q^n\Delta^{\frac{n}{2}}} \begin{bmatrix} -qU_{n-1} & qU_n \\ -U_n & U_{n+1} \end{bmatrix} & \text{if } n \text{ even ,} \\ \frac{-1}{q^n\Delta^{\frac{n+1}{2}}} \begin{bmatrix} -qV_{n-1} & qV_n \\ -V_n & V_{n+1} \end{bmatrix} & \text{if } n \text{ odd .} \end{cases} \tag{3.5}$$

Similarly, by computing the equality $V^{m-n}(p,q) = V^m(p,q)V^{-n}(p,q)$ the desired results are obtained. Indeed, for the case of odd m and n,

$$\Delta U_{m-n} = -q^{-n}(V_m V_{n+1} - V_{m+1}V_n),$$

for the case of even m and n,

$$U_{m-n} = q^{-n}(U_m U_{n+1} - U_{m+1}U_n).$$

Finally, for the cases of odd n and even m, odd m and even n,

$$V_{m-n} = -q^{-n}(U_m V_{n+1} - U_{m+1}V_n).$$

□

3.2. Theorem. *If A is a square matrix with $A^2 = pA - qI$ and I matrix identity of order 2. Then, $A^n = U_n A - qU_{n-1}I$, for all $n \in \mathbb{Z}$.*

Proof. If $n = 0$, the proof is obvious because $U_{-1} = -q^{-1}$ by (1.2). It can be shown by induction that $A^n = U_n A - qU_{n-1}I$, for every positive integer n. We now show that $A^{-n} = U_{-n}A - qU_{-n-1}I$. Let $B = pI - A = qA^{-1}$, then

$$B^2 = (pI - A)^2 = p^2 I - 2pA + A^2 = p(pI - A) - qI = pB - qI,$$

this shows that $B^n = U_n B - qU_{n-1}I$. That is, $(qA^{-1})^n = U_n(pI - A) - qU_{n-1}I$. Therefore $q^n A^{-n} = -U_n A + (pU_n - qU_{n-1})I = -U_n A + U_{n+1}I$. Thus,

$$A^{-n} = -q^{-n}U_n A + q^{-n}U_{n+1}I = U_{-n}A - qU_{-n-1}I.$$

Thus, the proof is completed. □

The well-known identity

$$U_{n+1}^2 - qU_n^2 = U_{2n+1} \tag{3.6}$$

has as its Lucas counterpart

$$V_{n+1}^2 - qV_n^2 = \Delta U_{2n+1}. \tag{3.7}$$

Indeed, since $V_{n+1} = U_{n+2} - qU_n = pU_{n+1} - 2qU_n$ and $V_n = 2U_{n+1} - pU_n$, the equation (3.7) follows from (3.6). We define $R(p,q)$ be the 2×2 matrix

$$R(p,q) = \frac{1}{2}\begin{bmatrix} p & \Delta \\ 1 & p \end{bmatrix}, \tag{3.8}$$

then for an integer n, $R^n(p,q)$ has the form

$$R^n(p,q) = \frac{1}{2}\begin{bmatrix} V_n & \Delta U_n \\ U_n & V_n \end{bmatrix}. \tag{3.9}$$

3.3. Theorem. $V_n^2 - \Delta U_n^2 = 4q^n$, *for all $n \in \mathbb{Z}$.*

Proof. Since $\det(R(p,q)) = q$, $\det(R^n(p,q)) = (\det(R(p,q)))^n = q^n$. Moreover, since (3.9), we get $\det(R^n(p,q)) = \frac{1}{4}(V_n^2 - \Delta U_n^2)$. The proof is completed. □

Let us give a different proof of one of the fundamental identities of Generalized Fibonacci and Lucas numbers, by using the matrix $R(p,q)$.

3.4. Theorem. *For all integers m and n, the following equalities are valid:*

(i) $2V_{m+n} = V_m V_n + \Delta U_m U_n$,
(ii) $2U_{m+n} = U_m V_n + V_m U_n$,
(iii) $2q^n V_{m-n} = V_n V_m - \Delta U_n U_m$,
(iv) $2q^n U_{m-n} = U_n V_m - V_n U_m$.

Proof. Since

$$R^m(p,q)R^n(p,q) = \frac{1}{4}\begin{bmatrix} V_m & \Delta U_m \\ U_m & V_m \end{bmatrix}\begin{bmatrix} V_n & \Delta U_n \\ U_n & V_n \end{bmatrix}$$
$$= \frac{1}{4}\begin{bmatrix} V_mV_n + \Delta U_mU_n & \Delta(U_mV_n + V_mU_n) \\ U_mV_n + V_mU_n & V_mV_n + \Delta U_mU_n \end{bmatrix}$$

and

$$R^{m+n}(p,q) = \frac{1}{2}\begin{bmatrix} V_{m+n} & \Delta U_{m+n} \\ U_{m+n} & V_{m+n} \end{bmatrix}. \tag{3.10}$$

Comparing the entries $(1,1)$ and $(2,1)$ of the matrix (3.10), we obtain the equations

$$2V_{m+n} = V_mV_n + \Delta U_mU_n,$$

and

$$2U_{m+n} = U_mV_n + V_mU_n.$$

Furthermore,

$$R^m(p,q)R^{-n}(p,q) = R^m(p,q)(R^n(p,q))^{-1}$$
$$= \frac{1}{4q^n}\begin{bmatrix} V_m & \Delta U_m \\ U_m & V_m \end{bmatrix}\begin{bmatrix} V_n & -\Delta U_n \\ -U_n & V_n \end{bmatrix}$$
$$= \frac{1}{4q^n}\begin{bmatrix} V_mV_n - \Delta U_mU_n & \Delta(U_mV_n - V_mU_n) \\ U_mV_n - V_mU_n & V_mV_n - \Delta U_mU_n \end{bmatrix},$$

and

$$R^{m-n}(p,q) = \frac{1}{2}\begin{bmatrix} V_{m-n} & \Delta U_{m-n} \\ U_{m-n} & V_{m-n} \end{bmatrix}. \tag{3.11}$$

Comparing the entries $(1,1)$ and $(2,1)$ of the matrix (3.11), we obtain the equations

$$2q^nV_{m-n} = V_nV_m - \Delta U_nU_m, \tag{3.12}$$

and $2q^nU_{m-n} = U_nV_m - V_nU_m$. □

3.5. Theorem. *For all integers m and n, the following equalities are valid:*

(i) $V_mV_n = V_{m-n} + q^nV_{m-n}$,
(ii) $U_mV_n = U_{m-n} + q^nU_{m-n}$.

Proof. By the definition of the matrix $R^n(p,q)$, it can be seen that

$$R^{m+n}(p,q) + q^nR^{m-n}(p,q) = \frac{1}{2}\begin{bmatrix} V_{m-n} + q^nV_{m-n} & \Delta(U_{m-n} + q^nU_{m-n}) \\ U_{m-n} + q^nU_{m-n} & V_{m-n} + q^nV_{m-n} \end{bmatrix}.$$

On the other hand,

$$R^{m+n}(p,q) + q^nR^{m-n}(p,q) = R^m(p,q)(R^n(p,q) + q^nR^{-n}(p,q))$$
$$= \frac{1}{2}\begin{bmatrix} V_m & \Delta U_m \\ U_m & V_m \end{bmatrix}\begin{bmatrix} V_n & 0 \\ 0 & V_n \end{bmatrix}$$
$$= \frac{1}{2}\begin{bmatrix} V_mV_n & \Delta U_mV_n \\ U_mV_n & V_mV_n \end{bmatrix}.$$

Then, the results follow by comparing entries in the two matrices. □

References

[1] Gould, H.W.,*A history of the Fibonacci Q-matrix and a higher-dimensional problem*, Fibonacci Quart. **19**, 7-250, 1981.

[2] Horadam, A. F., *Jacobsthal representation numbers, Fibonacci Quarterly*, **34**, 40-54, 1996.

[3] Kalman, D. and Mena, R., *The Fibonacci numbers - exposed, Math. Mag.*, **76**, 81-167, 2003.

[4] Koshy, T.,*Fibonacci and Lucas Numbers with Applications*, Wiley - Interscience, New York, 2001.

[5] Silvester, J.R.,*Fibonacci properties by matrix methods*, Mathematical Gazette **63**, 188-191, 1979.