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LAGRANGE INVERSION AND STIRLING NUMBER CONVOLUTIONS

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Abstract

Recently Agoh and Dilcher proved a convolution identity involving Stirling numbers $S(n,r)$ of the second kind. We prove an identity where $S(n,r)$ is replaced by a more general doubly-indexed family $A(n,r)$. Another admissible choice for $A(n,r)$ is the family of Stirling numbers of the first kind.

1. Introduction

Let $S(n,r)$ denote a Stirling number of the second kind, the number of unordered partitions of an n-element set into r nonempty subsets. Recently Agoh and Dilcher [2, Theorem 1] proved the following identity holds

$$\frac{(r-1)!}{(n-1)!}S(n,r) = \sum_{\substack{r_1+\cdots+r_k=r\\ r_1,\ldots,r_k\geq 1}} \prod_{j=1}^{k} \frac{(r_j-1)!}{(n_j-1)!}S(n_j,r_j) \tag{1}$$

where $n = n_1 + \cdots + n_k$ and $r > \max_j(n - n_j)$.

We give a simpler proof of a more general identity. In particular our identity specializes not only to (1) but also to its analogue with the $S(n,r)$ replaced by Stirling numbers of the first kind. The key to our approach is the use of the Lagrange inversion theorem.

2. The Main Theorem

The Stirling numbers of the second kind have the exponential generating function

$$\sum_{n=r}^{\infty} S(n,r)\frac{x^n}{n!} = \frac{(e^x-1)^r}{r!}$$

[4, Chapter 1 (24b)]. Our generalization concerns any doubly-indexed family of numbers with exponential generating functions of this form. Let $f(x) = \sum_{n=1}^{\infty} c_k x^k$ be a formal power series over a field of characteristic zero and with $c_1 \neq 0$. Define $A(n,r)$ for $n \geq r \geq 1$ by

$$\frac{f(x)^r}{r!} = \sum_{n=r}^{\infty} A(n,r)\frac{x^n}{n!}.$$

Thus when $f(x) = e^x - 1$, $A(n,r) = S(n,r)$. Define

$$B(n,r) = \frac{(r-1)!}{(n-1)!}A(n,r).$$

We can now state and prove our main theorem.

Theorem 1 *With the above notation, let $n_1, \ldots, n_k$ be positive integers, $n = n_1 + \cdots + n_k$ and s be an integer with $0 \leq s < \min_j n_j$. Then*

$$B(n, n-s) = \sum_{\substack{s_1+\cdots+s_k=s \\ s_1,\ldots,s_k \geq 0}} \prod_{j=1}^{k} B(n_j, n_j - s_j). \tag{2}$$

Proof. The power series f has a compositional inverse F, also a power series with constant term zero, satisfying $F(f(x)) = f(F(x)) = x$. The Lagrange inversion theorem [5, Theorem 5.4.2] states that $n[x^n](f(x)^r) = r[x^{-r}](F(x)^{-n})$ where $[x^m](\phi(x))$ denotes the coefficient of x^m in the formal Laurent series $\phi(x)$. As

$$\frac{f(x)^r}{r} = \sum_{n=r}^{\infty} B(n,r)\frac{x^n}{n}$$

then $B(n,r) = \frac{n}{r}[x^n](f(x)^r) = [x^{-r}](F(x)^{-n})$. Define for $n \geq 1$, $g_n(x) = \sum_{s=0}^{n-1} B(n, n-s)x^s$. Then $g_n(x)$ consists of the terms up to that in x^{n-1} in the power series $(x/F(x))^n$. It is convenient to write this as a congruence

$$g_n(x) \equiv \left(\frac{x}{F(x)}\right)^n \pmod{x^n}$$

in the ring of formal power series. If $n = n_1 + \cdots + n_k$ then

$$\prod_{j=1}^{k} g_{n_j}(x) \equiv \prod_{j=1}^{k} \left(\frac{x}{F(x)}\right)^{n_j} = \left(\frac{x}{F(x)}\right)^n \equiv g_n(x) \pmod{x^{\min_j n_j}}.$$

Comparing the coefficient of x^s when $0 \leq s < \min_j n_j$ gives

$$B(n, n-s) = \sum_{\substack{s_1+\cdots+s_k=s \\ s_1,\ldots,s_k \geq 0}} \prod_{j=1}^{k} B(n_j, n_j - s_j).$$

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Taking $f(x) = e^x - 1$ and setting $r = n - s$ and $r_j = n_j - s_j$, identity (2) reduces to (1).

Let $c(n,r)$ denote the (unsigned) Stirling number of the first kind, the number of permutations of $\{1,\dots,n\}$ with r cycles. Then

$$\sum_{n=r}^{\infty} c(n,r)\frac{x^n}{n!} = \frac{1}{r!}\left(\log\frac{1}{1-x}\right)^r$$

[3, §1.2.9 (26)]. By taking $f(x) = \log(1/(1-x))$ Theorem 1 gives

$$\frac{(r-1)!}{(n-1)!}c(n,r) = \sum_{\substack{r_1+\cdots+r_k=r\\ r_1,\dots,r_k\geq 1}} \prod_{j=1}^{k} \frac{(r_j-1)!}{(n_j-1)!}c(n_j,r_j)$$

whenever $n = n_1 + \cdots + n_k$ and $r > \max_j(n - n_j)$.

3. Another Identity of Agoh and Dilcher

We now give an alternative proof of Proposition 5.1 in [1].

Theorem 2 *For integers m, $n \geq 1$ and $1 \leq r \leq m+n$ the following identity holds:*

$$\begin{aligned}\frac{(m-1)!(n-1)!}{(m+n-1)!}S(m+n,r) &= \sum_{i=1}^{r-1}\frac{(i-1)!(r-i-1)!}{(r-1)!}S(m,i)S(n,r-i)\\ &\quad + (-1)^m\sum_{j=0}^{n-r}\frac{B_{j+m}}{j+m}\binom{n-1}{j}S(n-j,r)\\ &\quad + (-1)^n\sum_{j=0}^{m-r}\frac{B_{j+n}}{j+n}\binom{m-1}{j}S(m-j,r),\end{aligned}$$

where the B_s are the Bernoulli numbers.

Proof. Note that this is the same as [1, Proposition 5.1] on replacing their k and m by m and n, their d by $r-1$ and performing some rearrangement.

Let $f(x) = e^x - 1$. Then f has compositional inverse $F(x) = \log(1+x)$. Define $T(n,r)$ for integers $1 \leq r \leq n$ by

$$T(n,r) = \frac{(r-1)!}{(n-1)!}S(n,r).$$

Then

$$\frac{(e^x-1)^r}{r!} = \sum_{n=r}^{\infty} S(n,r)\frac{x^n}{n!} = \frac{1}{(r-1)!}\sum_{n=r}^{\infty} T(n,r)\frac{x^n}{n}.$$

Differentiating gives

$$e^x(e^x-1)^{r-1} = \sum_{n=r}^{\infty} T(n,r)x^{n-1}.$$

Thus

$$T(n,r) = [x^{n-1}]e^x(e^x-1)^{r-1}.$$

Now use this formula to define $T(n,r)$ for all integers n and r, noting that $T(n,r)=0$ if $n<r$.

By the Lagrange inversion formula $T(n,r)=[x^{-r}](\log(1+x))^{-n}$. Let $m,\ n\geq 1$ and $1\leq r\leq m+n$. Then

$$T(m+n,r)=[x^{-r}](\log(1+x))^{-m}(\log(1+x))^{-n}=\sum_{i=r-n}^{m}T(m,i)T(n,r-i).$$

We split this sum as follows: $T(m+n,r)=\Sigma_1+\Sigma_2-\Sigma_3$, where

$$\Sigma_1=\sum_{i=1}^{m}T(m,i)T(n,r-i),\quad \Sigma_2=\sum_{i=r-n}^{r-1}T(m,i)T(n,r-i),\ \text{ and }\ \Sigma_3=\sum_{i=1}^{r-1}T(m,i)T(n,r-i).$$

The first sum is

$$\begin{aligned}\Sigma_1 &= \sum_{i=1}^{m}T(m,i)T(n,r-i)\\ &= \frac{1}{(m-1)!}\sum_{i=1}^{m}(i-1)!S(m,i)T(n,r-i)\\ &= \frac{1}{(m-1)!}[x^{n-1}]\sum_{i=1}^{m}(i-1)!S(m,i)e^x(e^x-1)^{r-i-1}\\ &= \frac{1}{(m-1)!}[x^{n-1}]e^x(e^x-1)^{r-1}\sum_{i=1}^{m}(i-1)!S(m,i)\frac{1}{(e^x-1)^i}.\end{aligned}$$

By [1, Lemma 3.1],

$$\sum_{i=1}^{m}(i-1)!S(m,i)\frac{1}{(e^x-1)^i}=(-1)^{m-1}\frac{d^{m-1}}{dx^{m-1}}\frac{1}{e^x-1}.$$

By the definition of the Bernoulli numbers

$$\frac{1}{e^x-1}=\frac{1}{x}+\sum_{j=0}^{\infty}\frac{B_{j+1}}{j+1}\frac{x^j}{j!}.$$

Differentiating $m-1$ times gives

$$(-1)^{m-1}\frac{d^{m-1}}{dx^{m-1}}\frac{1}{e^x-1}=\frac{(m-1)!}{x^m}-(-1)^m\sum_{j=0}^{\infty}\frac{B_{j+m}}{j+m}\frac{x^j}{j!}.$$

Thus

$$\begin{aligned}\Sigma_1 &= [x^{m+n-1}]e^x(e^x-1)^{r-1}\\ &\quad -\frac{(-1)^m}{(m-1)!}\sum_{j=0}^{n-r}\frac{B_{j+m}}{(j+m)j!}[x^{n-j-1}]e^x(e^x-1)^{r-1}\\ &= T(m+n,r)-\frac{(-1)^m}{(m-1)!}\sum_{j=0}^{n-r}\frac{B_{j+m}}{(j+m)j!}T(n-j,r).\end{aligned}$$

Similarly

$$\Sigma_2 = \sum_{i=1}^{n} T(m, r-i)T(n,i) = T(m+n,r) - \frac{(-1)^n}{(n-1)!}\sum_{j=0}^{m-r}\frac{B_{j+n}}{(j+n)j!}T(m-j,r).$$

It follows that

$$\begin{aligned} T(m+n,r) &= 2T(m+n,r) - \Sigma_3 - \frac{(-1)^m}{(m-1)!}\sum_{j=0}^{n-r}\frac{B_{j+m}}{(j+m)j!}T(n-j,r) \\ &\quad - \frac{(-1)^n}{(n-1)!}\sum_{j=0}^{m-r}\frac{B_{j+n}}{(j+n)j!}T(m-j,r). \end{aligned}$$

Rearranging gives

$$\begin{aligned} T(m+n,r) &= \sum_{i=1}^{r-1} T(m,i)T(n,r-i) + \frac{(-1)^m}{(m-1)!}\sum_{j=0}^{n-r}\frac{B_{j+m}}{(j+m)j!}T(n-j,r) \\ &\quad + \frac{(-1)^n}{(n-1)!}\sum_{j=0}^{m-r}\frac{B_{j+n}}{(j+n)j!}T(m-j,r). \end{aligned}$$

Using the definition of T a further rearrangement yields

$$\begin{aligned} \frac{(m-1)!(n-1)!}{(m+n-1)!}S(m+n,r) &= \sum_{i=1}^{r-1}\frac{(i-1)!(r-i-1)!}{(r-1)!}S(m,i)S(n,r-i) \\ &\quad + (-1)^m\sum_{j=0}^{n-r}\frac{B_{j+m}}{j+m}\binom{n-1}{j}S(n-j,r) \\ &\quad + (-1)^n\sum_{j=0}^{m-r}\frac{B_{j+n}}{j+n}\binom{m-1}{j}S(m-j,r) \end{aligned}$$

as required. □

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