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Certain Classes of Generating Functions for the Jacobi and Related Hypergeometric Polynomials

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Abstract—For a certain class of generalized hypergeometric polynomials, the authors first derive a general theorem on bilinear, bilateral, and mixed multilateral generating functions and then apply these generating functions in order to deduce the corresponding results for the classical Jacobi and Laguerre polynomials. They also consider several linear generating functions for these polynomials as well as for some multivariable Jacobi and multivariable Laguerre polynomials which were investigated in recent years. Some of the linear generating functions, presented in this paper, are associated with the Stirling numbers of the second kind.

Keywords—Generating functions, Jacobi polynomials, Laguerre polynomials, Hypergeometric polynomials, Stirling numbers of the second kind, Lauricella functions.

1. INTRODUCTION, DEFINITIONS, AND PRELIMINARIES

The classical Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ of degree n in x (and with parameters or indices α and β) are defined usually by

$$\begin{aligned} P_n^{(\alpha,\beta)}(x) :&= \sum_{k=0}^{n} \binom{n+\alpha}{n-k}\binom{n+\beta}{k}\left(\frac{x+1}{2}\right)^{n-k}\left(\frac{x-1}{2}\right)^{k} \\ &= \binom{n+\alpha}{n} {}_2F_1\left(-n, \alpha+\beta+n+1; \alpha+1; \frac{1-x}{2}\right), \end{aligned} \tag{1.1}$$

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where ${}_2F_1$ denotes the familiar (Gauss's) hypergeometric function which corresponds to the special case

$$u - 1 = v = 1,$$

of the generalized hypergeometric function ${}_uF_v$ with u numerator and v denominator parameters. These polynomials are *orthogonal* over the interval $(-1, 1)$ with respect to the weight function

$$w(x) := (1-x)^\alpha (1+x)^\beta;$$

in fact, we have (cf., e.g., [1, p. 68, equation (4.3.3)])

$$\begin{gathered}\int_{-1}^{1} (1-x)^\alpha (1+x)^\beta P_m^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(x)\, dx \\ = \frac{2^{\alpha+\beta+1}\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{n!(\alpha+\beta+2n+1)\Gamma(\alpha+\beta+n+1)}\delta_{m,n}, \\ (\min\{\Re(\alpha), \Re(\beta)\} > -1;\ m, n \in \mathbb{N}_0 := \mathbb{N}\cup\{0\};\ \mathbb{N} := \{1, 2, 3, \dots\}),\end{gathered} \tag{1.2}$$

where $\delta_{m,n}$ is the Kronecker delta.

Many other members of the family of classical orthogonal polynomials, including (for example) the Hermite polynomials $H_n(x)$, the Laguerre polynomials $L_n^{(\alpha)}(x)$, the Bessel polynomials $y_n(x; \alpha, \beta)$, the Gegenbauer (or ultraspherical) polynomials $C_n^\nu(x)$, the Legendre (or spherical) polynomials $P_n(x)$, and the Chebyshev polynomials $T_n(x)$ and $U_n(x)$ of the first and second kind, respectively, are special or limit cases of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$. In particular, for the classical Laguerre polynomials $L_n^{(\alpha)}(x)$ defined by

$$L_n^{(\alpha)}(x) := \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!} = \binom{n+\alpha}{n} {}_1F_1(-n; \alpha+1; x), \tag{1.3}$$

it is easily observed that [1, p. 103, equation (5.3.4)]

$$L_n^{(\alpha)}(x) = \lim_{|\beta|\to\infty} \left\{ P_n^{(\alpha,\beta)}\left(1 - \frac{2x}{\beta}\right) \right\}, \tag{1.4}$$

which can indeed be applied to deduce properties and characteristics of the Laguerre polynomials from those of the Jacobi polynomials.

For the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, it is known that (cf., e.g., [2, p. 170, equation 10.8(17)])

$$\begin{aligned}&\frac{\partial^k}{\partial t^k}\left\{P_n^{(\alpha,\beta)}(x+2t)\right\} \\ &= \begin{cases} \binom{\alpha+\beta+n+k}{k} k!\, P_{n-k}^{(\alpha+k,\beta+k)}(x+2t), & (k = 0, 1, \dots, n), \\ 0, & (k = n+1, n+2, n+3, \dots). \end{cases}\end{aligned} \tag{1.5}$$

Thus, as an immediate consequence of the Taylor expansion of

$$P_n^{(\alpha,\beta)}(x+2t)$$

in powers of t, we obtain

$$\sum_{k=0}^{n} \binom{\alpha+\beta+n+k}{k} P_{n-k}^{(\alpha+k,\beta+k)}(x) t^k = P_n^{(\alpha,\beta)}(x+2t). \tag{1.6}$$

Other linear generating functions of the type (1.6), in which the summation index appears in the Jacobi polynomials' parameters α and β, include the known results (cf., e.g., [3,4])

$$\sum_{k=0}^{n} \binom{k-\alpha-n-1}{k} P_{n-k}^{(\alpha,\beta+k)}(x)t^k = (1-t)^n P_n^{(\alpha,\beta)}\left(\frac{x-t}{1-t}\right) \tag{1.7}$$

and

$$\sum_{k=0}^{n} \binom{k-\beta-n-1}{k} P_{n-k}^{(\alpha+k,\beta)}(x)t^k = (1+t)^n P_n^{(\alpha,\beta)}\left(\frac{x-t}{1+t}\right), \tag{1.8}$$

which are, in fact, *equivalent*, since [1, p. 59, equation (4.1.3)]

$$P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x). \tag{1.9}$$

As a matter of fact, by appealing to the known relationship [1, p. 64, equation (4.22.1)]

$$P_n^{(\alpha,\beta)}(x) = \left(\frac{1-x}{2}\right)^n P_n^{(-\alpha-\beta-2n-1,\beta)}\left(\frac{x+3}{x-1}\right), \tag{1.10}$$

which, in view of (1.9), can be rewritten in the form:

$$P_n^{(\alpha,\beta)}(x) = \left(\frac{1+x}{2}\right)^n P_n^{(\alpha,-\alpha-\beta-2n-1)}\left(\frac{3-x}{1+x}\right), \tag{1.11}$$

each of the formulas (1.7) and (1.8) can be shown to be equivalent also to the generating function (1.6). Moreover, upon reversing the order of the sum in (1.6)–(1.8), if we replace α, β, and t (wherever necessary) by $\alpha-n$, $\beta-n$, and t^{-1}, respectively, we obtain the following *further* equivalent forms of these generating functions:

$$\sum_{k=0}^{n} \binom{n}{k}\binom{\alpha+\beta}{k}^{-1} P_k^{(\alpha-k,\beta-k)}(x)t^k = \binom{\alpha+\beta}{n}^{-1} t^n\, P_n^{(\alpha-n,\beta-n)}\left(x+2t^{-1}\right), \tag{1.12}$$

$$\sum_{k=0}^{n} \binom{n}{k}\binom{\alpha+k}{k}^{-1} P_k^{(\alpha,\beta-k)}(x)t^k = \binom{\alpha+n}{n}^{-1} (1+t)^n P_n^{(\alpha,\beta-n)}\left(\frac{1+xt}{1+t}\right), \tag{1.13}$$

and

$$\sum_{k=0}^{n} \binom{n}{k}\binom{\beta+k}{k}^{-1} P_k^{(\alpha-k,\beta)}(x)t^k = (-1)^n\binom{\beta+n}{n}^{-1} (1-t)^n P_n^{(\alpha-n,\beta)}\left(-\frac{1+xt}{1-t}\right), \tag{1.14}$$

respectively.

In view of several known hypergeometric representations for the classical Jacobi polynomials (cf., e.g., [5, p. 91, Problem 16]), it is not difficult to show that each of these last results (1.12)–(1.14) (and hence, also (1.6)–(1.8)) is a special case of the familiar hypergeometric generating function (cf. [6, p. 62, equation (25)]; see also [5, p. 138, equation 2.6(8)])

$$\begin{aligned} &\sum_{k=0}^{\infty} \binom{\lambda}{k} {}_{u+1}F_v(-k, a_1,\dots,a_u; b_1,\dots,b_v; x)t^k \\ &= (1+t)^{\lambda}{}_{u+1}F_v\left(-\lambda, a_1,\dots,a_u; b_1,\dots,b_v; \frac{xt}{1+t}\right) \\ &\qquad (\lambda\in\mathbb{C};\ |t|<1), \end{aligned} \tag{1.15}$$

when

$$u=v=1 \quad\text{and}\quad \lambda=n \quad (n\in\mathbb{N}_0).$$

More generally, for the hypergeometric polynomials

$$\mathcal{B}_n^m [a_1, \ldots, a_u; b_1, \ldots, b_v : x] := {}_{m+u}F_v [\Delta(m; -n), a_1, \ldots, a_u; b_1, \ldots, b_v; x], \tag{1.16}$$

where, for convenience, $\Delta(m; \lambda)$ abbreviates the array of m parameters

$$\frac{\lambda}{m}, \frac{\lambda+1}{m}, \ldots, \frac{\lambda+m-1}{m} \qquad (m \in \mathbb{N}),$$

it is known that (cf. [7, p. 187, equation (55)]; see also [5, p. 136, equation 2.6(2)])

$$\begin{aligned} &\sum_{k=0}^{\infty} \binom{\lambda}{k} \mathcal{B}_k^m [a_1, \ldots, a_u; b_1, \ldots, b_v : x]\, t^k \\ &= (1+t)^{\lambda} {}_{m+u}F_v \left[\Delta(m; -\lambda), a_1, \ldots, a_u; b_1, \ldots, b_v; x \left(\frac{t}{1+t}\right)^m\right], \\ &\qquad (m \in \mathbb{N};\ |t| < 1;\ \lambda \in \mathbb{C}), \end{aligned} \tag{1.17}$$

which, for $m = 1$, reduces immediately to (1.15).

For $\lambda = n$ ($n \in \mathbb{N}_0$), the hypergeometric generating function (1.17) assumes the form:

$$\begin{aligned} &\sum_{k=0}^{n} \binom{n}{k} \mathcal{B}_{n-k}^m [a_1, \ldots, a_u; b_1, \ldots, b_v : x]\, t^k \\ &= (1+t)^n \mathcal{B}_n^m \left[a_1, \ldots, a_u; b_1, \ldots, b_v : \frac{x}{(1+t)^m}\right], \\ &\qquad (m \in \mathbb{N};\ n \in \mathbb{N}_0), \end{aligned} \tag{1.18}$$

which obviously contains, as its special cases, *numerous* generating functions for the Jacobi, Laguerre, and many other hypergeometric polynomials.

In this paper, we first make use of the general formula (1.18) with a view to deriving a class of bilinear, bilateral, and mixed multilateral generating functions for the hypergeometric polynomials defined by (1.16). We then apply these generating functions in order to deduce the corresponding results for the Jacobi and Laguerre polynomials. Several linear generating functions for these polynomials as well as for some *multivariable* extensions of the Jacobi and Laguerre polynomials, which were investigated in recent works, are also considered briefly.

2. APPLICATIONS OF THE GENERAL FORMULA (1.18)

Making use of the general formula (1.18), we first prove the following.

THEOREM 1. *Corresponding to an identically nonvanishing function $\Omega_\mu(y_1, \ldots, y_s)$ of s (real or complex) variables $y_1, \ldots, y_s$ ($s \in \mathbb{N}$) and of (complex) order μ, let*

$$\begin{aligned} \Lambda_{n,p,q}^{(1)} [x; y_1, \ldots, y_s; z] &:= \sum_{k=0}^{[n/q]} A_k\, \Omega_{\mu+pk} (y_1, \ldots, y_s) \\ &\cdot \mathcal{B}_{n-qk}^m [a_1 + \rho_1 k, \ldots, a_u + \rho_u k; b_1 + \sigma_1 k, \ldots, b_v + \sigma_v k : x]\, z^k, \\ &\qquad (A_k \neq 0;\ n, k \in \mathbb{N}_0;\ m, p, q \in \mathbb{N}), \end{aligned} \tag{2.1}$$

where $\rho_1, \ldots, \rho_u$ and $\sigma_1, \ldots, \sigma_u$ are suitable complex parameters. Suppose also that

$$\begin{aligned} \Theta_{k,n,p}^{m,\mu,q} (x; y_1, \ldots, y_s; z) &:= \sum_{l=0}^{[k/q]} \binom{n-ql}{k-ql} A_l\, \Omega_{\mu+pl} (y_1, \ldots, y_s) \\ &\cdot \mathcal{B}_{n-k}^m [a_1 + \rho_1 l, \ldots, a_u + \rho_u l; b_1 + \sigma_1 l, \ldots, b_v + \sigma_v l : x]\, z^l. \end{aligned} \tag{2.2}$$

Then

$$\sum_{k=0}^{n} \Theta_{k,n,p}^{m,\mu,q}(x;y_1,\ldots,y_s;z)\,t^k = (1+t)^n \Lambda_{n,p,q}^{(1)}\left[\frac{x}{(1+t)^m};y_1,\ldots,y_s;z\left(\frac{t}{1+t}\right)^q\right], \tag{2.3}$$

provided that each member of (2.3) exists.

REMARK 1. In each of our definitions (2.1) and (2.2), as well as in similar situations elsewhere in this paper, the product of the essentially arbitrary coefficients $A_k \neq 0$ $(k \in \mathbb{N}_0)$ and the identically nonvanishing functions

$$\Omega_{\mu+pk}(y_1,\ldots,y_s) \qquad (k \in \mathbb{N}_0;\ p,s \in \mathbb{N};\ \mu \in \mathbb{C})$$

can indeed be notationally merged into one set of essentially arbitrary (and nonvanishing) coefficients depending on the order μ *and* one, two, or more variables. However, with a view to applying such results as (2.3) above to derive *multilinear* and *multilateral* generating functions involving simpler special functions of one, two, or more variables, we find it to be convenient to specialize A_k and Ω_μ *individually* as well as *separately* (and in a manner dictated by the problem).

REMARK 2. The *additional* hypothesis surrounding assertion (2.3) of Theorem 1, as also such hypotheses occurring in conjunction with other assertions made in this paper, is meant to guarantee that *exceptional* parameter (and variable) values which would render either or both members of (2.3) invalid or undefined are *tacitly* excluded.

PROOF OF THEOREM 1. For convenience, let $\mathcal{S}(x,t)$ denote the first member of assertion (2.3). Then, upon substituting for the polynomials

$$\Theta_{k,n,p}^{m,\mu,q}(x;y_1,\ldots,y_s;z)$$

from definition (2.2) into the left-hand side of (2.3), we obtain

$$\mathcal{S}(x,t) = \sum_{k=0}^{n} t^k \sum_{l=0}^{[k/q]} \binom{n-ql}{k-ql} A_l\, \Omega_{\mu+pl}(y_1,\ldots,y_s) \cdot \mathcal{B}_{n-k}^{m}\left[a_1+\rho_1 l,\ldots,a_u+\rho_u l;b_1+\sigma_1 l,\ldots,b_v+\sigma_v l : x\right] t^k,$$

which readily yields

$$\mathcal{S}(x,t) = \sum_{l=0}^{[n/q]} A_l\, \Omega_{\mu+pl}(y_1,\ldots,y_s)\,\left(zt^q\right)^l \cdot \sum_{k=0}^{n-ql} \binom{n-ql}{k} \mathcal{B}_{n-k-ql}^{m}\left[a_1+\rho_1 l,\ldots,a_u+\rho_u l;b_1+\sigma_1 l,\ldots,b_v+\sigma_v l : x\right] t^k. \tag{2.4}$$

Now, by appealing to the general formula (1.18) *with*

$$n \longmapsto n-ql, \quad a_j \longmapsto a_j+\rho_j l \quad (j=1,\ldots,u), \qquad \text{and} \qquad b_j \longmapsto b_j+\sigma_j l \quad (j=1,\ldots,v),$$

we find from (2.4) that

$$\mathcal{S}(x,t) = (1+t)^n \sum_{l=0}^{[n/q]} A_l\, \Omega_{\mu+pl}(y_1,\ldots,y_s) \left(\frac{zt^q}{(1+t)^q}\right)^l \cdot \mathcal{B}_{n-ql}^{m}\left[a_1+\rho_1 l,\ldots,a_u+\rho_u l;b_1+\sigma_1 l,\ldots,b_v+\sigma_v l : \frac{x}{(1+t)^m}\right],$$

which leads us at once to assertion (2.3) by means of definition (2.1).

For the classical Jacobi polynomials, by suitably specializing Theorem 1 *with*

$$u = v = 1,$$

we obtain Corollaries 1 and 2 below.

COROLLARY 1. *Under the hypotheses of Theorem 1, let*

$$\Lambda^{(2)}_{n,p,q}\left[x;y_1,\ldots,y_s;z\right] := \sum_{k=0}^{[n/q]} A_k\, P_{n-qk}^{(\alpha-\rho qk,\beta+(\sigma+1)qk)}(x)\Omega_{\mu+pk}\left(y_1,\ldots,y_s\right)z^k, \qquad (A_k \neq 0;\; n,k\in\mathbb{N}_0; p,q\in\mathbb{N}) \tag{2.5}$$

and

$$\Phi^{\rho,\sigma,q}_{k,n,p}\left(x;y_1,\ldots,y_s;z\right) := \sum_{l=0}^{[k/q]} \binom{k-\alpha+\rho ql-n-1}{k-ql} A_l \cdot P_{n-k}^{(\alpha-\rho ql,\beta+k+\sigma ql)}(x)\Omega_{\mu+pl}\left(y_1,\ldots,y_s\right)z^l, \tag{2.6}$$

where ρ and σ are suitable complex parameters.

Then

$$\sum_{k=0}^{n} \Phi^{\rho,\sigma,q}_{k,n,p}\left(x;y_1,\ldots,y_s;z\right)t^k = (1-t)^n \Lambda^{(2)}_{n,p,q}\left[\frac{x-t}{1-t};y_1,\ldots,y_s;z\left(\frac{t}{1-t}\right)^q\right], \tag{2.7}$$

provided that each member of (2.7) exists.

COROLLARY 2. *Under the hypotheses of Theorem 1, let*

$$\Lambda^{(3)}_{n,p,q}\left[x;y_1,\ldots,y_s;z\right] := \sum_{k=0}^{[n/q]} A_k\, P_{n-qk}^{(\alpha+(\rho+1)qk,\beta-\sigma qk)}(x)\Omega_{\mu+pk}\left(y_1,\ldots,y_s\right)z^k, \qquad (A_k \neq 0;\; n,k\in\mathbb{N}_0;\; p,q\in\mathbb{N}) \tag{2.8}$$

and

$$\Psi^{\rho,\sigma,q}_{k,n,p}\left(x;y_1,\ldots,y_s;z\right) := \sum_{l=0}^{[k/q]} \binom{k-\beta+\sigma ql-n-1}{k-ql} A_l \cdot P_{n-k}^{(\alpha+k+\rho ql,\beta-\sigma ql)}(x)\Omega_{\mu+pl}\left(y_1,\ldots,y_s\right)z^l, \tag{2.9}$$

where ρ and σ are suitable complex parameters.

Then

$$\sum_{k=0}^{n} \Psi^{\rho,\sigma,q}_{k,n,p}\left(x;y_1,\ldots,y_s;z\right)t^k = (1+t)^n \Lambda^{(3)}_{n,p,q}\left[\frac{x-t}{1+t};y_1,\ldots,y_s;z\left(\frac{t}{1+t}\right)^q\right], \tag{2.10}$$

provided that each member of (2.10) exists.

Corollaries 1 and 2 can indeed be proven *directly* by making use of (1.7) and (1.8), respectively, each of which is a special case of the general formula (1.18). If, instead of (1.7) and (1.8), we apply the equivalent result (1.6), we similarly obtain the following.

THEOREM 2. *Under the hypotheses of Theorem 1, let*

$$\Lambda^{(4)}_{n,p,q}\left[x;y_1,\ldots,y_s;z\right] := \sum_{k=0}^{[n/q]} A_k\, P_{n-qk}^{(\alpha+(\rho+1)qk,\beta+(\sigma+1)qk)}(x)\Omega_{\mu+pk}\left(y_1,\ldots,y_s\right)z^k, \qquad (A_k \neq 0;\; n,k\in\mathbb{N}_0;\; p,q\in\mathbb{N}) \tag{2.11}$$

and

$$\Xi^{\rho,\sigma,q}_{k,n,p}\left(x;y_1,\ldots,y_s;z\right) := \sum_{l=0}^{[k/q]} \binom{\alpha+\beta+(\rho+\sigma)ql+n+k}{k-ql} A_l \cdot P_{n-k}^{(\alpha+k+\rho ql,\beta+k+\sigma ql)}(x)\Omega_{\mu+pl}\left(y_1,\ldots,y_s\right)z^l, \tag{2.12}$$

where ρ and σ are suitable complex parameters.

Then

$$\sum_{k=0}^{n} \Xi_{k,n,p}^{\rho,\sigma,q}(x;y_1,\ldots,y_s;z)\,t^k = \Lambda_{n,p,q}^{(4)}\left[x+2t;y_1,\ldots,y_s;zt^q\right], \tag{2.13}$$

provided that each member of (2.13) exists.

Next, we turn to the classical Laguerre polynomials $L_n^{(\alpha)}(x)$ defined by (1.3). As a matter of fact, in view of the limit relationship (1.4), if we first replace x and t in (1.6) or (1.8) by

$$1-\frac{2x}{\beta} \qquad \text{and} \qquad \frac{t}{\beta},$$

respectively, and then let $|\beta| \longrightarrow \infty$, we get the known result (cf., e.g., [8, p. 348, equation (27)], [9, p. 142, equation (18)], and [10, p. 319, Entry (48.19.2)])

$$\sum_{k=0}^{n} L_{n-k}^{(\alpha+k)}(x)\,\frac{t^k}{k!} = L_n^{(\alpha)}(x-t), \tag{2.14}$$

which is precisely the Taylor expansion of $L_n^{(\alpha)}(x-t)$ in powers of t, since (cf. equation (1.5) above)

$$\frac{\partial^k}{\partial t^k}\left\{L_n^{(\alpha)}(x-t)\right\} = \begin{cases} L_{n-k}^{(\alpha+k)}(x-t), & (k=0,1,\ldots,n), \\ 0, & (k=n+1,n+2,n+3,\ldots). \end{cases} \tag{2.15}$$

In case we replace x in (1.7) instead by

$$1-\frac{2x}{\beta}$$

and let $|\beta| \longrightarrow \infty$, we similarly obtain another known result for the classical Laguerre polynomials (cf. [11, p. 85, equation (9)] and [12, p. 35, equation (1)]):

$$\sum_{k=0}^{n} \binom{\alpha+n}{k} L_{n-k}^{(\alpha)}(x)t^k = (1+t)^n L_n^{(\alpha)}\left(\frac{x}{1+t}\right), \tag{2.16}$$

which, in view of the hypergeometric representation in (1.3), is an immediate special case of the general formula (1.18) when

$$m=1 \qquad \text{and} \qquad u+1=v=1.$$

Thus, by suitably applying Theorem 1, Corollary 1, and Corollary 2, or by appealing directly to the known results (2.14) and (2.16), we can deduce the following families of bilinear, bilateral, and mixed multilateral generating functions for the classical Laguerre polynomials.

COROLLARY 3. *Under the hypotheses of Theorem 1 (or Corollary 2), let*

$$\Lambda_{n,p,q}^{(5)}\left[x;y_1,\ldots,y_s;z\right] := \sum_{k=0}^{[n/q]} A_k\, L_{n-qk}^{(\alpha+(\rho+1)qk)}(x)\Omega_{\mu+pk}(y_1,\ldots,y_s)\,z^k, \tag{2.17}$$
$$(A_k \neq 0;\ n,k \in \mathbb{N}_0;\ p,q \in \mathbb{N})$$

and

$$\mathcal{U}_{k,n,p}^{\alpha,q,\rho}(x;y_1,\ldots,y_s;z) := \sum_{l=0}^{[k/q]} \frac{A_l}{(k-ql)!}\, L_{n-k}^{(\alpha+k+\rho ql)}(x)\Omega_{\mu+pl}(y_1,\ldots,y_s)\,z^l, \tag{2.18}$$

where ρ is a suitable complex parameter.

Then

$$\sum_{k=0}^{n} \mathcal{U}_{k,n,p}^{\alpha,q,\rho}(x;y_1,\ldots,y_s;x)\,t^k = \Lambda_{n,p,q}^{(5)}\left[x-t;y_1,\ldots,y_s;zt^q\right], \tag{2.19}$$

provided that each member of (2.19) exists.

COROLLARY 4. *Under the hypotheses of Theorem 1 (or Corollary 1), let*

$$\Lambda^{(6)}_{n,p,q}\left[x;y_1,\dots,y_s;z\right] := \sum_{k=0}^{[n/q]} A_k\, L^{(\alpha+\rho qk)}_{n-qk}(x)\Omega_{\mu+pk}\left(y_1,\dots,y_s\right) z^k, \qquad (A_k \neq 0;\ n,k \in \mathbb{N}_0;\ p,q \in \mathbb{N}) \tag{2.20}$$

and

$$\mathcal{V}^{\alpha,q,\rho}_{k,n,p}\left(x;y_1,\dots,y_s;z\right) := \sum_{l=0}^{[k/q]} \binom{\alpha+(\rho-1)ql+n}{k-ql} A_l\, L^{(\alpha+\rho ql)}_{n-k}(x)\Omega_{\mu+pl}\left(y_1,\dots,y_s\right) z^l, \tag{2.21}$$

where ρ is a suitable complex parameter.
Then

$$\sum_{k=0}^{n} \mathcal{V}^{\alpha,q,\rho}_{k,n,p}\left(x;y_1,\dots,y_s;z\right) t^k = (1+t)^n \Lambda^{(6)}_{n,p,q}\left[\frac{x}{1+t};y_1,\dots,y_s;z\left(\frac{t}{1+t}\right)^q\right], \tag{2.22}$$

provided that each member of (2.22) exists.

For several families of linear and bilinear generating functions for the Laguerre and related polynomials, including (for example) the *probabilistic* derivations of some of these generating functions, see the recent works by Lee *et al.* [13,14] and Pittaluga *et al.* [15].

Finally, for the classical Hermite polynomials defined by (cf. equation (1.16))

$$\begin{aligned} H_n(x) :&= \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k} \frac{(2k)!}{k!} (2x)^{n-2k} \\ &= (2x)^n\, {}_2F_0\left[\Delta(2;-n);—;-\frac{1}{x^2}\right], \end{aligned} \tag{2.23}$$

it is easily seen from the general result (1.18) with

$$m = 2, \qquad u = v = 0, \qquad \text{and} \qquad x \longmapsto -\frac{1}{x^2}$$

that

$$\sum_{k=0}^{n} \binom{n}{k} H_{n-k}(x) t^k = H_n\left(x + \frac{1}{2}t\right). \tag{2.24}$$

Thus, by applying Theorem 1 once again, we obtain the following.

COROLLARY 5. *Under the hypotheses of Theorem 1, let*

$$\Lambda^{(7)}_{n,p,q}\left[x;y_1,\dots,y_s;z\right] := \sum_{k=0}^{[n/q]} A_k\, H_{n-qk}(x)\Omega_{\mu+pk}\left(y_1,\dots,y_s\right) z^k, \qquad (A_k \neq 0;\ n,k \in \mathbb{N}_0;\ p,q \in \mathbb{N}) \tag{2.25}$$

and

$$\mathcal{W}^{p,q}_{k,n}\left(x;y_1,\dots,y_s;z\right) := \sum_{l=0}^{[k/q]} \binom{n-ql}{k-ql} A_l\, H_{n-k}(x)\Omega_{\mu+pl}\left(y_1,\dots,y_s\right) z^l. \tag{2.26}$$

Then

$$\sum_{k=0}^{n} \mathcal{W}^{p,q}_{k,n}\left(x;y_1,\dots,y_s;z\right) t^k = \Lambda^{(7)}_{n,p,q}\left[x + \frac{1}{2}t;y_1,\dots,y_s;zt^q\right], \tag{2.27}$$

provided that each member of (2.27) exists.

3. GENERATING FUNCTIONS ASSOCIATED WITH THE STIRLING NUMBERS OF THE SECOND KIND

For each of the polynomials considered in the preceding section, we now derive several (presumably new) generating functions associated with the Stirling numbers $S(n,k)$ of the second kind, which are defined by (cf., e.g., [16, p. 90 *et seq.*])

$$S(n,k) := \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n, \tag{3.1}$$

so that

$$S(n,0) = \delta_{n,0} \quad (n \in \mathbb{N}_0), \qquad S(n,1) = S(n,n) = 1, \qquad \text{and} \qquad S(n,k) = 0 \quad (k > n), \tag{3.2}$$

$\delta_{m,n}$ being the Kronecker delta involved also in the orthogonality property (1.2).

By applying the method of proof of Srivastava's result [17, p. 754, Theorem 1] *mutatis mutandis*, we can easily obtain the following general family of generating functions involving the Stirling numbers $S(n,k)$ defined by (3.1).

THEOREM 3. *Let the polynomial sequence $\{\mathcal{T}_n(x)\}_{n=0}^{\infty}$ be generated by*

$$\sum_{k=0}^{n}\mathcal{T}_{n-k}(x)\,\frac{t^k}{k!} = f(x,t)\{g(x,t)\}^n\mathcal{T}_n(h(x,t)), \qquad (n \in \mathbb{N}_0), \tag{3.3}$$

where f, g, and h are suitable functions of x and t.

Then, in terms of the Stirling numbers $S(n,k)$ defined by (3.1), the following family of generating functions holds true:

$$\begin{aligned}\sum_{k=0}^{m}\frac{k^n}{k!}\mathcal{T}_{m-k}(h(x,-z))\left(\frac{z}{g(x,-z)}\right)^k &= \{f(x,-z)\}^{-1}\{g(x,-z)\}^{-m}\\ &\cdot\sum_{k=0}^{\min(m,n)}S(n,k)\mathcal{T}_{m-k}(x)z^k, \qquad (m,n \in \mathbb{N}_0),\end{aligned} \tag{3.4}$$

provided that each member of (3.4) exists.

REMARK 3. The functions $f(x,t)$, $g(x,t)$, and $h(x,t)$ in (3.3) are suitably chosen such that the generating function (3.3) exists for a given polynomial sequence $\{\mathcal{T}_n(x)\}_{n=0}^{\infty}$. Thus, for example, for the generating function (1.8), we have

$$f(x,t) = 1, \qquad g(x,t) = 1-t, \qquad h(x,t) = \frac{x+t}{1-t},$$

and

$$\mathcal{T}_n(x) \longmapsto \binom{\beta+n}{n}^{-1}P_n^{(\alpha-n,\beta)}(x) \qquad (n \in \mathbb{N}_0),$$

in terms of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ defined by (1.1). Furthermore, with a view to avoiding any ambiguity, the n^{th} power of the function $g(x,t)$ has been denoted *consistently* by

$$\{g(x,y)\}^n \text{ instead of } g^n(x,y) \qquad \text{or} \qquad (g(x,y))^n \quad (n \in \mathbb{N}_0).$$

PROOF OF THEOREM 3. Denote, for convenience, the left-hand side of assertion (3.4) of Theorem 3 by $\Omega(x,z)$. Then, since hypothesis (3.3) implies that

$$\mathcal{T}_n(h(x,-z)) = \{f(x,-z)\}^{-1}\{g(x,-z)\}^{-n}\sum_{l=0}^{n}\mathcal{T}_{n-l}(x)\,\frac{(-z)^l}{l!}, \qquad (n \in \mathbb{N}_0),$$

by applying definition (3.1) *as well*, we readily have

$$\begin{aligned}\Omega(x,z) :&= \sum_{k=0}^{m} \frac{k^n}{k!} \mathcal{T}_{m-k}(h(x,-z)) \left(\frac{z}{g(x,-z)}\right)^k \\ &= \{f(x,-z)\}^{-1}\{g(x,-z)\}^{-m} \sum_{k=0}^{m} \frac{k^n}{k!} \sum_{l=0}^{m-k} \frac{(-1)^l}{l!} \mathcal{T}_{m-k-l}(x) z^{k+l} \\ &= \{f(x,-z)\}^{-1}\{g(x,-z)\}^{-m} \sum_{l=0}^{\min(m,n)} \mathcal{T}_{m-l}(x) \frac{z^l}{l!} \sum_{k=0}^{l} (-1)^{l-k} \binom{l}{k} k^n \\ &= \{f(x,-z)\}^{-1}\{g(x,-z)\}^{-m} \sum_{l=0}^{\min(m,n)} S(n,l) \mathcal{T}_{m-l}(x) z^l, \qquad (m,n \in \mathbb{N}_0),\end{aligned}$$

which evidently completes the proof of Theorem 3.

Theorem 3, when applied appropriately to the generating functions (1.6)–(1.8), (1.18), (2.14), (2.16), and (2.24), would yield the following generating functions associated with the Stirling numbers $S(n,k)$ defined by (3.1).

I. GENERALIZED HYPERGEOMETRIC POLYNOMIALS.

$$\begin{gathered}\sum_{k=0}^{N} \binom{N}{k} k^n \mathcal{B}_{N-k}^{m} \left(\frac{x}{(1-z)^m}\right) \left(\frac{z}{1-z}\right)^k \\ = (1-z)^{-N} \sum_{k=0}^{\min(n,N)} \binom{N}{k} k!\, S(n,k) \mathcal{B}_{N-k}^{m}(x) z^k, \\ (m \in \mathbb{N};\ n, N \in \mathbb{N}_0),\end{gathered} \tag{3.5}$$

which, for

$$z \longmapsto \frac{z}{1+z} \qquad \text{and} \qquad x \longmapsto \frac{x}{(1+z)^m},$$

assumes the form:

$$\begin{aligned}\sum_{k=0}^{N} \binom{N}{k} k^n \mathcal{B}_{N-k}^{m}(x) z^k &= (1+z)^N \sum_{k=0}^{\min(n,N)} \binom{N}{k} k!\, S(n,k) \\ &\cdot \mathcal{B}_{N-k}^{m} \left(\frac{x}{(1+z)^m}\right) \left(\frac{z}{1+z}\right)^k, \qquad (m \in \mathbb{N};\ n, N \in \mathbb{N}_0).\end{aligned} \tag{3.6}$$

II. JACOBI POLYNOMIALS.

$$\begin{gathered}\sum_{k=0}^{m} \binom{\alpha+\beta+m+k}{k} k^n P_{m-k}^{(\alpha+k,\beta+k)}(x) z^k \\ = \sum_{k=0}^{\min(m,n)} \binom{\alpha+\beta+m+k}{k} k!\, S(n,k) P_{m-k}^{(\alpha+k,\beta+k)}(x+2z) z^k, \\ (m,n \in \mathbb{N}_0).\end{gathered} \tag{3.7}$$

$$\begin{gathered}\sum_{k=0}^{m} \binom{k-\alpha-m-1}{k} k^n P_{m-k}^{(\alpha,\beta+k)}(x) z^k \\ = (1-z)^m \sum_{k=0}^{\min(m,n)} \binom{k-\alpha-m-1}{k} k!\, S(n,k) P_{m-k}^{(\alpha,\beta+k)} \left(\frac{x-z}{1-z}\right) \left(\frac{z}{1-z}\right)^k, \\ (m,n \in \mathbb{N}_0).\end{gathered} \tag{3.8}$$

$$\sum_{k=0}^{m} \binom{k-\beta-m-1}{k} k^n \, P_{m-k}^{(\alpha+k,\beta)}(x) z^k$$
$$= (1+z)^m \sum_{k=0}^{\min(m,n)} \binom{k-\beta-m-1}{k} k! \; S(n,k) P_{m-k}^{(\alpha+k,\beta)}\left(\frac{x-z}{1+z}\right)\left(\frac{z}{1+z}\right)^k, \tag{3.9}$$
$$(m, n \in \mathbb{N}_0).$$

III. LAGUERRE POLYNOMIALS.

$$\sum_{k=0}^{m} \frac{k^n}{k!} L_{m-k}^{(\alpha+k)}(x) z^k = \sum_{k=0}^{\min(m,n)} S(n,k) L_{m-k}^{(\alpha+k)}(x-z) z^k, \qquad (m, n \in \mathbb{N}_0). \tag{3.10}$$

$$\sum_{k=0}^{m} \binom{\alpha+m}{k} k^n \, L_{m-k}^{(\alpha)}(x) z^k = (1+z)^m \sum_{k=0}^{\min(m,n)} \binom{\alpha+m}{k} k! \; S(n,k)$$
$$\cdot L_{m-k}^{(\alpha)}\left(\frac{x}{1+z}\right)\left(\frac{z}{1+z}\right)^k, \qquad (m, n \in \mathbb{N}_0). \tag{3.11}$$

IV. HERMITE POLYNOMIALS.

$$\sum_{k=0}^{m} \binom{m}{k} k^n \, H_{m-k}(x) z^k = \sum_{k=0}^{\min(m,n)} \binom{m}{k} k! \; S(n,k) H_{m-k}\left(x + \frac{1}{2} z\right), \qquad (m, n \in \mathbb{N}_0). \tag{3.12}$$

Numerous *further* applications of Theorem 3, involving (for example) some of the aforementioned and other relatives of the classical Jacobi, Laguerre, and Hermite polynomials, can be presented in an analogous manner.

In view of the following recurrence relation for the Stirling numbers $S(n,k)$ defined by (3.1):

$$S(n+1,k) = k \; S(n,k) + S(n,k-1), \qquad (n \geqq k \geqq 1), \tag{3.13}$$

it is readily seen by the principle of mathematical induction on $n \in \mathbb{N}_0$ that (cf., e.g., [16, p. 218, equation (34)])

$$(zD_z)^n = \sum_{k=0}^{n} S(n,k) z^k \; D_z^k, \qquad \left(D_z := \frac{d}{dz};\; n \in \mathbb{N}_0\right). \tag{3.14}$$

By means of the operator identity (3.14), many of the particular cases (listed under I–IV above) can alternatively be obtained by applying the differential operator $(zD_z)^n$ to both sides of the corresponding generating functions. For the sake of illustration, we first rewrite the generating function (2.14) in the form (cf. Definition (1.3))

$$\sum_{k=0}^{m} L_{m-k}^{(\alpha+k)}(x) \frac{z^k}{k!} = \sum_{j=0}^{m} \binom{m+\alpha}{m-j} \frac{(z-x)^j}{j!}. \tag{3.15}$$

Then, operating upon both sides of (3.15) by $(zD_z)^n$ $(n \in \mathbb{N}_0)$ and making use of (3.14) on the right-hand side, we find that

$$\begin{aligned}
\sum_{k=0}^{m} \frac{k^n}{k!} L_{m-k}^{(\alpha+k)}(x) z^k &= \sum_{j=0}^{m} \binom{m+\alpha}{m-j} \frac{1}{j!} \; (zD_z)^n \left\{(z-x)^j\right\} \\
&= \sum_{j=0}^{m} \binom{m+\alpha}{m-j} \frac{1}{j!} \sum_{k=0}^{n} S(n,k) z^k \; D_z^k \left\{(z-x)^j\right\} \\
&= \sum_{j=0}^{m} \binom{m+\alpha}{m-j} \sum_{k=0}^{\min(n,j)} S(n,k) z^k \, \frac{(z-x)^{j-k}}{(j-k)!}
\end{aligned}$$

$$= \sum_{k=0}^{\min(m,n)} S(n,k) z^k \sum_{j=0}^{m-k} \binom{m+\alpha}{m-k-j} \frac{(z-x)^j}{j!}$$

$$= \sum_{k=0}^{\min(m,n)} S(n,k) L_{m-k}^{(\alpha+k)}(x-z) z^k,$$

by means of definition (1.3). This evidently completes the alternative (operational) derivation of the particular case (3.10) involving the Laguerre polynomials.

Such alternative (operational) derivations of the other particular cases (listed under I–IV above) can also be detailed similarly.

4. JACOBI AND LAGUERRE POLYNOMIALS IN SEVERAL VARIABLES

Motivated essentially by the works of Erdélyi [18] and Chak [19], Carlitz and Srivastava [20] considered two classes of multivariable hypergeometric polynomials associated with a particularly simple form of the (Srivastava-Daoust) generalized Lauricella function (cf. [21–23]; see also [24, p. 37 *et seq.*]). The work of Carlitz and Srivastava [20] is described fairly adequately in Section 9.4 of the monograph by Srivastava and Manocha [5, p. 462 *et seq.*].

A very specialized class of multivariable hypergeometric polynomials was investigated recently by Shrivastava [25] who considered the Jacobi polynomials in several variables, defined by (cf. [25, p. 65, equation (15)])

$$\begin{aligned} P_n^{(\alpha_1,\beta_1;\ldots;\alpha_r,\beta_r)}(x_1,\ldots,x_r) &:= \binom{\alpha_1+n}{n} \cdots \binom{\alpha_r+n}{n} \\ &\cdot F_A^{(r)} \Big[-n, \alpha_1+\beta_1+n+1, \ldots, \alpha_r+\beta_r+n+1; \\ &\alpha_1+1,\ldots,\alpha_r+1; \frac{1}{2}(1-x_1),\ldots,\frac{1}{2}(1-x_r) \Big], \end{aligned} \tag{4.1}$$

where $F_A^{(r)}$ denotes one of Lauricella's hypergeometric functions of r variables [26, p. 113]:

$$\begin{aligned} &F_A^{(r)}[a, b_1, \ldots, b_r; c_1, \ldots, c_r; z_1, \ldots, z_r] \\ &:= \sum_{k_1,\ldots,k_r=0}^{\infty} \frac{(a)_{k_1+\cdots+k_r} (b_1)_{k_1} \cdots (b_r)_{k_r}}{(c_1)_{k_1} \cdots (c_r)_{k_r}} \frac{z_1^{k_1}}{k_1!} \cdots \frac{z_r^{k_r}}{k_r!}, \\ &(|z_1| + \cdots + |z_r| < 1), \end{aligned} \tag{4.2}$$

$(\lambda)_k := \Gamma(\lambda+k)/\Gamma(\lambda)$ being the Pochhammer symbol (or the *shifted* factorial, since $(1)_k = k!$ for $k \in \mathbb{N}_0$).

Just as in the limit relationship (1.4), it is easily observed that

$$\begin{aligned} &\lim_{\min(|\beta_1|,\ldots,|\beta_r|)\longrightarrow\infty} \left\{ P_n^{(\alpha_1,\beta_1;\ldots;\alpha_r,\beta_r)} \left(1-\frac{2x_1}{\beta_1},\ldots,1-\frac{2x_r}{\beta_r}\right) \right\} \\ &= L_n^{(\alpha_1,\ldots,\alpha_r)}(x_1,\ldots,x_r) \end{aligned} \tag{4.3}$$

in terms of the Laguerre polynomials in r variables, defined by (cf., e.g., [27, p. 163, equation (7.3)]; see also [28, p. 113, equation (1.1)])

$$\begin{aligned} L_n^{(\alpha_1,\ldots,\alpha_r)}(x_1,\ldots,x_r) &:= \binom{\alpha_1+n}{n} \cdots \binom{\alpha_r+n}{n} \\ &\cdot \Psi_2^{(r)}[-n; \alpha_1+1,\ldots,\alpha_r+1; x_1,\ldots,x_r], \end{aligned} \tag{4.4}$$

where $\Psi_2^{(r)}$ denotes Humbert's confluent hypergeometric function of r variables (cf. [29, p. 429]; see also [30, p. 134, equation (34)] and [24, p. 35, equation 1.4 (11)]):

$$\begin{aligned}\Psi_2^{(r)}\left[a;c_1,\dots,c_r;z_1,\dots,z_r\right] &:= \sum_{k_1,\dots,k_r=0}^{\infty}\frac{(a)_{k_1+\cdots+k_r}}{(c_1)_{k_1}\cdots(c_r)_{k_r}}\frac{z_1^{k_1}}{k_1!}\cdots\frac{z_r^{k_r}}{k_r!}\\ &= \lim_{\min(|b_1|,\dots,|b_r|)\longrightarrow\infty}\left\{F_A^{(r)}\left[a,b_1,\dots,b_r;c_1,\dots,c_r;\frac{z_1}{b_1},\dots,\frac{z_r}{b_r}\right]\right\}.\end{aligned}\tag{4.5}$$

For the multivariable Jacobi polynomials defined by (4.1), it is not difficult to derive the following family of linear generating functions:

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(\lambda_1)_n\cdots(\lambda_p)_n}{(\mu_1)_n\cdots(\mu_q)_n}\prod_{j=1}^{r}\left\{\binom{\alpha_j+n}{n}^{-1}\right\}P_n^{(\alpha_1,\beta_1;\dots;\alpha_r,\beta_r)}(x_1,\dots,x_r)\frac{t^n}{n!}\\ &\quad= F_{q+r:1;\dots;1;0}^{p+r:0;\dots;0;0}\left[\begin{matrix}(\lambda_1:1,\dots,1),\dots,(\lambda_p:1,\dots,1),\\ (\mu_1:1,\dots,1),\dots,(\mu_q:1,\dots,1),\end{matrix}\right.\\ &\qquad\begin{matrix}(\alpha_1+\beta_1+1:2,1,\dots,1),\dots,(\alpha_r+\beta_r+1:1,\dots,1,2,1):\\ (\alpha_1+\beta_1+1:1,\dots,1),\dots,(\alpha_r+\beta_r+1:1,\dots,1):\end{matrix}\\ &\qquad\left.\begin{matrix}\text{———};\cdots;\text{———};\text{——};\\ (\alpha_1+1:1);\dots;(\alpha_r+1:1);\text{——};\end{matrix}\ \frac{1}{2}(x_1-1)t,\dots,\frac{1}{2}(x_r-1)t,t\right],\end{aligned}\tag{4.6}$$

where we have made use of a special case of the aforementioned (Srivastava-Daoust) generalized Lauricella function (cf. [21, p. 454]; see also [22,23], and [24, p. 37 *et seq.*]).

In its special case when

$$p=r,\quad q=1,\quad \lambda_j=\alpha_j+\beta_j+1\quad(j=1,\dots,r),\qquad\text{and}\qquad \mu_1=\mu,$$

the generating function (4.6) reduces immediately to the form:

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(\alpha_1+\beta_1+1)_n\cdots(\alpha_r+\beta_r+1)_n}{(\mu)_n(\alpha_1+1)_n\cdots(\alpha_r+1)_n}(n!)^{r-1}P_n^{(\alpha_1,\beta_1;\dots;\alpha_r,\beta_r)}(x_1,\dots,x_r)t^n\\ &\quad= F_{1:1;\dots;1;0}^{r:0;\dots;0;0}\left[\begin{matrix}(\alpha_1+\beta_1+1:2,1,\dots,1),\dots,(\alpha_r+\beta_r+1:1,\dots,1,2,1):\\ (\mu:1,\dots,1):\end{matrix}\right.\\ &\qquad\left.\begin{matrix}\text{———};\dots;\text{———};\text{——};\\ (\alpha_1+1:1);\dots;(\alpha_r+1:1);\text{——};\end{matrix}\ \frac{1}{2}(x_1-1)t,\dots,\frac{1}{2}(x_r-1)t,t\right],\end{aligned}\tag{4.7}$$

which provides the *corrected* version of a known result [25, p. 66, equation (19)].

Next, by appealing to the limit relationship (4.3), we find from (4.6) *with*

$$x_j\longmapsto 1-\frac{2x_j}{\beta_j}\quad(j=1,\dots,r)\qquad\text{and}\qquad\min(|\beta_1|,\dots,|\beta_r|)\longrightarrow\infty$$

that

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(\lambda_1)_n\cdots(\lambda_p)_n}{(\mu_1)_n\cdots(\mu_q)_n}\prod_{j=1}^{r}\left\{\binom{\alpha_j+n}{n}^{-1}\right\}L_n^{(\alpha_1,\dots,\alpha_r)}(x_1,\dots,x_r)\frac{t^n}{n!}\\ &\quad= F_{q:1;\dots;1;0}^{p:0;\dots;0;0}\left[\begin{matrix}(\lambda_1:1,\dots,1),\dots,(\lambda_p:1,\dots,1):\\ (\mu_1:1,\dots,1),\dots,(\mu_q:1,\dots,1):\end{matrix}\right.\\ &\qquad\left.\begin{matrix}\text{———};\dots;\text{———};\text{——};\\ (\alpha_1+1:1);\dots;(\alpha_r+1:1);\text{——};\end{matrix}\ -x_1t,\dots,-x_rt,t\right],\end{aligned}\tag{4.8}$$

which, for

$$p-1=q=1, \qquad \lambda_1=\lambda, \qquad \lambda_2=\mu, \qquad \text{and} \qquad \mu_1=\nu,$$

yields the generating function

$$\begin{aligned}\sum_{n=0}^{\infty} & \frac{(n!)^{r-1}(\lambda)_n(\mu)_n}{(\nu)_n(\alpha_1+1)_n\cdots(\alpha_r+1)_n} L_n^{(\alpha_1,\ldots,\alpha_r)}(x_1,\ldots,x_r)\,t^n \\ &= (1-t)^{-\lambda} F_{1:1;\ldots;1;0}^{2:0;\ldots;0;1}\left[\begin{array}{l}(\lambda:1,\ldots,1),(\mu:1,\ldots,1,0): \\ (\nu:1,\ldots,1):\end{array}\right. \\ & \left.\begin{array}{lllll}\text{———};\ldots; & \text{———}; & (\nu-\mu:1); & \\ (\alpha_1+1:1);\ldots; & (\alpha_r+1:1); & \text{———}; \end{array} -\frac{x_1t}{1-t},\ldots,-\frac{x_rt}{1-t},-\frac{t}{1-t}\right], \qquad (|t|<1),\end{aligned} \tag{4.9}$$

where we have made use of the Pfaff-Kummer transformation (cf., e.g., [5, p. 33, equation 1.2 (19)]).

An obvious *further* special case of (4.9) when $\mu=\nu$ happens to be another known result [27, p. 164, equation (7.6)]:

$$\begin{aligned}\sum_{n=0}^{\infty} & \frac{(n!)^{r-1}(\lambda)_n}{(\alpha_1+1)_n\cdots(\alpha_r+1)_n} L_n^{(\alpha_1,\ldots,\alpha_r)}(x_1,\ldots,x_r)\,t^n \\ &= (1-t)^{-\lambda}\Psi_2^{(r)}\left[\lambda;\alpha_1+1,\ldots,\alpha_r+1;-\frac{x_1t}{1-t},\ldots,-\frac{x_rt}{1-t}\right], \qquad (|t|<1),\end{aligned} \tag{4.10}$$

which is actually a very specialized case of some general multivariable generating functions considered, *over three decades ago*, by Srivastava (cf., e.g., [5, p. 455, Theorem 1]; see also [5, p. 489, Problem 1]). The generating function (4.10) *and* its limit case when

$$t \longmapsto \frac{t}{\lambda} \qquad \text{and} \qquad |\lambda| \longrightarrow \infty$$

are stated also in [28, p. 114, Section 2] indicating their derivation from one of the much more general known results referred to above [5, p. 490, Problem 1 (iii)].

Some further properties of the multivariable Jacobi polynomials are worthy of note here. First of all, it readily follows from definition (4.1) that

$$\begin{aligned}P_n^{(\alpha_1,\beta_1;\ldots;\alpha_r,\beta_r)}(x_1,\ldots,x_r) &= \prod_{j=1}^{r}\left\{\binom{\alpha_j+n}{n}\right\} \\ \cdot\sum_{k=0}^{n}\binom{n}{k}\frac{(\alpha_r+\beta_r+n+1)_k}{(\alpha_r+1)_k}&\left(\frac{x_r-1}{2}\right)^k\prod_{j=1}^{r-1}\left\{\binom{\alpha_j+n-k}{n-k}^{-1}\right\} \\ &\cdot P_{n-k}^{(\alpha_1,\beta_1+k;\ldots;\alpha_{r-1},\beta_{r-1}+k)}(x_1,\ldots,x_{r-1}),\end{aligned} \tag{4.11}$$

which provides the *corrected* version of a known formula [25, p. 66, equation (17)]. Secondly, by making use of some familiar linear transformations of Lauricella's multivariable function $F_A^{(r)}$ (cf. [26, p. 148]; see also [30, p. 116]), we obtain an analogue of the relationship (1.11) in the form:

$$\begin{aligned}P_n^{(\alpha_1,\beta_1;\ldots;\alpha_r,\beta_r)}(x_1,\ldots,x_r) &= \left(\frac{1+x_1}{2}\right)^n \\ \cdot P_n^{(\alpha_1,-\alpha_1-\beta_1-2n-1;\alpha_2,\beta_2;\ldots;\alpha_r,\beta_r)}&\left(\frac{3-x_1}{1+x_1},\frac{x_1+2x_2-1}{1+x_1},\ldots,\frac{x_1+2x_r-1}{1+x_1}\right),\end{aligned} \tag{4.12}$$

with similar results involving one, two, or all of the parameters $\beta_1,\ldots,\beta_r$.

Next, for *nonpositive* integer values of the parameter λ, we find from the generating function (4.10) *and* definition (4.4) that

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}^{1-r}\prod_{j=1}^{r}\left\{\binom{\alpha_j+n}{k}\right\}L_{n-k}^{(\alpha_1,\ldots,\alpha_r)}(x_1,\ldots,x_r)\,t^k\\ =(1+t)^n L_n^{(\alpha_1,\ldots,\alpha_r)}\left(\frac{x_1}{1+t},\ldots,\frac{x_r}{1+t}\right),\end{aligned}\tag{4.13}$$

which obviously provides a multivariable extension of the known result (2.16). By suitably applying Theorem 3 to the generating function (4.13), we obtain

$$\begin{aligned}\sum_{k=0}^{m}\binom{m}{k}^{1-r}k^n\prod_{j=1}^{r}\left\{\binom{\alpha_j+m}{k}\right\}L_{m-k}^{(\alpha_1,\ldots,\alpha_r)}\left(\frac{x_1}{1-z},\ldots,\frac{x_r}{1-z}\right)\left(\frac{z}{1-z}\right)^k\\ =(1-z)^{-m}\sum_{k=0}^{\min(m,n)}\binom{m}{k}^{1-r}\prod_{j=1}^{r}\left\{\binom{\alpha_j+m}{k}\right\}k!\,S(n,k)\\ \cdot L_{m-k}^{(\alpha_1,\ldots,\alpha_r)}(x_1,\ldots,x_r)\,z^k,\qquad(m,n\in\mathbb{N}),\end{aligned}\tag{4.14}$$

which, for

$$z\longmapsto\frac{z}{1+z}\quad\text{and}\quad x_j\longmapsto\frac{x_j}{1+z},\qquad(j=1,\ldots,r),$$

assumes the form

$$\begin{aligned}\sum_{k=0}^{m}\binom{m}{k}^{1-r}k^n\prod_{j=1}^{r}\left\{\binom{\alpha_j+m}{k}\right\}L_{m-k}^{(\alpha_1,\ldots,\alpha_r)}(x_1,\ldots,x_r)\,z^k\\ =(1+z)^m\sum_{k=0}^{\min(m,n)}\binom{m}{k}^{1-r}\prod_{j=1}^{r}\left\{\binom{\alpha_j+m}{k}\right\}k!\,S(n,k)\\ \cdot L_{m-k}^{(\alpha_1,\ldots,\alpha_r)}\left(\frac{x_1}{1+z},\ldots,\frac{x_r}{1+z}\right)\left(\frac{z}{1+z}\right)^k,\qquad(m,n\in\mathbb{N})\end{aligned}\tag{4.15}$$

involving the Stirling numbers $S(n,k)$ defined by (3.1).

Finally, we recall another very specialized case of the aforementioned multivariable generating functions which, just as we remarked with (4.10), were considered by Srivastava *over three decades ago* (cf. [5, p. 455, Theorem 1]; see also [5, p. 490, Problem 1 (iii)])

$$\begin{aligned}\sum_{n=0}^{\infty}\binom{\lambda}{n}\mathcal{F}_n^{(m_1,\ldots,m_r)}(x_1,\ldots,x_r)\,t^n\\ =(1+t)^{\lambda}\mathcal{F}_{\lambda}^{(m_1,\ldots,m_r)}\left(x_1\left(\frac{t}{1+t}\right)^{m_1},\ldots,x_r\left(\frac{t}{1+t}\right)^{m_1}\right),\end{aligned}\tag{4.16}$$

where

$$\begin{aligned}\mathcal{F}_{\lambda}^{(m_1,\ldots,m_r)}(x_1,\ldots,x_r):=\sum_{k_1,\ldots,k_r=0}^{\infty}(-\lambda)_{m_1k_1+\cdots+m_rk_r}C(k_1,\ldots,k_r)\,x_1^{k_1}\ldots x_r^{k_r},\\ (m_j\in\mathbb{N};\ j=1,\ldots,r;\ \lambda\in\mathbb{C})\end{aligned}\tag{4.17}$$

in terms of a bounded multiple sequence $\{C(k_1,\ldots,k_r)\}$ of complex numbers.

The multivariable generating function (4.16) is itself a generalization of (4.10) as well as (1.17). Furthermore, for *nonnegative* integer values of the parameter λ, (4.16) yields the following generalization of (1.18):

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}\mathcal{F}_{n-k}^{(m_1,\ldots,m_r)}(x_1,\ldots,x_r)\,t^k=(1+t)^n\mathcal{F}_n^{(m_1,\ldots,m_r)}\left(\frac{x_1}{(1+t)^{m_1}},\ldots,\frac{x_r}{(1+t)^{m_r}}\right),\\ (m_j\in\mathbb{N};\ j=1,\ldots,r;\ n\in\mathbb{N}_0).\end{aligned}\tag{4.18}$$

In terms of the Stirling numbers $S(n,k)$ defined by (3.1), we find from (4.18) that

$$\begin{gathered}\sum_{k=0}^{N}\binom{N}{k}k^n\,\mathcal{F}_{N-k}^{(m_1,\ldots,m_r)}\left(\frac{x_1}{(1-z)^{m_1}},\ldots,\frac{x_r}{(1-z)^{m_r}}\right)\left(\frac{z}{1-z}\right)^k\\ =(1-z)^{-N}\sum_{k=0}^{\min(n,N)}\binom{N}{k}k!\;S(n,k)\mathcal{F}_{N-k}^{(m_1,\ldots,m_r)}(x_1,\ldots,x_r)\,z^k,\\ (m_j\in\mathbb{N};\ j=1,\ldots,r;\ n,N\in\mathbb{N}_0),\end{gathered}\tag{4.19}$$

which, for

$$z\longmapsto\frac{z}{1+z}\quad\text{and}\quad x_j\longmapsto\frac{x_j}{(1+z)^{m_j}}\qquad(j=1,\ldots,r),$$

assumes the form:

$$\begin{gathered}\sum_{k=0}^{N}\binom{N}{k}k^n\,\mathcal{F}_{N-k}^{(m_1,\ldots,m_r)}(x_1,\ldots,x_r)\,z^k=(1+z)^N\sum_{k=0}^{\min(n,N)}\binom{N}{k}k!\;S(n,k)\\ \cdot\mathcal{F}_{N-k}^{(m_1,\ldots,m_r)}\left(\frac{x_1}{(1+z)^{m_1}},\ldots,\frac{x_r}{(1+z)^{m_r}}\right)\left(\frac{z}{1+z}\right)^k,\\ (m_j\in\mathbb{N};\ j=1,\ldots,r;\ n,N\in\mathbb{N}_0).\end{gathered}\tag{4.20}$$

Upon setting $r=1$, if we *also* let [cf. equation (4.17)]

$$C(k)=\frac{(a_1)_k\cdots(a_u)_k}{(b_1)_k\cdots(b_v)_k},\qquad(k\in\mathbb{N}_0),\tag{4.21}$$

these last results (4.19) and (4.20) would reduce to the generating functions (3.5) and (3.6), respectively.

We conclude this paper by remarking that, by making use of the generating function (4.18) *directly* (or, alternatively, by appropriately specializing a general result due to Chen and Srivastava [3, p. 183, Theorem 3 *with* $\lambda=-N$ $(N\in\mathbb{N}_0)$]), we can deduce a multivariable generalization of Theorem 1 above, which would apply relatively more easily to derive bilinear, bilateral, and mixed multilateral generating functions for numerous multivariable polynomials including (for example) the Laguerre polynomials $L_n^{(\alpha_1,\ldots,\alpha_r)}(x_1,\ldots,x_r)$ defined by (4.4). For the sake of completeness, however, we choose to state these interesting consequences of the general result of Chen and Srivastava [3, p. 183, Theorem 3] as Theorem 4 and Corollary 6 below.

THEOREM 4. *Under the applicable hypotheses of Theorem 1, let*

$$\begin{gathered}\Lambda_{n,p,q}^{(8)}[x_1,\ldots,x_r;y_1,\ldots,y_s;z]:=\sum_{k=0}^{[n/q]}A_k\,\mathcal{F}_{n-qk}^{(m_1,\ldots,m_r)}(x_1,\ldots,x_r)\,\Omega_{\mu+pk}(y_1,\ldots,y_s)\,z^k,\\ (A_k\neq0;\ n,k\in\mathbb{N}_0;\ m_j\in\mathbb{N}\ (j=1,\ldots,r);\ p,q\in\mathbb{N})\end{gathered}\tag{4.22}$$

and

$$\begin{gathered}\mathcal{P}_{k,n}^{p,q}(x_1,\ldots,x_r;y_1,\ldots,y_s;z):=\sum_{l=0}^{[k/q]}\binom{n-ql}{k-ql}A_l\,\mathcal{F}_{n-k}^{(m_1,\ldots,m_r)}(x_1,\ldots,x_r)\\ \cdot\Omega_{\mu+pl}(y_1,\ldots,y_s)\,z^l.\end{gathered}\tag{4.23}$$

Then

$$\begin{gathered}\sum_{k=0}^{n}\mathcal{P}_{k,n}^{p,q}(x_1,\ldots,x_r;y_1,\ldots,y_s;z)\,t^k\\ =(1+t)^n\Lambda_{n,p,q}^{(8)}\left[\frac{x_1}{(1+t)^{m_1}},\ldots,\frac{x_r}{(1+t)^{m_r}};y_1,\ldots,y_s;z\left(\frac{t}{1+t}\right)^q\right],\end{gathered}\tag{4.24}$$

provided that each member of (4.24) exists.

COROLLARY 6. *Under the applicable hypotheses of Theorem 1, let*

$$\begin{aligned}\Lambda^{(9)}_{n,p,q}[x_1,\ldots,x_r;y_1,\ldots,y_s;z] := \sum_{k=0}^{[n/q]} A_k\, L^{(\alpha_1+\rho_1 k,\ldots,\alpha_r+\rho_r k)}_{n-qk}(x_1,\ldots,x_r) \\ \cdot\Omega_{\mu+pk}(y_1,\ldots,y_s)\, z^k, \\ (A_k \neq 0;\ n,k\in\mathbb{N}_0;\ p,q\in\mathbb{N})\end{aligned} \tag{4.25}$$

and

$$\begin{aligned}\mathcal{Q}^{\mu,q,r}_{k,n,p}(x_1,\ldots,x_r;y_1,\ldots,y_s;z) := \sum_{l=0}^{[k/q]} \binom{n-ql}{k-ql}^{1-r} A_l\, L^{(\alpha_1+\rho_1 l,\ldots,\alpha_r+\rho_r l)}_{n-k}(x_1,\ldots,x_r) \\ \cdot\prod_{j=1}^{r}\left\{\binom{\alpha_j+(\rho_j-1)ql+n}{k-ql}\right\}\Omega_{\mu+pl}(y_1,\ldots,y_s)\, z^l,\end{aligned} \tag{4.26}$$

where $\rho_1,\ldots,\rho_r$ are suitable complex parameters.

Then

$$\begin{aligned}\sum_{k=0}^{n}\mathcal{Q}^{\mu,q,r}_{k,n,p}(x_1,\ldots,x_r;y_1,\ldots,y_s;z)\,t^k \\ = (1+t)^n\Lambda^{(9)}_{n,p,q}\left[\frac{x_1}{1+t},\ldots,\frac{x_r}{1+t};y_1,\ldots,y_s;z\left(\frac{t}{1+t}\right)^q\right],\end{aligned} \tag{4.27}$$

provided that each member of (4.27) exists.

Evidently, Corollary 6 provides a multivariable extension of Corollary 4 involving the classical Laguerre polynomials $L^{(\alpha)}_n(x)$ defined by (1.3). On the other hand, in the definitions (4.22) and (4.23) of Theorem 4, the coefficients of the multivariable $\mathcal{F}$-polynomials are tacitly assumed to depend suitably upon the summation indices k and l, respectively.

REFERENCES

1. G. Szegö, *Orthogonal Polynomials, American Mathematical Society Colloquium Publications, Volume 23*, Fourth Edition, American Mathematical Society, Providence, RI, (1975).
2. E. Erdélyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, *Higher Transcendental Functions, Volume II*, McGraw-Hill, New York, (1953).
3. M.-P. Chen and H.M. Srivastava, Orthogonality relations and generating functions for Jacobi polynomials and related hypergeometric functions, *Appl. Math. Comput.* **68**, 153–188, (1995).
4. G. Pittaluga, L. Sacripante and H.M. Srivastava, Some families of generating functions for the Jacobi and related orthogonal polynomials, *J. Math. Anal. Appl.* **238**, 385–417, (1999).
5. H.M. Srivastava and H.L. Manocha, *A Treatise on Generating Functions*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, (1984).
6. T.W. Chaundy, An extension of hypergeometric functions (I), *Quart. J. Math. Oxford Ser.* **14**, 55–78, (1943).
7. F. Brafman, Some generating functions of Laguerre and Hermite polynomials, *Canad. J. Math.* **9**, 180–187, (1957).
8. A. Erdélyi, Transformation of a certain series of products of confluent hypergeometric functions: Applications to Laguerre and Charlier polynomials, *Compositio Math.* **7**, 340–352, (1939).
9. H. Buchholz, *The Confluent Hypergeometric Function, Springer Tracts in Natural Philosophy, Volume 15*, (Translated from the German by H. Lichtblau and K. Wetzel), Springer-Verlag, New York, (1969).
10. E.R. Hansen, *A Table of Series and Products*, Prentice-Hall, Englewood Cliffs, NJ, (1975).
11. C. Truesdell, *An Essay Toward a Unified Theory of Special Functions*, Princeton University Press, Princeton, NJ, (1948); Reprinted by Kraus Reprint Corporation, New York, (1965).
12. E.B. McBride, *Obtaining Generating Functions, Springer Tracts in Natural Philosophy, Volume 21*, Springer-Verlag, New York, (1971).
13. P.-A. Lee, S.-H. Ong and H.M. Srivastava, Probabilistic derivation of some generating functions for the Laguerre polynomials, *Comput. Math. Appl.* **38** (7–8), 21–29, (1999).
14. P.-A. Lee, S.-H. Ong and H.M. Srivastava, Some generating functions for the Laguerre and related polynomials, *Appl. Math. Comput.* **108**, 129–138, (2000).
15. G. Pittaluga, L. Sacripante and H.M. Srivastava, Some generating functions of the Laguerre and modified Laguerre polynomials, *Appl. Math. Comput.* **113**, 141–160, (2000).

16. J. Riordan, *Combinatorial Identities, Wiley Tracts on Probability and Statistics*, John Wiley and Sons, New York, (1968).
17. H.M. Srivastava, Some families of generating functions associated with the Stirling numbers of the second kind, *J. Math. Anal. Appl.* **251**, 752–769, (2000).
18. A. Erdélyi, Beitrag zur Theorie der konfluenten hypergeometrischen Funktionen von mehreren Veränderlichen, *S.-B. Akad. Wiss. Wien Abt. IIa Math.-Natur. Kl.* **146**, 431–467, (1937).
19. A.M. Chak, Some generalizations of Laguerre polynomials. I and II, *Mat. Vesnik* **7** (22), 7–18, (1970).
20. L. Carlitz and H.M. Srivastava, Some hypergeometric polynomials associated with the Lauricella function F_D of several variables. I and II, *Mat. Vesnik* **13** (28), 41–47 and 143–152, (1976).
21. H.M. Srivastava and M.C. Daoust, Certain generalized Neumann expansions associated with the Kampé de Fériet function, *Nederl. Akad. Wetensch. Indag. Math.* **31**, 449–457, (1969).
22. H.M. Srivastava and M.C. Daoust, A note on the convergence of Kampé de Fériet's double hypergeometric series, *Math. Nachr.* **53**, 151–157, (1972).
23. N.T. Hài, O.I. Marichev and H.M. Srivastava, A note on the convergence of certain families of multiple hypergeometric series, *J. Math. Anal. Appl.* **164**, 104–115, (1992).
24. H.M. Srivastava and P.W. Karlsson, *Multiple Gaussian Hypergeometric Series*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, (1985).
25. H.S.P. Shrivastava, On Jacobi polynomials of several variables, *Integral Transform. Spec. Funct.* **10**, 61–70, (2000).
26. G. Lauricella, Sulle funzioni ipergeometriche a più variabili, *Rend. Circ. Mat. Palermo* **7**, 111–158, (1893).
27. M.A. Khan and A.K. Shukla, On Laguerre polynomials of several variables, *Bull. Calcutta Math. Soc.* **89**, 155–164, (1997).
28. M.A. Khan and A.K. Shukla, A note on Laguerre polynomials of m-variables, *Bull. Greek Math. Soc.* **40**, 113–117, (1998).
29. P. Humbert, La fonction $W_{k,\mu_1,\mu_2,\ldots,\mu_n}(x_1, x_2, \ldots, x_n)$, *C.R. Acad. Sci. Paris* **171**, 428–430, (1920).
30. P. Appell et J. Kampé de Fériet, *Fonctions Hypergéométriques et Hypersphériques; Polynômes d'Hermite*, Gauthier-Villars, Paris, (1926).