



On linearization and connection coefficients for generalized Hermite polynomials

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ABSTRACT

We consider the problem of finding explicit formulas, recurrence relations and sign properties for both connection and linearization coefficients for generalized Hermite polynomials. Most of the computations are carried out by the computer algebra system *Maple* using appropriate algorithms.

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1. Introduction

In this paper, we deal with the connection and linearization problems which are defined as follows:

Given two polynomial sets $\{S_n\}_{n \geq 0}$ and $\{P_n\}_{n \geq 0}$ such that $\deg(S_n) = \deg(P_n) = n$. The so-called *connection problem* between them asks to find the coefficients $C_m(n)$ in the expression:

$$S_n(x) = \sum_{m=0}^n C_m(n) P_m(x). \quad (1.1)$$

When $S_{i+j}(x) = Q_i(x)R_j(x)$ in (1.1), $\{Q_n\}_n$ and $\{R_n\}_n$ being two more polynomial sets with $\deg(Q_n) = \deg(R_n) = n$, we are faced with the general *linearization problem*

$$Q_i(x)R_j(x) = \sum_{k=0}^{i+j} L_{ij}(k) P_k(x). \quad (1.2)$$

A particular case of this problem is the *standard linearization problem* or *Clebsch–Gordan-type problem* if $Q_n = R_n = P_n$.

The computation of the connection and linearization coefficients plays an important role in many situations of pure and applied mathematics and also in physical and quantum chemical applications. The study of the linearization problem has gained an increasing interest in the past years. In particular, the study of positivity conditions of the connection and linearization coefficients has received special attention [1, Lecture 5]. Many problems in harmonic analysis related to nontrigonometric orthogonal expansions depend on the nonnegativity of certain connection coefficients (see [1, Lecture 7]), while the nonnegativity of the linearization coefficients gives rise to a convolution structure associated with orthogonal polynomials [1,2].

The literature on this topic is extremely vast and a wide variety of methods, based on specific properties of the involved polynomials, have been devised for computing the linearization coefficients either in explicit form or by means of recursive relations (see e.g. [3–6] and the references therein).

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In a series of papers [7,8], Ronveaux et al. designed the so-called NAVIMA algorithm which allows us to calculate recursively the connection and linearization coefficients. There exists an alternative approach to building recurrence relations for both connection and linearization coefficients due to Lewanowicz [4,9].

In some cases (classical orthogonal polynomials), many results concerning the positivity of the connection and linearization coefficients and the recurrence relations satisfied by $C_m(n)$ and $L_{i,j}(k)$ are known. Hounkonnou et al. [6] proved that for a family of classical orthogonal polynomials the coefficients $L_{i,j}(k)$ satisfy a linear second-order recurrence relation involving only the index k . The explicit coefficients of the second-order recurrence relation were obtained in [9], rewriting for this purpose the fourth-order differential equation for the product $P_i P_j$.

A further, computer algebra based, method was proposed in [10]. Using several structure formulas of the classical systems these authors derive generic recurrence equations for the connection coefficients of these systems.

If $\{P_n\}_n$ is a semi-classical orthogonal family, the corresponding standard linearization coefficients, defined in (1.2), satisfy a linear recurrence relation involving only the index k . This property also extends to the linearization coefficients arising from an arbitrary number of products of semi-classical orthogonal polynomials [5].

A general method, based on suitable operators, generating functions and a simple manipulation of formal power series, was developed to solve connection and linearization problems. The coefficients are given explicitly, very often in terms of hypergeometric terms and/or terminating hypergeometric functions [11–13].

For the sign properties, the nonnegativity of the connection and linearization coefficients has many important consequences. Several criteria to get sign properties for the aforementioned coefficients have been investigated by many authors. Some of them are given in terms of corresponding spectral measures [14], the others impose conditions on coefficients in the recurrence formula satisfied by the polynomials [15,1,2,16]. Moreover, alternation of signs in the $C_m(n)$ sequence is linked to the relative position of the zeros of Q_n as compared to those of P_n [17].

In a recent paper [18], we solved a special case of the connection problem, called the *duplication problem*, which asks to find the connection coefficients in

$$P_n(ax) = \sum_{m=0}^n C_m(n, a) P_m(x),$$

where $\{P_n\}_{n \geq 0}$ belongs to a wide class of polynomials, including the classical orthogonal polynomials (Hermite, Laguerre, Jacobi) as well as the classical discrete orthogonal polynomials (Charlier, Meixner, Krawtchouk), the latter for the specific case $a = -1$. We gave explicit expressions as well as recurrence relations satisfied by these coefficients. The essential computations were done completely automatically by some packages of the *Maple* system [19,20]. The only prerequisite is the knowledge of a suitable generating function of the involved polynomials.

In this work we extend these results to the so-called generalized Hermite polynomials (see definition below).

The main aim of this paper is to study connection and linearization problems associated to generalized Hermite polynomials. We give an explicit expression of the connection and linearization coefficients as well as recurrence relations for these coefficients. Sign properties of both connection and linearization coefficients will be also given. As application, we consider the *classical Hermite* and the *classical Laguerre polynomials*.

The *generalized Hermite polynomial* set was introduced in [21] as a set of real polynomials orthogonal with respect to the weight $|x|^{2\mu} e^{-x^2}$, $\mu > -\frac{1}{2}$, then investigated by Chihara in his Ph.D. Thesis [22]. This family reduces to the ordinary Hermite polynomial set for $\mu = 0$.

The generalized Hermite polynomials have been mentioned in connection with the Gauss quadrature formulas in [23]. They were also studied by Rosenblum in connection with a Bose-like oscillator calculus [24] as a polynomial set generated by [24]

$$e^{-t^2} e_\mu(2xt) = \sum_{n=0}^{\infty} \mathcal{H}_n^\mu(x) \frac{t^n}{n!}, \quad \mu \in \mathbb{C}, \quad \mu \neq -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \dots, \quad (1.3)$$

where

$$e_\mu(x) = \sum_{n=0}^{\infty} \frac{x^n}{\gamma_\mu(n)},$$

with

$$\gamma_\mu(2m + \epsilon) = 2^{2m+\epsilon} m! \left(\mu + \frac{1}{2} \right)_{m+\epsilon}, \quad \epsilon = 0, 1, \quad (1.4)$$

where $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$.

Many other authors investigated properties of these polynomials, using classical methods well known in the theory of special functions. For instance, some characterization problems related to this polynomial set were given in [25,26]. Recently, many characteristic properties and operational rules associated to this family were given in [27]. In particular, it was shown

that $\{\mathcal{H}_n^\mu\}_n$ is a $(\frac{1}{2}\mathcal{D}_\mu)$ -Appell polynomial set of transfer power series $A(t) = e^{-t^2}$. This means that $\mathcal{D}_\mu(f)(\mathcal{H}_n^\mu) = 2n\mathcal{H}_{n-1}^\mu$ and

$$e^{-\frac{\mathcal{D}_\mu^2}{4}(x^n)} = \frac{\gamma_\mu(n)}{2^n n!} \mathcal{H}_n^\mu(x). \tag{1.5}$$

\mathcal{D}_μ is the well-known Dunkl operator associated with the parameter μ on the real line [24]:

$$\mathcal{D}_\mu(f)(x) = \frac{d}{dx}f(x) + \frac{\mu}{x}(f(x) - f(-x)).$$

2. Connection coefficients

In this section, we are interested in finding explicit formulas, recurrence relations and sign properties of the connection coefficients relating two generalized Hermite polynomials with different parameters. The obtained explicit connection formula (Eq. (2.11) below) generalizes a well-known connection formula for Laguerre polynomials and appears to be new.

We begin by recalling a result giving the connection coefficients between two σ -Appell polynomials. That is to say $\sigma P_n = nP_{n-1}$, $n = 0, 1, \dots, n$, where σ is a linear operator, not depending on n , and called lowering operator. (For more details, we refer the reader to [28,11] and the references therein.)

Lemma 1 ([11, Corollary 3.4]). *Let $\{P_n\}_{n \geq 0}$ and $\{Q_n\}_{n \geq 0}$ be two σ -Appell polynomial sets of transfer power series, respectively, A_1 and A_2 . Then*

$$Q_n(x) = \sum_{m=0}^n \frac{n!}{m!} \alpha_{n-m} P_m(x), \quad \text{where } \frac{A_2(t)}{A_1(t)} = \sum_{k=0}^{\infty} \alpha_k t^k. \tag{2.1}$$

It was shown in [27], that $\{\mathcal{H}_n^\mu\}_{n \geq 0}$ and $\{B_n(x) = \frac{2^n n!}{\gamma_\mu(n)} x^n\}_{n \geq 0}$ are two $(\frac{1}{2}\mathcal{D}_\mu)$ -Appell sets of transfer power series, respectively, e^{-t^2} and 1. As application, we obtain the following expansion formulas

$$\frac{\mathcal{H}_n^\mu(x)}{n!} = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^m (2x)^{n-2m}}{m! \gamma_\mu(n-2m)}, \tag{2.2}$$

$$\frac{(2x)^n}{\gamma_\mu(n)} = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{\mathcal{H}_{n-2m}^\mu(x)}{m!(n-2m)!}, \tag{2.3}$$

and, by composition, we get the explicit connection relation

$$\frac{\mathcal{H}_n^{\mu_2}(x)}{n!} = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \left(\sum_{p=0}^m \frac{(-1)^{m-p} \gamma_{\mu_1}(n-2m+2p)}{(m-p)! p! \gamma_{\mu_2}(n-2m+2p)} \right) \frac{\mathcal{H}_{n-2m}^{\mu_1}(x)}{(n-2m)!}. \tag{2.4}$$

To obtain a pure recurrence relation for $C_{n-2m}(n)$, with respect to m , we use Zeilberger's algorithm (see e.g. [19], Chapter 7) via the Maple sumrecursion command, and get, with the notation $D_m := C_{n-2m}(n)$:

• For n even,

$$(m+1)(2\mu_1 - 1 + n - 2m)D_{m+1} + 2(n - 2m - 1)(n - 2m)(-\mu_1 + m + \mu_2)D_m = 0. \tag{2.5}$$

• For n odd,

$$(m+1)(2\mu_1 + n - 2m)D_{m+1} + 2(n - 2m - 1)(n - 2m)(-\mu_1 + m + \mu_2)D_m = 0. \tag{2.6}$$

A unified form of the above recurrence relations can be written as follows

$$(m+1)(2\mu_1 + n - 2m - \theta_n)D_{m+1} + 2(n - 2m - 1)(n - 2m)(-\mu_1 + m + \mu_2)D_m = 0, \tag{2.7}$$

where $\theta_n = \frac{1+(-1)^n}{2}$.

Using Eq. (2.4) with the useful identities

$$\frac{(-1)^m}{(n-m)!} = \frac{(-n)_m}{n!} \quad \text{and} \quad (\delta)_{n+m} = (\delta)_n (\delta+m)_m, \quad 0 \leq m \leq n, \tag{2.8}$$

and the Chu–Vandermonde reduction formula [29],

$${}_2F_1 \left(\begin{matrix} -k, b \\ c \end{matrix}; 1 \right) = \frac{(c-b)_k}{(c)_k}, \quad c \neq 0, -1, -2, \dots, \tag{2.9}$$

we get the following simple and explicit form of the connection coefficients for $0 \leq m \leq \lfloor \frac{n}{2} \rfloor$,

$$C_{n-2m}(n) = \frac{n!}{m!(n-2m)!} \frac{4^m \lfloor \frac{n}{2} \rfloor!}{\lfloor \frac{n}{2} - m \rfloor!} \frac{\gamma_{\mu_1}(n-2m)}{\gamma_{\mu_2}(n)} (-1)^m (\mu_2 - \mu_1)_m. \tag{2.10}$$

The explicit formula (2.10) can be also obtained by induction using, for this purpose, the recurrence relation (2.7).

Putting $\widehat{\mathcal{H}}_n^{\mu_2}(x) = \frac{\mathcal{H}_n^{\mu_2}(x)}{\lfloor \frac{n}{2} \rfloor!}$, we get a simple form of the connection between two suitable generalized Hermite polynomials

$$\widehat{\mathcal{H}}_n^{\mu_2}(x) = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^m \frac{4^m}{m!} \frac{\gamma_{\mu_1}(n-2m)}{\gamma_{\mu_2}(n)} (\mu_2 - \mu_1)_m \widehat{\mathcal{H}}_{n-2m}^{\mu_1}(x). \tag{2.11}$$

So, when $\mu_2 > \mu_1$, the corresponding connection coefficients alternate in sign, while this coefficient is nonnegative if $\mu_2 - \mu_1$ is a negative integer. On the other hand, if $\mu_2 - \mu_1 < 0$ and it is not an integer, then the connection coefficient is always nonnegative provided that $\mu_2 - \mu_1 \geq m - 1$.

3. Linearization problem

3.1. Explicit expression and recurrence relation

To study the linearization problem, we begin by recalling the following result, which gives an explicit expression of the linearization coefficients associated to three polynomial sets of Brenke type, generalizing a product formula associated to Appell and q -Appell polynomials given by Carlitz in [30].

Corollary 2 ([13, Corollary 2.8]). Let $\{P_n\}_{n \geq 0}$, $\{Q_n\}_{n \geq 0}$ and $\{R_n\}_{n \geq 0}$ be three polynomial sets of Brenke type, i.e. polynomial sets generated, respectively, by

$$A_1(t)B_1(xt) = \sum_{n=0}^{\infty} \frac{P_n(x)}{n!} t^n, \quad A_2(t)B_2(xt) = \sum_{n=0}^{\infty} \frac{Q_n(x)}{n!} t^n \quad \text{and} \quad A_3(t)B_3(xt) = \sum_{n=0}^{\infty} \frac{R_n(x)}{n!} t^n, \tag{3.1}$$

where

$$A_p(t) = \sum_{k=0}^{\infty} a_k^{(p)} t^k, \quad \text{and} \quad B_p(t) = \sum_{k=0}^{\infty} b_k^{(p)} t^k, \quad a_0^{(p)} b_k^{(p)} \neq 0, \quad p = 1, 2, 3. \tag{3.2}$$

Then the linearization coefficients in (1.2) are given by

$$L_{ij}(k) = \frac{i!j!}{k!} \sum_{r=0}^i \sum_{s=0}^j \frac{b_r^{(2)} b_s^{(3)}}{b_{r+s}^{(1)}} a_{i-r}^{(2)} a_{j-s}^{(3)} \widehat{a}_{r+s-k}^{(1)}, \quad k = 0, 1, \dots, i+j, \tag{3.3}$$

where $\widehat{A}_1(t) = \frac{1}{A_1(t)} = \sum_{k=0}^{\infty} \widehat{a}_k^{(1)} t^k$.

According to (1.3), the generalized Hermite family is a Brenke type polynomial set.

The application of Corollary 2 allows us to solve the linearization problem for the generalized Hermite polynomials.

Taking into account the orthogonality of this family, we obtain

$$\mathcal{H}_i^{\mu}(x) \mathcal{H}_j^{\mu}(x) = \sum_{k=|i-j|}^{i+j} L_{ij}(k) \mathcal{H}_k^{\mu}(x). \tag{3.4}$$

Here the sum range is given by $k = |i - j|$ because if $k < |i - j|$ then $k + j < i$ or $k + i < j$. Hence $\deg(\mathcal{H}_k^{\mu} \mathcal{H}_j^{\mu}) < \deg \mathcal{H}_i^{\mu}$ or $\deg(\mathcal{H}_k^{\mu} \mathcal{H}_i^{\mu}) < \deg \mathcal{H}_j^{\mu}$. In both cases the coefficient

$$L_{ij}(k) = \int_{\mathbb{R}} \mathcal{H}_i^{\mu}(x) \mathcal{H}_j^{\mu}(x) \mathcal{H}_k^{\mu}(x) |x|^{2\mu} e^{-x^2} dx,$$

vanishes.

On the other hand and according to the symmetry property, $\mathcal{H}_n^{\mu}(-x) = (-1)^n \mathcal{H}_n^{\mu}(x)$, the associated standard linearization Eq. (3.4) can be reduced to

$$\mathcal{H}_i^{\mu}(x) \mathcal{H}_j^{\mu}(x) = \sum_{k=0}^{\min(i,j)} L_{ij}(i+j-2k) \mathcal{H}_{i+j-2k}^{\mu}(x).$$

By virtue of (3.3) and the generating function (1.3), we obtain the explicit expression

$$L_{ij}(i + j - 2k) = \frac{i!j!}{(i + j - 2k)!k!} \sum_{r=0}^{\lfloor \frac{i}{2} \rfloor} \sum_{s=0}^{\lfloor \frac{j}{2} \rfloor} \frac{\gamma_{\mu}(i + j - 2(r + s))}{\gamma_{\mu}(i - 2r)\gamma_{\mu}(j - 2s)} \frac{(-k)_{r+s}}{r!s!}. \tag{3.5}$$

Using the explicit formula (3.5), and a *Fasenmyer type algorithm* [19] to deduce recurrence equations for multiple hypergeometric series ([20]; see also [31]) we get – using Sprenger’s *multsum package* – the following recurrence relations (on one index) for the standard linearization coefficient of generalized Hermite polynomials.

Denote by $S(k) := L_{ij}(i + j - 2k)$ and consider the following three cases.

- For i even and j even:

$$-(k + 2)(-2k + 2\mu + i - 3 + j)S(k + 2) + 2(ij - 2i - 2ik + 5k + 3k^2 + 2 - 2j - 2jk) \times (i + j - 2k - 3)S(k + 1) + 4(j - k)(-k + i)(i + j - 2k - 1)(i + j - 2k - 3)S(k) = 0. \tag{3.6}$$

- For i even and j odd:

$$\begin{aligned} &-(k + 3)(k - i)(2k + 4 - i - j - 2\mu)(2k + 2 - i - j - 2\mu)S(k + 3) \\ &+ 2(-7jk - 22ik + 14k - 12i - 2j + 17k^2 + 5k^3 + 6i^2 + 4jik + 6ij \\ &- 8k^2i - 3jk^2 + 3i^2k - ji^2)(-i - j + 2k + 4)(2k + 2 - i - j - 2\mu)S(k + 2) \\ &- 4(k + 1 - i)(-i - j + 2k + 4)(-i - j + 2k + 2) \\ &\times (-13jk - 15ik + 7k - 5i - 2j + 15k^2 + 3j^2k + 2j\mu k + 8k^3 + 4i\mu + 2j^2 + 3i^2 + 10jik \\ &+ 2i\mu k - 2\mu k^2 + 8ij - 10k^2i - 10jk^2 + 3i^2k - 2j\mu i - 4\mu k - 2j^2i - 2ji^2)S(k + 1) \\ &+ 8(k - j)(k + 1 - i)(k - i)(-i - j + 2k) \\ &\times (-i - j + 2k + 1)(-i - j + 2k + 2)(-i - j + 2k + 4)S(k) = 0. \end{aligned} \tag{3.7}$$

- For i odd and j odd:

$$\begin{aligned} &-(k + 3)(2k + 5 - j - i - 2\mu)S(k + 3) + 2(i + j - 2k - 5) \\ &\times (-ji + 5k^2 - 18k - 16 + 2\mu k + 2\mu + 3ki + 6i + 3jk + 6j)S(k + 2) - 4(i + j - 2k - 5) \\ &\times (3ki + 3i - 2ji - 4k^2 - 7k - 3 + 3jk + 3j)(i + j - 2k - 3)S(k + 1) \\ &+ 8(k - j)(k - i)(i + j - 2k - 3)(i + j - 2k - 1)(i + j - 2k - 5)S(k) = 0. \end{aligned} \tag{3.8}$$

Note here that the linearization problem associated to generalized Hermite polynomials was already studied by Ronveaux et al. in [5] in the context of semi-classical polynomials. In fact, it was shown that the linearization coefficients $L_{ij}(k)$ satisfy a linear recurrence relation involving only the k index. The coefficients of this recurrence relation are very complicated and can only be obtained using a symbolic manipulation system like *Maple* or *Mathematica*, the obtained coefficients filled many pages [5].

Next, we consider two interesting particular cases involving classical Hermite and Laguerre polynomials.

Classical Hermite polynomials: The generalized Hermite polynomials reduce to the classical Hermite polynomials if $\mu = 0$.

To obtain an explicit recurrence relation satisfied by the linearization coefficients associated to classical Hermite polynomials, we use Sprenger’s *multsum package* by applying the *multsumrecursion* command to formula (3.5). The recurrence relation is given by

$$(-6k^2 - 4 + 4ik - 2ji + 4i - 10k + 4j + 4jk)S(k + 1) + 4(-j + k)(i - k)(i + j - 2k - 1)S(k) + (k + 2)S(k + 2) = 0, \tag{3.9}$$

where, as usual, $S(k) := L_{ij}(i + j - 2k)$.

The explicit linearization formula for Hermite polynomials is known as the Feldheim formula and is given by [1]

$$H_i(x)H_j(x) = \sum_{k=0}^{\min(i,j)} \binom{i}{k} \binom{j}{k} 2^k k! H_{i+j-2k}(x). \tag{3.10}$$

Note that (3.10) follows directly from (3.9) by using the Petkovšek–van Hoeij algorithm (see e.g. [19], Chapter 9), implemented in *Maple* by Mark van Hoeij as `LREtools[hypergeomsols]` [32].

Classical Laguerre polynomials: Now, we consider the classical Laguerre polynomials L_n^α defined by [33]

$$L_n^\alpha(x) = \frac{(\alpha + 1)_n}{n!} {}_1F_1\left(\begin{matrix} -n \\ \alpha + 1 \end{matrix}; x\right). \tag{3.11}$$

The generalized Hermite polynomials are related to the Laguerre polynomials by the following formula

$$H_{2n+\epsilon}^\mu(x) = \frac{(-1)^n (2n + \epsilon)!}{\left(\mu + \frac{1}{2}\right)_{n+\epsilon}} x^\epsilon L_n^{\mu-\frac{1}{2}+\epsilon}(x^2), \quad \epsilon = 0, 1. \tag{3.12}$$

This can be obtained according to (1.4) and the explicit formula (2.2).

The obtained expansion formulas can be used to recover some known expansions associated to Laguerre polynomials. For instance, combining (2.11) with (3.12) and using (1.4), we get the well-known connection formula [33] relating two families of Laguerre polynomials with different parameters

$$L_n^\beta(x) = \sum_{m=0}^n \frac{(\beta - \alpha)_m}{m!} L_{n-m}^\alpha(x). \quad (3.13)$$

For the linearization coefficients, applying formula (3.5) to the Laguerre polynomial set, we obtain, in view of (1.4) and (3.12),

$$L_i^\beta(x)L_j^\gamma(x) = \binom{i+j}{j} \sum_{k=0}^{i+j} F_{2:0}^{1:2} \left(\begin{matrix} -k : -\beta - i, -i; -\gamma - j, -j; 1, 1 \\ -\alpha - i - j, -i - j : -; -; \end{matrix} \right) \times (-1)^k \frac{(-\alpha - i - j)_k}{k!} L_{i+j-k}^\alpha(x), \quad (3.14)$$

where $F_{q:s}^{p:r}$ denotes the Kampé de Fériet function defined as follows [29]:

$$F_{q:s}^{p:r} \left(\begin{matrix} (a_p) : (b_r); (c_r); \\ (\alpha_q) : (\beta_s); (\gamma_s); \end{matrix} ; x, y \right) = \sum_{n,m=0}^{\infty} \frac{[a_p]_{n+m} [b_r]_n [c_r]_m}{[\alpha_q]_{n+m} [\beta_s]_n [\gamma_s]_m} \frac{x^n y^m}{n! m!}, \quad (3.15)$$

where $[a_p]_n = \prod_{j=1}^p (a_j)_n$.

For the standard case ($\alpha = \beta = \gamma$), using the explicit representation given by (3.14) and Sprenger's `multsum` package, we obtain the following second-order recurrence relation,

$$(-k + 2j)(2i - k)(i + j - k + \alpha)S(k) - (4ji - 4ki - 4i - 4kj - 4j + 3k^2 + 5k + 2) \times (i + j - k)S(k + 1) - 2(k + 2)(i + j - k)(i + j - k - 1)S(k + 2) = 0 \quad (3.16)$$

where $S(k) = L_{ij}(i + j - k)$.

Note here that this result corresponds to the fact that the standard linearization coefficients $L_{ij}(k)$ for classical orthogonal polynomials satisfy a second-order linear recurrence relation on the index k .

The explicit expression for the standard linearization coefficients for Laguerre polynomials was first given in [34] by means of a terminating hypergeometric function ${}_3F_2$:

$$L_{ij}(i + j - k) = \frac{(-2)^k}{k!} \frac{(i + j - k)!}{(i - k)!(j - k)!} {}_3F_2 \left(\begin{matrix} i + j + 1 - k + \alpha, -\frac{k}{2}, -\frac{k - 1}{2} \\ i - k + 1, j - k + 1 \end{matrix} ; 1 \right). \quad (3.17)$$

Unfortunately, this result cannot be automatically discovered from (3.16). However, *a posteriori*, one can prove that (3.17) is correct since it satisfies the same recurrence equation (proved by Zeilberger's algorithm) and has the same initial values.

For the general case (3.14), to deduce the recurrence relation associated to three Laguerre polynomials with arbitrary parameters we used the *Mathematica* package `MultSum` [31] to deduce for $S(k) = L_{ij}(i + j - k)$ a very complicated recurrence equation which can be found in the Appendix and is put for download on www.mathematik.uni-kassel.de/koepf/CA/MultSumLaguerre.nb.

3.2. Sign properties

Next, we will be concerned with the sign property of the linearization coefficients associated to generalized Hermite polynomials.

A generating function manipulation permits to show that the integral involving three Laguerre polynomials (with same parameters) is always nonnegative, we have [1],

$$(-1)^{i+j+k} \int_0^{+\infty} L_i^\alpha(x)L_j^\alpha(x)L_k^\alpha(x)x^\alpha e^{-x} dx \geq 0, \quad \alpha > -1. \quad (3.18)$$

This property can be useful to study the sign behavior of the linearization coefficients associated with the generalized Hermite polynomial set. This family is orthogonal with respect to the weight $|x|^{2\mu} e^{-x^2}$, $\mu > -\frac{1}{2}$, therefore to state the sign of the corresponding linearization coefficients it is sufficient to consider the sign behavior of

$$L_{ij}(k) = \int_{\mathbb{R}} H_i^\mu(x)H_j^\mu(x)H_k^\mu(x)|x|^{2\mu} e^{-x^2} dx.$$

To this end, we consider the following two cases.

- For $i = 2i'$ and $j = 2j'$, we have, in view of (3.12),

$$L_{ij}(i + j - 2k) = (-1)^k \alpha_{ijk} \int_0^\infty L_{i'}^{\mu-\frac{1}{2}}(x) L_{j'}^{\mu-\frac{1}{2}}(x) L_{i'+j'-k}^{\mu-\frac{1}{2}}(x) x^{\mu-\frac{1}{2}} e^{-x} dx, \tag{3.19}$$

where $\alpha_{ijk} = \frac{i!j!(i+j-2k)!}{(\mu+\frac{1}{2})_{i'}(\mu+\frac{1}{2})_{j'}(\mu+\frac{1}{2})_{i'+j'-k}}$.

According to (3.18), we deduce that for, $\mu > -\frac{1}{2}$, the linearization coefficient given by Eq. (3.19) is nonnegative.

- For $i = 2i' + 1$ and $j = 2j'$, we have

$$L_{ij}(i + j - 2k) = (-1)^k \beta_{ijk} \int_0^\infty L_{i'}^{\mu+\frac{1}{2}}(x) L_{j'}^{\mu-\frac{1}{2}}(x) L_{i'+j'-k}^{\mu+\frac{1}{2}}(x) x^{\mu+\frac{1}{2}} e^{-x} dx.$$

The sign of the previous integral cannot be obtained directly from (3.18).

Using the connection relation (3.13) for standard Laguerre polynomials, with $\beta = \mu - \frac{1}{2}$ and $\alpha = \mu + \frac{1}{2}$, we get

$$L_{j'}^{\mu-\frac{1}{2}}(x) = L_{j'}^{\mu+\frac{1}{2}}(x) - L_{j'-1}^{\mu+\frac{1}{2}}(x).$$

It follows that

$$\begin{aligned} L_{ij}(i + j - 2k) &= (-1)^k \beta_{ijk} \int_0^\infty L_{i'}^{\mu+\frac{1}{2}}(x) L_{j'}^{\mu+\frac{1}{2}}(x) L_{i'+j'-k}^{\mu+\frac{1}{2}}(x) x^{\mu+\frac{1}{2}} e^{-x} dx \\ &\quad + (-1)^{k+1} \beta_{ijk} \int_0^\infty L_{i'}^{\mu+\frac{1}{2}}(x) L_{j'-1}^{\mu+\frac{1}{2}}(x) L_{i'+j'-k}^{\mu+\frac{1}{2}}(x) x^{\mu+\frac{1}{2}} e^{-x} dx, \end{aligned} \tag{3.20}$$

where $\beta_{ijk} = \frac{i!j!(i+j-2k)!}{(\mu+\frac{1}{2})_{i'+1}(\mu+\frac{1}{2})_{j'}(\mu+\frac{1}{2})_{i'+j'+1-k}}$.

Then we conclude that the coefficient given by (3.20) is also nonnegative as a sum of two nonnegative integrals.

For odd values of i and j , taking into account the symmetry property $\mathcal{H}_n^\mu(-x) = (-1)^n \mathcal{H}_n^\mu(x)$ of the generalized Hermite polynomials, the corresponding linearization is always zero.

Finally, we conclude that the considered polynomials admit nonnegative linearization coefficients.

Note that sign properties of generalized Hermite polynomials have been already investigated in [2], using for this purpose, criterion based on the three term recurrence relation satisfied by the polynomials.

Appendix

The following is the recurrence for $\mathbf{S}(\mathbf{k}) = L_{ij}(i + j - k)$ in (3.14).

$$\begin{aligned} &-(i + j - k + \alpha)(2j - k + \alpha - \beta)(2i - k + \alpha - \gamma)(2\alpha i - 2\beta i + 2i - 2j - 2j\alpha + k\beta + 2\beta + 2j\gamma - k\gamma - 2\gamma)\mathbf{S}(\mathbf{k}) \\ &- (-5\beta k^4 + 5\gamma k^4 - 5\beta^2 k^3 + 5\gamma^2 k^3 - 10ik^3 + 10jk^3 - 10i\alpha k^3 + 10j\alpha k^3 + 23i\beta k^3 + 13j\beta k^3 + 12\alpha\beta k^3 \\ &- 20\beta k^3 - 13i\gamma k^3 - 23j\gamma k^3 - 12\alpha\gamma k^3 + 20\gamma k^3 - \beta^3 k^2 + \gamma^3 k^2 + 26i^2 k^2 - 26j^2 k^2 + 24i\alpha^2 k^2 - 24j\alpha^2 k^2 \\ &+ 21i\beta^2 k^2 + 5j\beta^2 k^2 + 8\alpha\beta^2 k^2 - 17\beta^2 k^2 - 5i\gamma^2 k^2 - 21j\gamma^2 k^2 - 8\alpha\gamma^2 k^2 + 4\beta\gamma^2 k^2 + 17\gamma^2 k^2 - 22ik^2 + 22jk^2 \\ &+ 26i^2\alpha k^2 - 26j^2\alpha k^2 + 2i\alpha k^2 - 2j\alpha k^2 - 34i^2\beta k^2 - 8j^2\beta k^2 - 9\alpha^2\beta k^2 + 56i\beta k^2 - 54ij\beta k^2 + 54j\beta k^2 \\ &- 54i\alpha\beta k^2 - 10j\alpha\beta k^2 + 41\alpha\beta k^2 - 26\beta k^2 + 8i^2\gamma k^2 + 34j^2\gamma k^2 + 9\alpha^2\gamma k^2 - 4\beta^2\gamma k^2 - 54i\gamma k^2 \\ &+ 54ij\gamma k^2 - 56j\gamma k^2 + 10i\alpha\gamma k^2 + 54j\alpha\gamma k^2 - 41\alpha\gamma k^2 + 4i\beta\gamma k^2 - 4j\beta\gamma k^2 + 26\gamma k^2 - 16i^3 k \\ &+ 16j^3 k - 18i\alpha^3 k + 18j\alpha^3 k + 3i\beta^3 k + \alpha\beta^3 k - 3\beta^3 k - 3j\gamma^3 k - \alpha\gamma^3 k + \beta\gamma^3 k + 3\gamma^3 k + 42i^2 k + 40ij^2 k - 42j^2 k \\ &- 40i^2\alpha^2 k + 40j^2\alpha^2 k + 20i\alpha^2 k - 20j\alpha^2 k - 28i^2\beta^2 k - 3\alpha^2\beta^2 k + 42i\beta^2 k - 16ij\beta^2 k + 17j\beta^2 k \\ &- 26i\alpha\beta^2 k - 2j\alpha\beta^2 k + 22\alpha\beta^2 k - 16\beta^2 k + 28j^2\gamma^2 k + 3\alpha^2\gamma^2 k - 17i\gamma^2 k + 16ij\gamma^2 k - 42j\gamma^2 k + 2i\alpha\gamma^2 k \\ &+ 26j\alpha\gamma^2 k - 22\alpha\gamma^2 k - 2i\beta\gamma^2 k - 13j\beta\gamma^2 k - 3\alpha\beta\gamma^2 k + 10\beta\gamma^2 k + 16\gamma^2 k - 16ik - 40i^2jk + 16jk - 16i^3\alpha k \\ &+ 16j^3\alpha k + 2i^2\alpha k + 40ij^2\alpha k - 2j^2\alpha k + 22i\alpha k - 40i^2j\alpha k - 22j\alpha k + 16i^3\beta k + 2\alpha^3\beta k - 42i^2\beta k + 28ij^2\beta k \\ &- 32j^2\beta k + 41i\alpha^2\beta k - 9j\alpha^2\beta k - 25\alpha^2\beta k + 43i\beta k + 68i^2j\beta k - 126ij\beta k + 57j\beta k + 68i^2\alpha\beta k - 4j^2\alpha\beta k \\ &- 91i\alpha\beta k + 48ij\alpha\beta k - 55j\alpha\beta k + 39\alpha\beta k - 13\beta k - 16j^3\gamma k - 2\alpha^3\gamma k - \beta^3\gamma k + 32i^2\gamma k - 68ij^2\gamma k + 42j^2\gamma k \\ &+ 9i\alpha^2\gamma k - 41j\alpha^2\gamma k + 25\alpha^2\gamma k + 13i\beta^2\gamma k + 2j\beta^2\gamma k + 3\alpha\beta^2\gamma k - 10\beta^2\gamma k - 57i\gamma k - 28i^2j\gamma k + 126ij\gamma k \\ &- 43j\gamma k + 4i^2\alpha\gamma k - 68j^2\alpha\gamma k + 55i\alpha\gamma k - 48ij\alpha\gamma k + 91j\alpha\gamma k - 39\alpha\gamma k - 4i^2\beta\gamma k + 4j^2\beta\gamma k - 9i\beta\gamma k \\ &+ 9j\beta\gamma k - 22i\alpha\beta\gamma k + 22j\alpha\beta\gamma k + 13\gamma k + 4i\alpha^4 - 4j\alpha^4 - 16i^3 - 24ij^3 + 16j^3 + 14i^2\alpha^3 - 14j^2\alpha^3 - 12i\alpha^3 \\ &+ 12j\alpha^3 - 2i^2\beta^3 + 4i\beta^3 - 2i\alpha\beta^3 + 2\alpha\beta^3 - 2\beta^3 + 2j^2\gamma^3 - 4j\gamma^3 + 2j\alpha\gamma^3 - 2\alpha\gamma^3 - 2j\beta\gamma^3 + 2\beta\gamma^3 + 2\gamma^3 \\ &+ 16i^2 + 32ij^2 - 16j^2 + 12i^3\alpha^2 - 12j^3\alpha^2 - 22i^2\alpha^2 - 28ij^2\alpha^2 + 22j^2\alpha^2 - 2i\alpha^2 + 28i^2j\alpha^2 + 2j\alpha^2 + 12i^3\beta^2 \end{aligned}$$

$$\begin{aligned}
& -26i^2\beta^2 + 8i\alpha^2\beta^2 - 2j\alpha^2\beta^2 - 6\alpha^2\beta^2 + 18i\beta^2 + 12i^2j\beta^2 - 24ij\beta^2 + 12j\beta^2 + 18i^2\alpha\beta^2 - 28i\alpha\beta^2 \\
& + 6ij\alpha\beta^2 - 8j\alpha\beta^2 + 12\alpha\beta^2 - 4\beta^2 - 12j^3\gamma^2 - 12ij^2\gamma^2 + 26j^2\gamma^2 + 2i\alpha^2\gamma^2 - 8j\alpha^2\gamma^2 + 6\alpha^2\gamma^2 + 2i\beta^2\gamma^2 \\
& - 2j\beta^2\gamma^2 - 12i\gamma^2 + 24ij\gamma^2 - 18j\gamma^2 - 18j^2\alpha\gamma^2 + 8i\alpha\gamma^2 - 6ij\alpha\gamma^2 + 28j\alpha\gamma^2 - 12\alpha\gamma^2 + 8j^2\beta\gamma^2 \\
& - 8i\beta\gamma^2 + 6ij\beta\gamma^2 - 10j\beta\gamma^2 - 4i\alpha\beta\gamma^2 + 10j\alpha\beta\gamma^2 - 6\alpha\beta\gamma^2 + 4\beta\gamma^2 + 4\gamma^2 - 4i + 24i^3j - 32i^2j + 4j - 4i^3\alpha \\
& - 24ij^3\alpha + 4j^3\alpha - 20i^2\alpha + 4ij^2\alpha + 20j^2\alpha + 10i\alpha + 24i^3j\alpha - 4i^2j\alpha - 10j\alpha + 4i^3\beta - 10i\alpha^3\beta + 6j\alpha^3\beta + 4\alpha^3\beta \\
& - 12i^2\beta - 24i^2j^2\beta + 48ij^2\beta - 24j^2\beta - 30i^2\alpha^2\beta + 8j^2\alpha^2\beta + 36i\alpha^2\beta - 6ij\alpha^2\beta + 6j\alpha^2\beta - 14\alpha^2\beta + 10i\beta \\
& - 24i^3j\beta + 68i^2j\beta - 60ij\beta + 16j\beta - 24i^3\alpha\beta + 48i^2\alpha\beta + 8j^2\alpha\beta - 36i\alpha\beta - 40i^2j\alpha\beta + 72ij\alpha\beta - 38j\alpha\beta + 10\alpha\beta \\
& - 2\beta + 24ij^3\gamma - 4j^3\gamma - 6i\alpha^3\gamma + 10j\alpha^3\gamma - 4\alpha^3\gamma + 2i\beta^3\gamma - 2\beta^3\gamma + 24i^2\gamma + 24i^2j^2\gamma - 68ij^2\gamma + 12j^2\gamma \\
& - 8i^2\alpha^2\gamma + 30j^2\alpha^2\gamma - 6i\alpha^2\gamma + 6ij\alpha^2\gamma - 36j\alpha^2\gamma + 14\alpha^2\gamma - 8i^2\beta^2\gamma + 10i\beta^2\gamma - 6ij\beta^2\gamma + 8j\beta^2\gamma - 10i\alpha\beta^2\gamma \\
& + 4j\alpha\beta^2\gamma + 6\alpha\beta^2\gamma - 4\beta^2\gamma - 16i\gamma - 48i^2j\gamma + 60ij\gamma - 10j\gamma + 24j^3\alpha\gamma - 8i^2\alpha\gamma + 40ij^2\alpha\gamma - 48j^2\alpha\gamma + 38i\alpha\gamma \\
& - 72ij\alpha\gamma + 36j\alpha\gamma - 10\alpha\gamma + 8i^2\beta\gamma - 8j^2\beta\gamma + 14i\alpha^2\beta\gamma - 14j\alpha^2\beta\gamma - 6i\beta\gamma + 6j\beta\gamma + 16i^2\alpha\beta\gamma - 16j^2\alpha\beta\gamma \\
& - 4i\alpha\beta\gamma + 4j\alpha\beta\gamma + 2\gamma)\mathbf{S}(\mathbf{k} + \mathbf{1}) - (-9\beta k^4 + 9\gamma k^4 - 7\beta^2 k^3 + 7\gamma^2 k^3 - 18ik^3 + 18jk^3 - 18i\alpha k^3 + 18j\alpha k^3 \\
& + 39i\beta k^3 + 21j\beta k^3 + 15\alpha\beta k^3 - 54\beta k^3 - 21i\gamma k^3 - 39j\gamma k^3 - 15\alpha\gamma k^3 + 54\gamma k^3 - \beta^3 k^2 + \gamma^3 k^2 + 42i^2 k^2 \\
& - 42j^2 k^2 + 30i\alpha^2 k^2 - 30j\alpha^2 k^2 + 27i\beta^2 k^2 + 7j\beta^2 k^2 + 6\alpha\beta^2 k^2 - 34\beta^2 k^2 - 7i\gamma^2 k^2 - 27j\gamma^2 k^2 - 6\alpha\gamma^2 k^2 \\
& + 3\beta\gamma^2 k^2 + 34\gamma^2 k^2 - 80ik^2 + 80jk^2 + 42i^2\alpha k^2 - 42j^2\alpha k^2 - 50i\alpha k^2 + 50j\alpha k^2 - 54i^2\beta k^2 - 12j^2\beta k^2 - 6\alpha^2\beta k^2 \\
& + 166i\beta k^2 - 78ij\beta k^2 + 114j\beta k^2 - 66i\alpha\beta k^2 - 8j\alpha\beta k^2 + 73\alpha\beta k^2 - 115\beta k^2 + 12i^2\gamma k^2 + 54j^2\gamma k^2 + 6\alpha^2\gamma k^2 \\
& - 3\beta^2\gamma k^2 - 114i\gamma k^2 + 78ij\gamma k^2 - 166j\gamma k^2 + 8i\alpha\gamma k^2 + 66j\alpha\gamma k^2 - 73\alpha\gamma k^2 + 8i\beta\gamma k^2 - 8j\beta\gamma k^2 + 115\gamma k^2 \\
& - 24i^3k + 24j^3k - 12i\alpha^3k + 12j\alpha^3k + 3i\beta^3k - 4\beta^3k - 3j\gamma^3k + 4\gamma^3k + 136i^2k + 48ij^2k - 136j^2k \\
& - 44i^2\alpha^2k + 44j^2\alpha^2k + 86i\alpha^2k - 86j\alpha^2k - 32i^2\beta^2k + 85i\beta^2k - 20ij\beta^2k + 28j\beta^2k - 19i\alpha\beta^2k - j\alpha\beta^2k \\
& + 22\alpha\beta^2k - 52\beta^2k + 32j^2\gamma^2k - 28i\gamma^2k + 20ij\gamma^2k - 85j\gamma^2k + i\alpha\gamma^2k + 19j\alpha\gamma^2k - 22\alpha\gamma^2k - i\beta\gamma^2k \\
& - 10j\beta\gamma^2k + 10\beta\gamma^2k + 52\gamma^2k - 116ik - 48i^2jk + 116jk - 24i^3\alpha k + 24j^3\alpha k + 92i^2\alpha k + 48ij^2\alpha k - 92j^2\alpha k \\
& - 18i\alpha k - 48i^2j\alpha k + 18j\alpha k + 24i^3\beta k - 152i^2\beta k + 36ij^2\beta k - 56j^2\beta k + 28i\alpha^2\beta k - 8j\alpha^2\beta k - 22\alpha^2\beta k + 220i\beta k \\
& + 84i^2j\beta k - 264ij\beta k + 194j\beta k + 76i^2\alpha\beta k - 8j^2\alpha\beta k - 205i\alpha\beta k + 48ij\alpha\beta k - 41j\alpha\beta k + 112\alpha\beta k - 100\beta k \\
& - 24j^3\gamma k + 56i^2\gamma k - 84ij^2\gamma k + 152j^2\gamma k + 8i\alpha^2\gamma k - 28j\alpha^2\gamma k + 22\alpha^2\gamma k + 10i\beta^2\gamma k + j\beta^2\gamma k - 10\beta^2\gamma k \\
& - 194i\gamma k - 36i^2j\gamma k + 264ij\gamma k - 220j\gamma k + 8i^2\alpha\gamma k - 76j^2\alpha\gamma k + 41i\alpha\gamma k - 48ij\alpha\gamma k + 205j\alpha\gamma k - 112\alpha\gamma k \\
& - 8i^2\beta\gamma k + 8j^2\beta\gamma k + 21i\beta\gamma k - 21j\beta\gamma k - 18i\alpha\beta\gamma k + 18j\alpha\beta\gamma k + 100\gamma k - 48i^3 - 24ij^3 + 48j^3 + 8i^2\alpha^3 \\
& - 8j^2\alpha^3 - 24i\alpha^3 + 24j\alpha^3 - 2i^2\beta^3 + 6i\beta^3 - 4\beta^3 + 2j^2\gamma^3 - 6j\gamma^3 + 4\gamma^3 + 104i^2 + 80ij^2 - 104j^2 + 12i^3\alpha^2 \\
& - 12j^3\alpha^2 - 72i^2\alpha^2 - 20ij^2\alpha^2 + 72j^2\alpha^2 + 52i\alpha^2 + 20i^2j\alpha^2 - 52j\alpha^2 + 12i^3\beta^2 - 50i^2\beta^2 + 62i\beta^2 + 12i^2\beta^2 \\
& - 40ij\beta^2 + 28j\beta^2 + 12i^2\alpha\beta^2 - 36i\alpha\beta^2 + 4ij\alpha\beta^2 + 20\alpha\beta^2 - 24\beta^2 - 12j^3\gamma^2 - 12ij^2\gamma^2 + 50j^2\gamma^2 - 28i\gamma^2 \\
& + 40ij\gamma^2 - 62j\gamma^2 - 12j^2\alpha\gamma^2 - 4ij\alpha\gamma^2 + 36j\alpha\gamma^2 - 20\alpha\gamma^2 + 6j^2\beta\gamma^2 + 4ij\beta\gamma^2 - 18j\beta\gamma^2 + 8\beta\gamma^2 + 24\gamma^2 - 56i \\
& + 24i^3j - 80i^2j + 56j - 36i^3\alpha - 24ij^3\alpha + 36j^3\alpha + 24i^2\alpha + 60ij^2\alpha - 24j^2\alpha + 20i\alpha + 24i^3j\alpha - 60i^2j\alpha - 20j\alpha \\
& + 36i^3\beta - 96i^2\beta - 24i^2j^2\beta + 76ij^2\beta - 64j^2\beta - 18i^2\alpha^2\beta + 6j^2\alpha^2\beta + 54i\alpha^2\beta - 4ij\alpha^2\beta - 18j\alpha^2\beta - 20\alpha^2\beta \\
& + 88i\beta - 24i^3j\beta + 144i^2j\beta - 208ij\beta + 100j\beta - 24i^3\alpha\beta + 122i^2\alpha\beta - 14j^2\alpha\beta - 146i\alpha\beta - 32i^2j\alpha\beta + 92ij\alpha\beta \\
& - 50j\alpha\beta + 52\alpha\beta - 28\beta + 24ij^3\gamma - 36j^3\gamma + 64i^2\gamma + 24i^2j^2\gamma - 144ij^2\gamma + 96j^2\gamma - 6i^2\alpha^2\gamma + 18j^2\alpha^2\gamma \\
& + 18i\alpha^2\gamma + 4ij\alpha^2\gamma - 54j\alpha^2\gamma + 20\alpha^2\gamma - 6i^2\beta^2\gamma + 18i\beta^2\gamma - 4i\beta^2\gamma - 8\beta^2\gamma - 100i\gamma - 76i^2j\gamma + 208ij\gamma \\
& - 88j\gamma + 24j^3\alpha\gamma + 14i^2\alpha\gamma + 32ij^2\alpha\gamma - 122j^2\alpha\gamma + 50i\alpha\gamma - 92ij\alpha\gamma + 146j\alpha\gamma - 52\alpha\gamma - 14i^2\beta\gamma + 14j^2\beta\gamma \\
& + 10i\beta\gamma - 10j\beta\gamma + 12i^2\alpha\beta\gamma - 12j^2\alpha\beta\gamma - 36i\alpha\beta\gamma + 36j\alpha\beta\gamma + 28\gamma)\mathbf{S}(\mathbf{k} + \mathbf{2}) - (i + j - k - 2)(7\beta k^3 \\
& - 7\gamma k^3 + 3\beta^2 k^2 - 3\gamma^2 k^2 + 14ik^2 - 14jk^2 + 14i\alpha k^2 - 14j\alpha k^2 - 22i\beta k^2 - 8j\beta k^2 - 6\alpha\beta k^2 + 42\beta k^2 + 8i\gamma k^2 \\
& + 22j\gamma k^2 + 6\alpha\gamma k^2 - 42\gamma k^2 - 16i^2k + 16j^2k - 12i\alpha^2k + 12j\alpha^2k - 8i\beta^2k + 13\beta^2k + 8j\gamma^2k - 13\gamma^2k + 66ik \\
& - 66jk - 16i^2\alpha k + 16j^2\alpha k + 54i\alpha k - 54j\alpha k + 16i^2\beta k - 94i\beta k + 20ij\beta k - 40j\beta k + 20i\alpha\beta k - 4j\alpha\beta k - 26\alpha\beta k \\
& + 75\beta k - 16j^2\gamma k + 40i\gamma k - 20ij\gamma k + 94j\gamma k + 4i\alpha\gamma k - 20j\alpha\gamma k + 26\alpha\gamma k - 4i\beta\gamma k + 4j\beta\gamma k - 75\gamma k - 48i^2 \\
& - 8ij^2 + 48j^2 + 4i^2\alpha^2 - 4j^2\alpha^2 - 36i\alpha^2 + 36j\alpha^2 + 4i^2\beta^2 - 20i\beta^2 + 12\beta^2 - 4j^2\gamma^2 + 20j\gamma^2 - 12\gamma^2 + 72i + 8i^2j \\
& - 72j - 44i^2\alpha - 8ij^2\alpha + 44j^2\alpha + 36i\alpha + 8i^2j\alpha - 36j\alpha + 44i^2\beta - 84i\beta - 8i^2j\beta + 56ij\beta - 48j\beta - 8i^2\alpha\beta \\
& + 56i\alpha\beta - 16j\alpha\beta - 24\alpha\beta + 36\beta + 8ij^2\gamma - 44j^2\gamma + 48i\gamma - 56ij\gamma + 84j\gamma + 8j^2\alpha\gamma + 16i\alpha\gamma - 56j\alpha\gamma + 24\alpha\gamma \\
& - 16i\beta\gamma + 16j\beta\gamma - 36\gamma)\mathbf{S}(\mathbf{k} + \mathbf{3}) + 2(i + j - k - 3)(i + j - k - 2)(k + 4)(2\alpha i - 2\beta i + 2i - 2j - 2j\alpha \\
& + k\beta + \beta + 2j\gamma - k\gamma - \gamma)\mathbf{S}(\mathbf{k} + \mathbf{4}) = 0.
\end{aligned}$$

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