A note on Generalized Hermite polynomials

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***Abstract*—** By starting from the general theory of the one-variable Hermite polynomials, we will introduce a bi-dimensional generalization of them that is useful to obtain a different approach with the harmonic oscillator functions. We will see some interesting properties of this class of Hermite polynomials and we also discuss the related applications on the particular partial differential equations.

***Keywords*—** Hermite Polynomials, Generating Functions, Orthogonal Polynomials, Special Functions, Harmonic oscillator.

I. INTRODUCTION

IN previous papers [1,2] we have discussed different kinds of the Kampé de Feriét Hermite polynomials and some examples of their applications [3,4]. The approach we have used to introduce the different families of these important orthogonal polynomials has been extremely varied. We want now introduce the ordinary one-variable Hermite polynomials and the related generalization of two-variable by using the formalism and the techniques of the exponential operators. If we consider a function $f(x)$, which is analytic in a neighborhood of the origin, it can be expanded in Taylor series and, in particular, we can write:

$$f(x+\lambda)=\sum_{n=0}^{+\infty}\frac{\lambda^n}{n!}f^{(n)}(x), \tag{1}$$

where λ is a continuous parameter.
The so-called “shift” or “translation” operator can be defined as:

$$e^{\lambda\frac{d}{dx}}f(x)=\sum_{n=0}^{+\infty}\frac{\lambda^n}{n!}\frac{d^n}{dx^n}f(x)=\sum_{n=0}^{+\infty}\frac{\lambda^n}{n!}\frac{d^n}{dx^n}f^{(n)}(x)=f(x+\lambda) \tag{2}$$

limiting ourselves to real domain and by assuming that λ is a real number and $f(x)$ is also analytic in $x+\lambda$, without any other restriction. The action of the exponential operator, on an analytic function $f(x)$, produces a shift of the variable x by λ. The two-variable Hermite polynomials can be defined by using the relation stated in (2). After noting that:

$$e^{yD}f(x)=f(x+y)=\sum_{n=0}^{+\infty}\frac{y^n}{n!}f^{(n)}(x), \tag{3}$$

we have:

$$f(x)=x^m \text{ implies } e^{yD}x^m=(x+y)^m, \tag{4}$$

$$f(x)=\sum_{m=0}^{+\infty}a_m x^m \text{ implies } e^{yD}f(x)=\sum_{m=0}^{+\infty}a_m(x+y)^m \tag{5}$$

The above procedure can be easily generalized to exponential operators containing higher derivatives. In fact by considering the second derivative, we get:

$$e^{yD^2}f(x)=\sum_{n=0}^{+\infty}\frac{y^n}{n!}f^{(2n)}(x), \tag{6}$$

and by noting that:

$$D^{2n}x^m=\frac{m!}{(m-2n)!}x^{m-2n}, \tag{7}$$

we have:

$$e^{yD^2}x^m=\sum_{n=0}^{[m/2]}\frac{y^n}{n!}\frac{m!}{(m-2n)!}x^{m-2n}. \tag{8}$$

We can now define the two-variable Hermite polynomials $H_n^{(2)}(x,y)$ of Kampé de Fériet form [5,6] by the following formula:

$$H_n^{(2)}(x,y)=n!\sum_{r=0}^{[n/2]}\frac{y^r x^{n-2r}}{r!(n-2r)!}. \tag{9}$$

It is important to note that, assuming:

$$f(x)=\sum_{m=0}^{+\infty}a_m x^m, \tag{10}$$

the general identity reads:

$$e^{yD^2}f(x)=\sum_{n=0}^{+\infty}a_n H_n^{(2)}(x,y), \tag{11}$$

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and then, we can immediately obtain:

$$H_n^{(1)}(x,y) = (x+y)^n , \tag{12}$$

which can also be recast in the form:

$$e^{yD} f(x) = \sum_{n=0}^{+\infty} a_n H_n^{(1)}(x,y) . \tag{13}$$

In the following we will indicate the two-variable Hermite polynomials of Kampé de Fériet form by using the symbol $H_n(x,y)$ instead than $H_n^{(2)}(x,y)$.

The two-variable Hermite polynomials $H_n(x,y)$ are linked to the ordinary Hermite polynomials by the following relations [5]:

$$H_n\left(x,-\frac{1}{2}\right) = He_n(x) \tag{14}$$

where:

$$He_n(x) = n! \sum_{r=0}^{[n/2]} \frac{(-1)^r x^{n-2r}}{2^r r!(n-2r)!} , \tag{15}$$

and:

$$H_n\left(2x,-1\right) = H_n(x) , \tag{16}$$

where:

$$H_n(x) = n! \sum_{r=0}^{[n/2]} \frac{(-1)^r (2x)^{n-2r}}{r!(n-2r)!} . \tag{17}$$

Finally, it is also important to note that the Hermite polynomials $H_n(x,y)$ satisfy the relation:

$$H_n(x,0) = x^n . \tag{18}$$

From the above relations, we can deduce that the generalized Hermite polynomials satisfy the following partial differential equation:

$$\frac{\partial}{\partial y} H_n(x,y) = \frac{\partial^2}{\partial x^2} H_n(x,y) . \tag{19}$$

This result help us to derive an important operational rule for the two-variable Hermite polynomials. In fact, by considering the differential equation (19) as linear ordinary in the variable *y* and by remanding the equation (18) we can immediately state the following relation:

$$H_n(x,y) = e^{y\frac{\partial^2}{\partial x^2}} H_n(x,0) = e^{y\frac{\partial^2}{\partial x^2}} x^n . \tag{20}$$

The generating function [7] of the above two-variable Hermite polynomials can be state in many ways, we have for instance, after noting that they solved the following differential-difference equation:

$$\begin{aligned} &\frac{d}{dz} Y_n(z) = anY_n(z) + bn(n-1)Y_{n-2}(z) \\ &Y_n(0) = \delta_{n,0} \end{aligned} \tag{21}$$

where *a* and *b* are real numbers, that by exploiting the generating function method, setting:

$$G(z;t) = \sum_{n=0}^{+\infty} \frac{t^n}{n!} Y_n(z) \tag{22}$$

with *t* continuous variable, we can rewrite the equation (21) in the form:

$$\begin{aligned} &\frac{d}{dz} G(z;t) = (at+bt^2)G(z;t) \\ &G(0,t) = 1 \end{aligned} . \tag{23}$$

that is a linear ordinary differential equation and then its solution reads:

$$G(x;t) = \exp(xt + yt^2) , \tag{24}$$

where we have putted $az = x$ and $bz = y$.

Finally, by exploiting the r.h.s of the previous relation we find the generating function of the generalized Hermite polynomials $H_n(x,y)$, that is:

$$\exp(xt + yt^2) = \sum_{n=0}^{+\infty} \frac{t^n}{n!} H_n(x,y) . \tag{25}$$

II. HERMITE POLYNOMIALS WITH TWO INDICES AND TWO VARIABLES

In the previous section we have introduced the one-variable, one-index Hermite polynomials $He_n(x)$ as a particular case of the polynomials $H_n(x,y)$. It is possible to use these polynomials to introduce a different class of Hermite polynomials with two indices and two variables, which are a vectorial extension of the polynomials $He_n(x)$; this means that from an index acts on a one-dimensional variable, we will have a couple of indices acting on a two-dimensional variable. Let the positive quadratic form:

$$\begin{aligned} &q(x,y) = ax^2 + 2bxy + cy^2 \\ &a, c > 0 \\ &\Delta = ac - b^2 > 0 \end{aligned} \tag{26}$$

where *a*, *b*, *c* are real numbers. The associated matrix reads:

$$\hat{M} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \tag{27}$$

and since we have putted $ac - b^2 > 0$, $|\hat{M}| > 0$, that is an invertible matrix. Let now a vector:

$$\underline{z} = \begin{pmatrix} x \\ y \end{pmatrix},$$

in the space $\mathbb{R}^2$, it immediately follows that:

$$\begin{aligned} &q\left(\underline{z}\right) = \underline{z}^t \hat{M} \underline{z} \\ &q\left(\underline{z}\right) = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = ax^2 + 2bxy + cy^2. \end{aligned} \tag{28}$$

Let:

$$\underline{z} = \begin{pmatrix} x \\ y \end{pmatrix} \text{ and } \underline{h} = \begin{pmatrix} t \\ u \end{pmatrix}$$

two vectors of the space $\mathbb{R}^2$ such that:

$$t \neq u \;,\; (|t|, |u|) < +\infty \;.$$

We will called two-index, two-variable Hermite polynomials [6] and we will indicate with the symbol $He_{m,n}(x,y)$, the polynomials defined by the following generating function:

$$e^{\underline{h}^t \hat{M} \underline{z} - \frac{1}{2} \underline{h}^t \hat{M} \underline{h}} = \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} \frac{t^m}{m!} \frac{u^n}{n!} He_{m,n}(x,y) \;. \tag{29}$$

(the superscript "*t*" denotes transpose).

These polynomials are exploited in many fields of pure and applied mathematics [8,9]. They are very useful in description of the quantum treatment [10] of coupled harmonic oscillator.

By using the definition of the quadratic form (eq. (26)), we can introduce the adjoint class of the two-index, two-variable Hermite polynomials $He_{m,n}(x,y)$.

Since we have stated that the associated matrix $\hat{M}$ at the quadratic form is invertible (eq. (26),(27)), we can define the quadratic form adjoint, by setting:

$$\bar{q}\left(\underline{z}\right) = \underline{z}^t \hat{M}^{-1} \underline{z} \;. \tag{30}$$

We define the adjoint polynomials of the two-index, two-variable Hermite polynomials, the polynomials expressed from the following relation:

$$e^{\underline{v}^t \hat{M}^{-1} \underline{k} - \frac{1}{2} \underline{k}^t \hat{M}^{-1} \underline{k}} = \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} \frac{r^m}{m!} \frac{s^n}{n!} G_{m,n}(x,y) \;, \tag{31}$$

where the vectors:

$$\underline{v} = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \;,\; \underline{k} = \begin{pmatrix} r \\ s \end{pmatrix} \in \mathbb{R}^2 \tag{32}$$

and such that:

$$\underline{v} = \hat{M} \underline{z} \;,\; \underline{k} = \hat{M} \underline{h} \;. \tag{33}$$

It is easy to note that, from the second of the relations contained in the above equation, it immediately follows that:

$$r \neq s \;,\; . \;(|r|, |s|) < +\infty \;.$$

By using the relations in the equation (33), the expression of the generating function (eq. (31)) defining the adjoint Hermite polynomials of two-index and two-variable, could be recast in the following form:

$$e^{\underline{z}^t \underline{k} - \frac{1}{2} \underline{k}^t \hat{M}^{-1} \underline{k}} = \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} \frac{r^m}{m!} \frac{s^n}{n!} G_{m,n}(x,y) \;. \tag{34}$$

In the following section(s) we will derive a number of identitie regarding the two-index, two-variable Hermite polynomials and their adjoint.

III. Generalized Identities Involving Two-Index, Two-Variable Hermite Polynomials

Before to proceed, it is necessary to remind some basic rules of the vectorial differential calculus. By the positions stated in the previous section, we have:

$$\frac{\partial}{\partial x}\left[\underline{z}^t \hat{M} \underline{z}\right] = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \;, \tag{35}$$

$$\frac{\partial^2}{\partial x^2}\left[\underline{z}^t \hat{M} \underline{z}\right] = 2 \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{36}$$

$$\frac{\partial^n}{\partial x^n}\left[\underset{-}{z}^t \hat{M} \underset{-}{z}\right]=0 \text{ , for } n \geq 3 . \tag{37}$$

Similar relations could be stated by deriving respect to y. The Hermite polynomials of the type $He_{m,n}(x,y)$ [5,6] satisfy the following recurrence relations:

$$\begin{aligned} &He_{m+1,n}(x,y)=(ax+by)He_{m,n}(x,y)-amHe_{m-1,n}(x,y)+ \\ &-bnHe_{m,n-1}(x,y) \end{aligned}, \tag{38}$$

$$\begin{aligned} &He_{m,n+1}(x,y)=(bx+cy)He_{m,n}(x,y)-bmHe_{m-1,n}(x,y)+ \\ &-cnHe_{m,n-1}(x,y) \end{aligned}, \tag{39}$$

where a, b, c are real numbers defined in the relations (26).

To prove the first of the above recurrence relations, it is enough to note that, by deriving with respect to t in the equation (29) [7], we have:

$$\begin{aligned} &\frac{\partial}{\partial t}\left[\underset{-}{z}^t \hat{M} \underset{-}{h}-\frac{1}{2}\underset{-}{h}^t \hat{M} \underset{-}{h}\right]\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\frac{t^m}{m!}\frac{u^n}{n!}He_{m,n}(x,y)= \\ &=\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}m\frac{t^{m-1}}{m!}\frac{u^n}{n!}He_{m,n}(x,y) \end{aligned} \tag{40}$$

and by exploiting the l.h.s, using the results stated in the equations (35-37), we get:

$$\begin{aligned} &\frac{\partial}{\partial t}\left[\underset{-}{z}^t \hat{M} \underset{-}{h}-\frac{1}{2}\underset{-}{h}^t \hat{M} \underset{-}{h}\right]\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\frac{t^m}{m!}\frac{u^n}{n!}He_{m,n}(x,y)=\begin{pmatrix}x & t\end{pmatrix}\hat{M}\begin{pmatrix}1\\0\end{pmatrix}+ \\ &-\frac{1}{2}\left[\begin{pmatrix}1 & 0\end{pmatrix}\hat{M}\begin{pmatrix}t\\u\end{pmatrix}+\begin{pmatrix}t & u\end{pmatrix}\hat{M}\begin{pmatrix}1\\0\end{pmatrix}\right]\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\frac{t^m}{m!}\frac{u^n}{n!}He_{m,n}(x,y), \end{aligned} \tag{41}$$

that is:

$$\begin{aligned} &(ax+by-at-bu)\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\frac{t^m}{m!}\frac{u^n}{n!}He_{m,n}(x,y)= \\ &=\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}m\frac{t^{m-1}}{m!}\frac{u^n}{n!}He_{m,n}(x,y). \end{aligned} \tag{42}$$

After manipulating the l.h.s of the above equation and by equating the like t-power, we immediately obtain the relation (38). By following the same procedure, but by deriving with respect to u in the equation (29), we have:

$$\begin{aligned} &(bx+cy)-\frac{1}{2}(2bt-2cu)\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\frac{t^m}{m!}\frac{u^n}{n!}He_{m,n}(x,y)= \\ &=\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}n\frac{t^m}{m!}\frac{u^{n-1}}{n!}He_{m,n}(x,y), \end{aligned} \tag{43}$$

and then the recurrence relation (39) follows.

By using the same procedure and the related formalism, it is possible to state others important relations involving the two-index, two-variable Hermite polynomials. By omitting the proof, since it completely similar to the above procedure, we can state the relations:

$$\frac{\partial}{\partial x}He_{m,n}(x,y)=amHe_{m-1,n}(x,y)bnHe_{m,n-1}(x,y)\,, \tag{44}$$

$$\frac{\partial}{\partial y}He_{m,n}(x,y)=bmHe_{m-1,n}(x,y)cnHe_{m,n-1}(x,y)\ . \tag{45}$$

The four relations explicated in the equations (38-39) and (44-45) can be combined to define the shift operators related to the Hermite polynomials of the type $He_{m,n}(x,y)$.

For instance, bi putting the relation (38) in the (44), we get:

$$\frac{\partial}{\partial x}He_{m,n}(x,y)=(ax+by)He_{m,n}(x,y)-He_{m+1,n}(x,y)\,, \tag{46}$$

and then:

$$\left[(ax+by)-\frac{\partial}{\partial x}\right]He_{m,n}(x,y)=He_{m+1,n}(x,y)\,. \tag{47}$$

By combining the other relations, using the same procedure of the above, we finally obtain:

$$\left[(bx+cy)-\frac{\partial}{\partial y}\right]He_{m,n}(x,y)=He_{m,n+1}(x,y)\,, \tag{48}$$

$$-\frac{1}{n\Delta}\left[b\frac{\partial}{\partial x}-a\frac{\partial}{\partial y}\right]He_{m,n}(x,y)=He_{m,n-1}(x,y)\,, \tag{49}$$

$$\frac{1}{m\Delta}\left[c\frac{\partial}{\partial x}-b\frac{\partial}{\partial y}\right]He_{m,n}(x,y)=He_{m-1,n}(x,y)\,. \tag{50}$$

It is now natural introduce the shift operators, by putting:

$$\hat{E}_{+,0}=(ax+by)-\frac{\partial}{\partial x}\,, \tag{51}$$

$$\hat{E}_{0,+}=(bx+cy)-\frac{\partial}{\partial y}\,, \tag{52}$$

$$\hat{E}_{-,0}=\frac{1}{m\Delta}\left(c\frac{\partial}{\partial x}-b\frac{\partial}{\partial y}\right), \tag{53}$$

$$\hat{E}_{0,-}=-\frac{1}{n\Delta}\left(b\frac{\partial}{\partial x}-a\frac{\partial}{\partial y}\right). \tag{54}$$

If we indicate with the symbol $f_{m,n}$ the generic Hermite polynomial function, it is possible to read the action of the shift operators in a more compact way; that is:

$$\begin{cases} \hat{E}_{\pm,0}\left[f_{m,n} \right] = f_{m\pm1,n} \\ \hat{E}_{0,\pm}\left[f_{m,n} \right] = f_{m,n\pm1}. \end{cases} \tag{55}$$

The operators defined above are discrete operator, in the sense that they depend by the indices *m* and *n*, that as we said are positive integer.

IV. DIFFERENTIAL EQUATIONS RELATED TO THE POLYNOMIALS $He_{m,n}(x,y)$

The relations stated in previous sections and in particular the action of the shift operators defined in the equations (51-54), allow us to state an important result for the two-index, two-variable Hermite polynomials discussed in this paper [6]. The polynomials $He_{m,n}(x,y)$ satisfied the following partial differential equation:

$$\left[-\underset{-z}{\partial^t}\, \hat{M}^{-1} \underset{-z}{\partial} + \underset{-}{z}^t \underset{-z}{\partial} \right] He_{m,n}(x,y) = (m+n) He_{m,n}(x,y)\,, \tag{56}$$

where we have indicated the partial derivative vector with:

$$\underset{-z}{\partial} = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \end{pmatrix}.$$

From the relations (55), it is evident that the following equations hold:

$$\hat{E}_{-,0}\left[\hat{E}_{+,0}\, He_{m,n}(x,y) \right] = He_{m,n}(x,y)\,, \tag{57}$$

$$\hat{E}_{0,-}\left[\hat{E}_{0,+}\, He_{m,n}(x,y) \right] = He_{m,n}(x,y)\,. \tag{58}$$

By exploiting these relations, we easily get:

$$\left[\frac{1}{(m+1)\Delta}\left(c\frac{\partial}{\partial x} - b\frac{\partial}{\partial y} \right) \right]\left[(ax+by) - \frac{\partial}{\partial x} \right] He_{m,n}(x,y) = \\ = He_{m,n}(x,y), \tag{59}$$

$$\frac{1}{\Delta}\left(-c\frac{\partial^2}{\partial x^2} + 2b\frac{\partial^2}{\partial x \partial y} - a\frac{\partial^2}{\partial y^2} \right) + \frac{1}{\Delta}\left[\Delta x \frac{\partial}{\partial x} + \right. \\ \left. + \Delta y \frac{\partial}{\partial y} \right] He_{m,n}(x,y) = (m+n) He_{m,n}(x,y). \tag{60}$$

By summing up these last expressions, we have:

$$\left[acx\frac{\partial}{\partial x} - b^2 y \frac{\partial}{\partial y} - c\frac{\partial^2}{\partial x^2} + 2b\frac{\partial^2}{\partial x \partial y} + \right. \\ \left. - b^2 x \frac{\partial}{\partial x} + acy\frac{\partial}{\partial y} - a\frac{\partial^2}{\partial y^2} \right] He_{m,n}(x,y) = \Delta(m+n) He_{m,n}(x,y), \tag{61}$$

and once we rearrange the terms in the l.h.s., we can write:

$$\left\{ \frac{1}{\Delta}\left(-c\frac{\partial^2}{\partial x^2} + 2b\frac{\partial^2}{\partial x \partial y} - a\frac{\partial^2}{\partial y^2} \right) + \frac{1}{\Delta}\left[\Delta x \frac{\partial}{\partial x} + \right. \right. \\ \left. \Delta y \frac{\partial}{\partial y} \right] He_{m,n}(x,y) = (m+n) He_{m,n}(x,y). \tag{62}$$

By noting that:

$$\hat{M}^{-1} = \frac{1}{\Delta}\begin{pmatrix} c & -b \\ -b & a \end{pmatrix} \text{ and } \underset{-z}{\partial^t} = \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{pmatrix}, \tag{63}$$

we immediately get:

$$\frac{1}{\Delta}\left(-c\frac{\partial^2}{\partial x^2} + 2b\frac{\partial^2}{\partial x \partial y} - a\frac{\partial^2}{\partial y^2} \right) = -\underset{-z}{\partial^t}\, \hat{M}^{-1} \underset{-z}{\partial}\,, \tag{64}$$

and:

$$x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} = \underset{-}{z}^t \underset{-z}{\partial}\,. \tag{65}$$

By substituting these last identities in the equation (62), we obtain:

$$\left[-\underset{-z}{\partial^t}\, \hat{M}^{-1} \underset{-z}{\partial} + \underset{-}{z}^t \underset{-z}{\partial} \right] He_{m,n}(x,y) = (m+n) He_{m,n}(x,y)\,, \tag{66}$$

which is exactly the partial differential equation defined in the equation (56).

By using an analogous procedure, it is also possible to prove that the adjoint two-index, two-variable Hermite polynomials, namely $G_{m,n}(x,y)$, satisfy a similar partial differential equation.

In a forthcoming paper we will discuss more aspects related to this family of Hermite polynomials and we will see, in particular, some applications to the harmonic oscillator functions [10,13] by using the property of bi-orthogonality satisfied by the Hermite functions derived from the Hermite polynomials treated in the present paper. Moreover, these polynomials are a very useful tool to investigate many problems connected with the theory of special polynomials as Chebyshev and Genebauer [3,11] and the field of special functions to derive relevant operational results [12,14].

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