q-analogues of Ehrhart polynomials

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Abstract

One considers weighted sums over points of lattice polytopes, where the weight of a point v is the monomial $q^{\lambda(v)}$ for some linear form λ. One proposes a q-analogue of the classical theory of Ehrhart series and Ehrhart polynomials, including Ehrhart reciprocity and involving evaluation at the q-integers.

Introduction

The theory of Ehrhart polynomials, which was introduced by Eugène Ehrhart in the 1960s [6], has now become a classical subject. Let us recall it very briefly. If Q is a lattice polytope, meaning a polytope with vertices in a lattice, one can count the number of lattice points inside Q. It turns out that the number of lattice points in the dilated lattice polytope nQ for some integer n is a polynomial function of n. This is called the Ehrhart polynomial of the lattice polytope Q. Moreover, the value of the Ehrhart polynomial at a negative integer $-n$ is (up to sign) the number of interior lattice points in nQ. This phenomenon is called Ehrhart reciprocity. This classical theory is for example detailed in the book [1].

In this article, one introduces a q-analogue of this theory, in which the number of lattice points is replaced by a weighted sum, which is a polynomial in the indeterminate q. One proves that these weighted sums for dilated polytopes are values at q-integers of a polynomial in x with coefficients in $\mathbb{Q}(q)$. Let us present this in more detail.

Let Q be a lattice polytope and let λ be a linear form on the ambient lattice of Q, assumed to take positive values on Q. One considers the weighted sum

$$W_\lambda(Q, q) = \sum_{x \in Q} q^{\lambda(x)}, \tag{1}$$

running over lattice points in Q, where q is an indeterminate. This is a polynomial q-analogue of the number of lattice points in Q, which is the value at $q = 1$.

Under two hypotheses of positivity and genericity on the pair (Q, λ), one proves that the polynomials $W_\lambda(nQ, q)$ for integers $n \geq 0$ are the values at q-integers $[n]_q$ of a polynomial in x with coefficients in $\mathbb{Q}(q)$, which is called the q-Ehrhart polynomial.

One also obtains a reciprocity theorem, which relates the value of the q-Ehrhart polynomial at the negative q-integer $[-n]_q$ to the weighted sum over interior points in nQ.

In the special case where the lattice polytope is the order polytope of a partially ordered set P, the theory presented here is closely related to the well-known theory of P-partitions, introduced by Richard P. Stanley in [10]. One can find there q-analogues of Ehrhart series, which coincide with the one used here. It seems though that the existence of q-Ehrhart polynomials is new even in this setting.

The q-Ehrhart polynomials seem to have interesting properties in the special case of empty polytopes. In particular, they vanish at $x = -1/q$ by reciprocity, and the derivative at this point may have a geometric meaning. One also presents an umbral property, which involves some q-analogues of Bernoulli numbers that were introduced by Carlitz in [4].

Our original motivation for this theory came from the study of some tree-indexed series, involving order polytopes of rooted trees, in the article [5]. In this study appeared some polynomials in x with coefficients in $\mathbb{Q}(q)$, who become Ehrhart polynomials when $q = 1$. Understanding this has led us to the results presented here.

Some other generalisations of the classical Ehrhart theory have been considered in [13, 12], but they do not seem to involve evaluation at q-integers.

The article is organised as follows. In section 1, one introduces the general setting and hypotheses and then studies the q-Ehrhart series. In section 2, one proves the existence of the q-Ehrhart polynomial and obtains the reciprocity theorem. In section 3, various general properties of q-Ehrhart series and polynomials are described. In section 4, the special case of order polytopes and the relationship with P-partitions are considered. Section 5 deals with the special case of empty polytopes.

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1   q-Ehrhart series

In this section, one introduces the q-Ehrhart series, which is a generating series for some weighted sums over dilatations of a polytope, and describes this series as a rational function.

This section should not be considered as completely original: similar series have been considered in many places, including [2, 3]. It is therefore essentially a brief account in our own notations of more or less classical material, adapted to a specific context.

Let M be a lattice, let Q be a lattice polytope in M and let $\lambda \in M^*$ be a linear form.

One will always assume that the pair (Q, λ) satisfies the following conditions:

Positivity For every vertex x of Q, $\lambda(x) \geq 0$.

Genericity For every edge x—y of Q, $\lambda(x) \neq \lambda(y)$.

Let q be a variable. Let us define the weighted sum over lattice points

$$W_\lambda(Q, q) = \sum_{x \in Q} q^{\lambda(x)}, \qquad (2)$$

and the q-Ehrhart series

$$\mathrm{Ehr}_{Q,\lambda}(t, q) = \sum_{n \geq 0} W_\lambda(nQ, q) t^n, \qquad (3)$$

where nQ is the dilatation of Q by a factor n.

When $q = 1$, the weighted sum becomes the number of lattice points and the q-Ehrhart series becomes the classical Ehrhart series.

Proposition 1.1 *The q-Ehrhart series $\mathrm{Ehr}_{Q,\lambda}$ is a rational function in t and q. Its denominator is a product without multiplicities of factors $1 - tq^j$ for some integers j with $0 \leq j \leq \max_Q(\lambda)$. The factor with index j can appear only if there is a vertex v of q such that $\lambda(v) = j$.*

The proof will be as follows. The first step is to define a special triangulation of Q, depending on λ. Then the q-Ehrhart series of Q is an alternating sum of q-Ehrhart series of simplices of the special triangulation. The last step is to prove that the q-Ehrhart series of every simplex of the chosen triangulation has the expected properties of $\mathrm{Ehr}_{Q,\lambda}$. This implies the proposition, as these properties are stable by linear combinations.

It is well-known, see for example [1, Theorem 3.1], that every polytope can be triangulated using no new vertices. One will need here a special triangulation, which depends on the linear form λ.

Remark that the Genericity condition implies that every face F of Q contains a unique vertex where λ is minimal. Let us call it the minimal vertex of F.

Proposition 1.2 *There exists a unique triangulation of Q with vertices the vertices of Q, such that every simplex contained in a face F of Q contains the minimal vertex of F.*

In this triangulation, for every edge x—y of every simplex, $\lambda(x) \neq \lambda(y)$.

Proof. By induction on the dimension of Q.

The statement is clear if the dimension is 0.

Assume that the dimension is at least 1. Let x_0 be the minimal vertex of Q.

By induction, there exists a triangulation of every facet F of Q not containing x_0, with the stated properties.

If two facets F and F' of Q share a face G, then the restrictions of their triangulations give two triangulations of G, both with the stated properties. By uniqueness, they must be the same.

Hence there exists a triangulation of the union of all facets of Q not containing x_0. One can define a triangulation of Q by adding x_0 to every simplex.

Conversely, this is the unique triangulation with the stated properties. Indeed, any maximal simplex of such a triangulation must contain the vertex x_0, hence the triangulation must be induced by triangulations of faces not containing x_0. By restriction, these triangulations must have the stated properties. Uniqueness follows.

The second property of this triangulation follows from its inductive construction. Indeed, the value of λ at x_0 is strictly less than the value of λ at every other vertex of Q. ∎

By the principle of inclusion-exclusion, one can then write the q-Ehrhart series of Q as an alternating sum of q-Ehrhart series of all simplices of the special triangulation. Let us now describe these summands.

Proposition 1.3 *Let S be a lattice simplex, such that for every pair of distinct vertices x, y of S, one has $\lambda(x) \neq \lambda(y)$. Then the q-Ehrhart series $\mathrm{Ehr}_{Q,\lambda}$ is a rational function in t and q whose denominator is the product of $1 - tq^j$ for all integers $j = \lambda(v)$ where v is a vertex of S. The integers j satisfy $0 \leq j \leq \max_S(\lambda)$.*

Proof. This is a special case of a classical result, see for instance [1, Theorem 3.5].

Consider the cone C over the simplex $\{1\} \times S$, in the product space $\mathbb{Z} \times M$. The generating rays of C are exactly the vectors $(1, v)$ for vertices v of S. According to the cited theorem, the poles of the generating series for C, which is also the q-Ehrhart series for S, are given by a factor $1 - tq^{\lambda(v)}$ for every vertex v of S. By the hypothesis, all these poles are distinct. By the Positivity condition, the exponents of q are positive. ∎

From all this, one immediately obtains the statement of proposition 1.1.

Proposition 1.4 *The factor $1 - q^{\min_Q(\lambda)}t$ and the factor $1 - q^{\max_Q(\lambda)}t$ are always present in the denominator of $\mathrm{Ehr}_{Q,\lambda}$.*

Proof. Let m be $\max_Q(\lambda)$. Then for every $n \geq 0$, there is exactly one term q^{nm} in the weighted sum $W_\lambda(nQ, q)$, corresponding to the unique maximal vertex of nQ.

On the other hand, let ℓ be the maximal j such that $1 - q^j t$ is a pole of $\mathrm{Ehr}_{Q,\lambda}$. In the Taylor expansion of this fraction, the coefficient of t^n is a polynomial in q with degree at most $n\ell + k$, for some $k \geq 0$ which does not depend on n. It follows that $\ell \geq m$.

By proposition 1.1, it is already known that $\ell \leq m$. Therefore $\ell = m$ and the result follows for the pole $1 - q^m t$.

The proof for the minimal case is similar. ∎

Remark 1.5 *The value of the q-Ehrhart series at $t = 0$ is 1, because the unique point in $0Q$ is $\{0\}$.*

Remark 1.6 *Contrary to classical Ehrhart series, the numerator does not always have only positive coefficients, see example 1.9 below.*

1.1 Examples

Example 1.7 *Consider the polytope in $\mathbb{Z}$ with vertices $(0), (1)$ and the linear form (1). The q-Ehrhart series is*

$$\frac{1}{(1-t)(1-qt)} = 1 + (1+q)t + (1+q+q^2)t^2 + \cdots. \tag{4}$$

Example 1.8 *Consider the polytope in $\mathbb{Z}^2$ with vertices $(0,0),(1,0),(1,1)$ and the linear form $(1,1)$. The q-Ehrhart series is*

$$\frac{1}{(1-t)(1-qt)(1-q^2t)} = 1+(1+q+q^2)t+(q^4+q^3+2q^2+q+1)t^2+\ldots \quad (5)$$

Example 1.9 *Consider the polytope in $\mathbb{Z}^2$ with vertices $(0,0),(1,0),(1,1),(2,1)$ and the linear form $(1,1)$. The q-Ehrhart series is*

$$\frac{1-q^3t^2}{(1-t)(1-qt)(1-q^2t)(1-q^3t)} = 1+(1+q+q^2+q^3)t+\ldots \quad (6)$$

Note that its numerator has a negative coefficient.

Example 1.10 *Consider the polytope in $\mathbb{Z}^2$ with vertices $(0,0),(1,0),(1,1),(0,3)$ and the linear form $(1,1)$. The q-Ehrhart series is*

$$\frac{1+(q^2+q)t-(q^4+q^3+q^2)t^2}{(1-t)(1-qt)(1-q^2t)(1-q^3t)}. \quad (7)$$

2 q-Ehrhart polynomial

In this section, one proves the existence of the q-Ehrhart polynomial, and obtains a q-analog of Ehrhart reciprocity.

Let $[n]_q$ be the q-integer

$$[n]_q = \frac{q^n-1}{q-1}.$$

Let Q be a lattice polytope and λ be a linear form that satisfy the Positivity and Genericity conditions.

Let us write m for $\max_Q(\lambda)$.

Theorem 2.1 *There exists a polynomial $L_{Q,\lambda} \in \mathbb{Q}(q)[x]$ such that*

$$\forall n \in \mathbb{Z}_{\geq 0} \quad L_{Q,\lambda}([n]_q) = W_\lambda(nQ,q). \quad (8)$$

The degree of $L_{Q,\lambda}$ is m. The coefficients of $L_{Q,\lambda}$ have poles only at roots of unity of order less than m.

Proof. Consider the q-Ehrhart series $\mathrm{Ehr}_{Q,\lambda}$. By Proposition 1.1, it can be written as a sum

$$\sum_{j=0}^m c_j \frac{1}{1-q^jt}, \quad (9)$$

for some coefficients c_j in $\mathbb{Q}(q)$. Expanding one of the simple fractions, one gets

$$\frac{1}{1-q^jt} = \sum_{n\geq 0} q^{nj}t^n.$$

Because

$$(1+qx-x)\left.\right|_{x=[n]_q} = q^n, \quad (10)$$

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the value of the polynomial $(1 + qx - x)^j$ at the q-integer $[n]_q$ is given by q^{nj}.

Define the polynomial $L_{Q,\lambda}(x)$ by

$$\sum_{j=0}^{m} c_j (1 + qx - x)^j.$$

It follows that the value $L_{Q,\lambda}([n]_q)$ is exactly the coefficient of t^n in the q-Ehrhart series $\mathrm{Ehr}_{Q,\lambda}$. This is the expected property.

The statement about the degree of $L_{Q,\lambda}(x)$ is clear from the previous formula and proposition 1.4.

The polynomial $L_{Q,\lambda}(x)$ can therefore be recovered by interpolation at the q-integers between $[0]_q$ and $[m]_q$. The stated property of poles of its coefficients follows. ∎

Remark 2.2 *Contrary to the case of classical Ehrhart polynomials, whose degree is bounded by the ambient dimension, the degree here is the maximal value of the linear form on the polytope, and can be arbitrary large in any fixed dimension.*

Remark 2.3 *Obviously, letting $q = 1$ in the q-Ehrhart polynomial recovers the classical Ehrhart polynomial.*

Example 2.4 *Consider the four polytopes of examples 1.7, 1.8, 1.9 and 1.10. Their q-Ehrhart polynomials are*

$$qx + 1,$$
$$\frac{(qx + 1)(q^2x + q + 1)}{q + 1},$$
$$\frac{(qx + 1)^2(q^2x - qx + q + 1)}{q + 1},$$
$$(qx + 1)(q(q - 1)x^2 + 2qx + 1).$$

The reader can check the values at $x = 0, 1$ and the reduction to the classical Ehrhart polynomial at $q = 1$. All four examples being empty lattice polytopes, the values at $[-1]_q$ vanish.

2.1 q-Ehrhart reciprocity

If Q is a polytope, let us denote by $\mathrm{Int}(Q)$ the interior of Q.

Let

$$W_\lambda(\mathrm{Int}(nQ), q) = \sum_{x \in \mathrm{Int}(nQ)} q^{\lambda(x)} \tag{11}$$

be the weighted sum over interior lattice points in nQ.

Let $L_{Q,\lambda}$ be the q-Ehrhart polynomial of (Q, λ).

The following theorem is a q-analogue of Ehrhart reciprocity.

Theorem 2.5 *For every integer $n \in \mathbb{Z}_{>0}$, one has*

$$L_{Q,\lambda}([-n]_q) = (-1)^d W_\lambda(\mathrm{Int}(nQ), 1/q), \tag{12}$$

where d is the dimension of Q.

Proof. By Stanley's reciprocity theorem for rational cones [1, Theorem 4.3], applied to the cone over the polytope Q and with variables specialised to t and appropriate powers of q, one obtains

$$\mathrm{Ehr}_{Q,\lambda}(1/t, 1/q) = (-1)^{d+1}\,\mathrm{Ehr}_{\mathrm{Int}(Q),\lambda}(t, q), \tag{13}$$

where

$$\mathrm{Ehr}_{\mathrm{Int}(Q),\lambda}(t, q) = \sum_{n \geq 1} W_\lambda(\mathrm{Int}(nQ), q)t^n. \tag{14}$$

By definition of the q-Ehrhart series, one has

$$\mathrm{Ehr}_{Q,\lambda}(1/t, 1/q) = \sum_{n \leq 0} L_{Q,\lambda}([-n]_{1/q})t^n.$$

By lemma 2.6 below, this is the same as

$$-\sum_{n \geq 1} L_{Q,\lambda}([-n]_{1/q})t^n. \tag{15}$$

From this, one deduces that

$$\sum_{n \geq 1} W_\lambda(\mathrm{Int}(nQ), q)t^n = (-1)^d \sum_{n \geq 1} L_{Q,\lambda}([-n]_{1/q})t^n, \tag{16}$$

which is equivalent to the statement of the proposition. ∎

Lemma 2.6 *Let P be a polynomial in x with coefficients in $\mathbb{Q}(q)$. Then*

$$F^+ = \sum_{n \geq 0} P([n]_q)t^n \quad \text{and} \quad F^- = \sum_{n < 0} P([n]_q)t^n$$

are rational functions in t, q and $F^+ + F^- = 0$.

Proof. Every such polynomial can be written as a finite sum

$$\sum_j c_j(1 + qx - x)^j,$$

for some coefficients c_j in $\mathbb{Q}(x)$.

By linearity, it in enough to prove the lemma for the polynomial $(1+qx-x)^j$. In this case, using (10), one finds that $F^+ = 1/(1 - q^j t)$ and $F^- = q^{-j}t^{-1}/(1 - q^{-j}t^{-1})$. The statement is readily checked. ∎

3 Other properties

In this section, various general properties of the q-Ehrhart series and the q-Ehrhart polynomials are described.

3.1 Shifting the linear form

Let Q be a polytope. Let $s(Q)$ be the image of Q by a translation by a vector ν such that $\lambda(\nu) = N \geq 0$. The Positivity and Genericity conditions still hold for $s(Q)$.

At the level of q-Ehrhart series, it is immediate to see that

$$\mathrm{Ehr}_{s(Q),\lambda}(t, q) = \mathrm{Ehr}_{Q,\lambda}(q^N t, q). \tag{17}$$

and that

$$W_\lambda(n s(Q)) = q^{Nn} W_\lambda(nQ). \tag{18}$$

Using (10), one obtains that, at the level of q-Ehrhart polynomial,

$$L_{s(Q),\lambda} = (1 + qx - x)^N L_{Q,\lambda}. \tag{19}$$

Remark 3.1 *By using this kind of shift, one can always assume that $0 \in Q$.*

3.2 Reversal of polytopes

One defines here a duality on polytopes, depending on λ.

By the Genericity condition, there exists a unique vertex $\nu_{\max} \in Q$ where λ is maximal. Let us define a polytope $\overline{Q}$ as $\nu_{\max} - Q$. It is therefore the image of Q by an integer affine map which exchanges 0 and $\nu_{\max}$, hence Q and $\overline{Q}$ are equivalent as lattice polytopes.

The Positivity and Genericity conditions still hold for $\overline{Q}$.

In general, the pairs (Q, λ) and $(\overline{Q}, \lambda)$ are not equivalent under the action of the integral affine group, but some pairs (Q, λ) can be isomorphic to their dual for this duality. A necessary condition is that 0 is in Q.

Proposition 3.2 *The effect of this duality on q-Ehrhart series is given by*

$$\mathrm{Ehr}_{\overline{Q},\lambda} = \mathrm{Ehr}_{Q,\lambda}(tq^m, 1/q), \tag{20}$$

where $m = \lambda(\nu_{\max})$ is the maximal value of λ on Q.

Proof. One can see that $\overline{nQ}$ is just $n\overline{Q}$ for every $n \geq 0$. Therefore every point of weight q^j in some nQ corresponds to a point of weight q^{nm-j} in $n\overline{Q}$. This implies the statement. ∎

Remark 3.3 *The q-Ehrhart series of a polytope and its reversal are usually distinct, unless the polytope is self-dual. But they give the same classical Ehrhart series when $q = 1$.*

3.3 Many different pyramids

Let Q be a lattice polytope in the lattice M.

Define a new polytope $\mathrm{Pyr}(Q)$ in the lattice $\mathbb{Z} \times M$ as the pyramid with apex $(1, 0)$ based on $(0, Q)$. This is the convex hull of the polytope Q and a new vertex placed in a shifted parallel space.

Let us choose an integer $m \geq 0$ such that m is not among the values of λ on Q. For example, one can always choose $\max_Q(\lambda) + 1$.

Let us extend the linear form λ to a linear form $m \oplus \lambda$ on the lattice $\mathbb{Z} \times M$, whose value on a vector (k, ν) in $\mathbb{Z} \times M$ is $km + \lambda(\nu)$.

The Positivity and Genericity conditions still hold for $\mathrm{Pyr}(Q)$ with respect to $m \oplus \lambda$.

Proposition 3.4 *The q-Ehrhart series of $(\mathrm{Pyr}(Q), \mathfrak{m} \oplus \lambda)$ is given by*

$$\mathrm{Ehr}_{\mathrm{Pyr}(Q), \mathfrak{m} \oplus \lambda} = \mathrm{Ehr}_{Q, \lambda} / (1 - q^{\mathfrak{m}} t). \tag{21}$$

Proof. Let us compute

$$\mathrm{Ehr}_{\mathrm{Pyr}(Q), \mathfrak{m} \oplus \lambda} = \sum_{n \geq 0} W_{\mathfrak{m} \oplus \lambda}(n \, \mathrm{Pyr}(Q), q) t^n. \tag{22}$$

By the definitions of $\mathrm{Pyr}(Q)$ and $\mathfrak{m} \oplus \lambda$, this is

$$\sum_{n \geq 0} \sum_{i=0}^{n} q^{\mathfrak{m} i} W_\lambda((n-i)Q, q) t^n = \sum_{n \geq i \geq 0} q^{\mathfrak{m} i} t^i W_\lambda((n-i)Q, q) t^{n-i}, \tag{23}$$

which can be rewritten as the expected result. ∎

3.4 Periodicity of values at cyclotomic q

Let N be an integer such that $N > \max_Q(\lambda)$, and let ξ be a primitive root of unity of order N.

By theorem 2.1, one can let $q = \xi$ in the q-Ehrhart polynomial $L_{Q, \lambda}$.

Proposition 3.5 *The sequence of values $L_{Q, \lambda}([n]_q)|_{q=\xi}$ for $n \in \mathbb{Z}$ is periodic of period N.*

Proof. Indeed, the sequence $[n]_q$ itself is periodic of period N. ∎

Assume now that there is no lattice point in $\mathrm{Int}(nQ)$ for some integer n. By q-Ehrhart reciprocity, one has $L_{Q, \lambda}([-n]_q) = 0$. By the previous proposition, one deduces that

$$L_{Q, \lambda}([-n + kN]_q)|_{q=\xi} = 0, \tag{24}$$

for all $k \in \mathbb{Z}$. This means that the cyclotomic polynomial Φ_N divides the value $L_{Q, \lambda}([-n + kN]_q)$ for all $k \in \mathbb{Z}$.

This construction provides many cyclotomic factors in the values of some q-Ehrhart polynomials.

4 Posets and P-partitions

There exists a well-known theory of P-partitions, due to R. Stanley [10], which describes decreasing colourings of partially ordered sets (see also [7]). Part of this theory, namely its restriction to natural labellings, coincides exactly with a special case of the theory developed here, namely its application to the order polytope of the opposite of a poset. The theory of P-partitions does not include any analog of Ehrhart polynomials.

This section describes this common special case, and some specific properties of q-Ehrhart series and q-Ehrhart polynomials for posets.

Let P be a finite poset. The order polytope Q_P of the poset P is a lattice polytope in $\mathbb{Z}^P$ (with coordinates z_x for $x \in P$), defined by the inequalities

$$\begin{aligned} 0 \leq z_x \leq 1 \quad &\forall x \in P, \\ z_x \leq z_y \quad &\text{if} \quad x \leq y \in P. \end{aligned}$$

The polytope Q_P has vertices in $\mathbb{Z}^{\{0,1\}}$ and no interior lattice point [11].

Points in the dilated polytope nQ_P correspond to increasing colourings of the elements of P by the integers in $\{0, \ldots, n\}$.

In this section, the linear form λ will always be given by the sum of coordinates. The Positivity condition is clearly satisfied by Q_P and this linear form. One can also check that the Genericity condition holds, by using the known description of the vertices and edges of the order polytopes [11]. The minimal and maximal values of λ on Q_P are 0 and the cardinality of P.

For short, one will denote Ehr_P and L_P for the q-Ehrhart series and polynomial of Q_P.

According to [10, §8], the q-Ehrhart series Ehr_P can be written

$$\mathrm{Ehr}_P = \frac{W_P}{(1-t)(1-qt)\ldots(1-q^{\#P}t)}, \tag{25}$$

where W_P is a polynomial in q and t with nonnegative integer coefficients. This polynomial has a known combinatorial interpretation, using descents and major indices, as a sum over all linear extensions of the poset P.

From the general existence of the q-Ehrhart polynomial for polytopes (theorem 2.1), one deduces

Proposition 4.1 *There exists a polynomial L_P, of degree $\#P$, such that $L_P([n]_q)$ is the weighted sum over increasing colourings of P by $\{0, \ldots, n\}$, where the weight is q to the power the sum of colours.*

From the q-Ehrhart reciprocity (theorem 2.5), one obtains

Proposition 4.2 *For every integer $n \geq$, the polynomial $(-1)^{\#P}L_P([-n]_q)$ is the weighted sum over strictly increasing colourings of P by $\{1, \ldots, n-1\}$, where the weight is q to the power the sum of colours.*

One can find in [10, Prop. 10.4] a reciprocity formula for the q-Ehrhart series, closely related to the previous proposition.
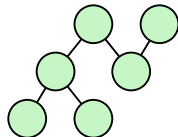

Figure 1: A poset P on 6 vertices, minima at the bottom
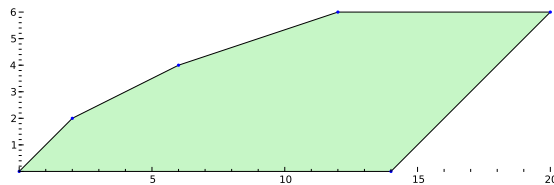

Figure 2: Newton polytope of the numerator of L_P

Based on experimental observations, one proposes the following conjecture, illustrated in figure 2.

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Conjecture 4.3 *The Newton polytope of the numerator of the q-Ehrhart polynomial L_P has the following shape. It has an horizontal top edge, an horizontal bottom edge and a diagonal right edge. Every element x of the poset P gives rise to a segment on the left border with inverse slope given by the length of the maximal chain of elements larger than x.*

For the trivial poset with just one element, the Ehrhart polynomial is $1 + qx$.

Example 4.4 *Let P be the partial order on the set $\{a, b, c, d\}$, where a is smaller than b, c, d. Then the Ehrhart polynomial L_P is $qx + 1$ times*

$$\frac{(q^2 x + q + 1)(\Phi_3 \Phi_4 + q(2q^4 + 4q^2 + q + 2)x + q^2(q^4 - q^3 + 3q^2 - q + 1)x^2)}{\Phi_2 \Phi_3 \Phi_4}, \tag{26}$$

where Φ_i is the cyclotomic polynomial of order i in the variable q.

For the opposite poset, one finds instead

$$\frac{(qx + 1)(q^2 x + q + 1)(\Phi_3 q^4 x^2 + (2q^4 + 2q^3 + 3q + 2)q^2 x + \Phi_4 \Phi_3)}{\Phi_2 \Phi_3 \Phi_4}. \tag{27}$$

4.1 Value at infinity

Given a poset P, one can evaluate the q-Ehrhart polynomial L_P at the limit (as a formal power series in q) of q-integers $[n]_q$ when n becomes infinity, namely at $1/(1 - q)$. This gives a rational function in q, which corresponds to the weighted sum of all increasing colourings of P.

For example, for the partial order on $\{a, b, c, d\}$ with $a \leq b, c, d$, one gets

$$\frac{1}{(q-1)^4 \Phi_2 \Phi_4}, \tag{28}$$

and for the opposite poset, one obtains

$$\frac{q^4 - q^3 + 3q^2 - q + 1}{(q-1)^4 \Phi_2 \Phi_3 \Phi_4}. \tag{29}$$

Let us compare this value to a limit at $t = 1$ of the q-Ehrhart series.

Proposition 4.5 *The value of L_p at $x = 1/(1 - q)$ is also the value at $t = 1$ of the product $(1 - t) \operatorname{Ehr}_P$.*

Proof. Indeed, the value of L_p at $x = 1/(1 - q)$ is the limit of the weighted sums over the dilated order polytopes of the poset P. This is just a weighted sum over the cone defined by the poset P.

On the other hand, the series Ehr_P has a simple pole at $t = 1$, hence the product $(1 - t) \operatorname{Ehr}_P$ has a well-defined value at $t = 1$. The coefficients of the series $(1 - t) \operatorname{Ehr}_P$ are the differences $L_p([n]_q) - L_P([n-1]_q)$. Their sum is also the weighted generating series of the cone associated with the poset P. ∎

4.2 Volume

Remark 4.6 *Let us note that the order polytope $Q_{\overline{P}}$ for the opposite $\overline{P}$ of a poset P is the reversal of the order polytope Q_P, as defined in section 3.2.*

Therefore, by proposition (3.2), one has

$$\text{Ehr}_{\overline{P}} = \text{Ehr}_P(tq^{\#P}, 1/q). \tag{30}$$

Let

$$\begin{bmatrix} n \\ m \end{bmatrix}_q \tag{31}$$

denote the q-binomial coefficients.

Lemma 4.7 *Let d be a nonnegative integer. Then*

$$\frac{1}{\prod_{j=0}^{d} 1 - q^j t} = \sum_{n \geq 0} \begin{bmatrix} d+n \\ n \end{bmatrix}_q t^n. \tag{32}$$

The coefficient of t^n is the value of the polynomial

$$\prod_{j=1}^{d} \frac{[j]_q + q^j x}{[j]_q} \tag{33}$$

at the q-integer $[n]_q$.

Proof. The first equation is classical and can be proved by an easy induction on d, using the definition of the q-binomial coefficients. The second statement is then clear. ∎

Lemma 4.8 *Let f be a polynomial in $\mathbb{Q}(q)[x]$ of degree d. Let g be the polynomial $f(1 + qx)$. Then $g([n]_q) = f([n+1]_q)$ for every integer n. The leading coefficient of g is q^d times the leading coefficient of f.*

Proof. Obvious. ∎

Let

$$[n]!_q = [1]_q [2]_q \cdots [n]_q$$

be the q-factorial of n.

Definition 1 *The q-volume of a poset P is the leading coefficient of the q-Ehrhart polynomial L_P times the q-factorial of $\#P$.*

For example, for the partial order on $\{a, b, c, d\}$ with $a \leq b, c, d$, the q-volume is

$$q^5(q+1)(q^4 - q^3 + 3q^2 - q + 1), \tag{34}$$

and for opposite poset, it is given by

$$q^7(q+1)(q^2 + q + 1). \tag{35}$$

Proposition 4.9 *The q-volume of P is equal to $q^{\binom{\#P+1}{2}}$ times the value at $t = 1$ and $q = 1/q$ of the numerator of the q-Ehrhart series of the opposite poset $\overline{P}$.*

Proof. Let us write

$$\mathrm{Ehr}_P = \frac{\sum_{k=0}^{\#P} h_k t^k}{\prod_{j=0}^{\#P} 1 - q^j t}, \tag{36}$$

for some coefficients h_k in $\mathbb{Q}[q]$.

According to (30), the value at $t = 1$ of the numerator of $\mathrm{Ehr}_{\overline{P}}(q = 1/q)$ is also the value at $t = 1$ of the numerator of $\mathrm{Ehr}_P(tq^{-\#P})$. This is given by

$$\sum_{k=0}^{\#P} h_k q^{-k\#P}.$$

By lemma 4.7 and lemma 4.8 applied to (36), the leading coefficient of the polynomial L_P is given by

$$\left(\sum_{k=0}^{\#P} h_k q^{-k\#P} \right) \prod_{j=1}^{\#P} \frac{q^j}{[j]_q} = \left(\sum_{k=0}^{\#P} h_k q^{-k\#P} \right) \frac{q^{\binom{\#P+1}{2}}}{[\#P]!_q}. \tag{37}$$

Comparing with the previous formula and using the definition of the q-volume concludes the proof. ∎

4.3 Pyramids for posets

As a special case of the general pyramid construction for polytopes described in section §3.3, one obtains the following results.

Proposition 4.10 *Let P be a poset. Consider the poset P^- with one minimal element added. Then the q-Ehrhart series are related by*

$$\mathrm{Ehr}_{P^-} = \mathrm{Ehr}_P /(1 - q^{1+\#P} t). \tag{38}$$

Proof. Indeed the order polytope of P^- is a pyramid over the product $\{0\} \times Q_P$, with one more vertex where every coordinate is 1. The sum-of-coordinates linear form takes the value $1 + \#P$ on this vertex. This pair (polytope, linear form) is equivalent as a pair to $(\mathrm{Pyr}(Q_P), m \oplus \lambda)$ with $m = 1 + \#P$ and λ the sum-of-coordinates linear form on Q_P. The result then follows from proposition 3.4. ∎

Proposition 4.11 *Consider the poset P^+ with one maximal element added. Then the q-Ehrhart series are related by*

$$\mathrm{Ehr}_{P^+}(t, q) = \mathrm{Ehr}_P(qt, q)/(1 - t). \tag{39}$$

Proof. This can be deduced from the previous proposition for the opposite poset, remark 4.6 and proposition 3.2. ∎

4.4 Vanishing at small negative q-integers

Let P be a poset, and let ℓ be the length of the longest increasing chain of P.

Because there are no strictly increasing colourings of P by integers $1, \ldots, n-1$ if $n \leq \ell$, the q-Ehrhart polynomial L_P vanishes at $[-n]_q$ for $1 \leq n \leq \ell$.

This implies that the q-Ehrhart polynomial L_P is divisible by $[n] + q^n x$ for every $1 \leq n \leq \ell$.

By the remarks of §3.4, this gives many cyclotomic factors in the values of L_P at q-integers.

4.5 q-Ehrhart polynomials of minuscule posets

For some posets, there are simple product formulas for the weighted sums over increasing colourings. One can deduce from them product formulas for the q-Ehrhart polynomials of these posets.

For example, consider the poset $P_{m,n} = A_m \times A_n$, where A_m is the total order of size m.

A famous result of MacMahon, usually described using plane partitions inside a box of size $m \times n \times k$, states that the weighted sum of decreasing colourings of the poset $P_{m,n}$ by integers in $\{0, \ldots, k\}$ is given by

$$\prod_{i=1}^{m} \prod_{j=1}^{n} \frac{[i+j-1+k]_q}{[i+j-1]_q}.$$

From this, one can easily deduce that

Proposition 4.12 *The q-Ehrhart polynomial of* $P_{m,n}$ *is*

$$\prod_{i=1}^{m} \prod_{j=1}^{n} \frac{[i+j-1]_q + x q^{i+j-1}}{[i+j-1]_q}. \tag{40}$$

More generally, this can be applied to the similar known formulas for other minuscule posets [9, 14], to obtain formulas for the q-Ehrhart polynomial as a product of linear polynomials.

5 Empty polytopes

An **empty lattice polytope** is a lattice polytope Q such that there is no lattice point in the interior $\mathrm{Int}(Q)$ of Q.

This section considers the q-Ehrhart theory in the special case of empty polytopes. For these polytopes, one can define a special evaluation of the q-Ehrhart polynomial, which seems to have interesting properties.

5.1 Special value for empty polytopes

Let Q be an empty lattice polytope. By q-Ehrhart reciprocity (theorem 2.5), the q-Ehrhart polynomial $L_{Q,\lambda}$ vanishes at $[-1]_q = -1/q$, hence is divisible by $1 + qx$.

Let us define the **special value** of Q by

$$v_{Q,\lambda} = \left(\frac{L_{Q,\lambda}(x)}{1 + qx} \right) \Bigg|_{x = -1/q} \tag{41}$$

By construction, the special value is a fraction in $\mathbb{Q}(q)$, with possibly poles at 0 and roots of unity.

The special value has the following property, which is not obviously true, because there exist polytopes with multiplicities in the denominator of their q-Ehrhart polynomials.

Proposition 5.1 *The poles of the special value $v_{Q,\lambda}$ at roots of unity are simple.*

Proof. Let $y = 1 + qx$ and let $L(y)$ be the polynomial $L_{Q,\lambda}((y-1)/q)$. One can write

$$L(y) = \sum_{i=1}^{d} c_i y^i,$$

for some integer d. The proposition is then equivalent to the statement that c_1 has only simple poles at root of unity.

The vector space over $\mathbb{Q}(q)$ spanned by y^i for $i = 1, \ldots, d$ has another basis, given by polynomials

$$P_i(y) = \frac{y}{[i]_q} \frac{\prod_{1 \le j \ne i} y - [j]_q}{\prod_{1 \le j \ne i} [i]_q - [j]_q},$$

for $i = 1, \ldots, d$.

The coefficients c_i are the coefficients of the polynomial $L(y)$ in the basis $(y^i)_{1 \le i \le d}$. The values of $L(y)$ at the q-integers $[i]_q$ are polynomials in q and give the coefficients of $L(y)$ in the basis $(P_i)_{1 \le i \le d}$.

The change of basis matrix from the basis $(P_i)_{1 \le i \le d}$ to the basis $(y^i)_{1 \le i \le d}$ is given by the expansion of the polynomials P_i in powers of y. In particular, the coefficient c_1 is computed using only the values of $L(y)$ at q-integers $[i]_q$ and the coefficients of y in the polynomials P_i, which are given by

$$(-1)^{d-1} \frac{1}{[i]_q} \frac{\prod_{1 \le j \ne i} [j]_q}{\prod_{1 \le j \ne i} [i]_q - [j]_q}. \tag{42}$$

This expression can be rewritten using the q-binomials (up to sign and a power of q) as

$$\frac{1}{[i]_q} \begin{bmatrix} d \\ i \end{bmatrix}_q, \tag{43}$$

and has therefore only simple poles at roots of unity. The statement follows. ∎

One may wonder whether this special value has any geometric meaning.

For the examples 1.7, 1.8, 1.9 and 1.10, the special values are 1, $1/(1+q)$, 0 and $-1/q$.

As the polytope Q_P associated with a poset P is empty, one can define the special value v_P. In the companion paper [5], it is proved using different methods that, for every poset P which is a rooted tree (with the root as maximum), the special value v_P has only simple poles at root of unity.

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5.2 Hahn operator

One defines the $\mathbb{Q}(q)$-linear operator Δ by

$$\Delta(f) = \frac{f(1 + qx) - f(x)}{1 + qx - x}, \tag{44}$$

acting on polynomials in x.

This is a q-analog of the derivative, which has been introduced by Hahn in [8]. The kernel of Δ is the space of constant polynomials. The restriction of Δ to the space of multiples of $1 + qx$ is an isomorphism with $\mathbb{Q}(q)[x]$.

One can translate this action of Δ on polynomials into an action on the values at $[n]_q$. Let f_n be the value $f([n]_q)$. Then

$$\Delta(f)([n]_q) = \frac{f_{n+1} - f_n}{q^n}. \tag{45}$$

For a lattice polytope Q in the lattice M, define a new polytope $B_+(Q)$ in the lattice $\mathbb{Z} \times M$ as the convex hull of the vertex $(0,0)$ and the product $\{1\} \times Q$. As a polytope, this is just the pyramid over Q. The linear form λ on M is extended to a linear form $B_+(\lambda)$ defined by

$$B_+(\lambda)(k, \nu) = k + \lambda(\nu).$$

The positivity and genericity conditions clearly hold for $(B_+(Q), B_+(\lambda))$.

Proposition 5.2 *One has*

$$\Delta(L_{B_+(Q), B_+(\lambda)}) = qL_{Q,\lambda}(1 + qx). \tag{46}$$

Proof. By (45), the value $q^n \Delta(L_{B_+(Q), B_+(\lambda)})([n]_q)$ is the weighted sum over $(n+1)B_+(Q)$ minus the weighted sum over $nB_+(Q)$. By the definition of $B_+(Q)$, this is nothing else than q^{n+1} times the weighted sum over $(n+1)Q$, namely $L_{Q,\lambda}([n+1]_q)$. The result follows. ∎

5.3 Umbral equalities

Recall that Carlitz q-Bernoulli numbers (introduced in [4]) are rational functions in q defined by $\beta_0 = 1$ and

$$q(q\beta + 1)^n - \beta_n = \begin{cases} 1 \text{ if } n = 1, \\ 0 \text{ if } n > 1, \end{cases} \tag{47}$$

where by convention one replaces β^k by β_k after expansion of the binomial power.

The Carlitz q-Bernoulli numbers have only simple poles at some roots of unity, and their value at $q = 1$ are the classical Bernoulli numbers.

Let P be a polynomial in x with coefficients in $\mathbb{Q}(q)$. Let us call q-umbra of P the value at P of the $\mathbb{Q}(q)$-linear form which maps x^n to the Carlitz q-Bernoulli number β_n. Let us denote by $\Psi(P)$ the q-umbra of P. It is a rational function in q.

Let Q be an empty polytope. The polytope $B_+(Q)$, which is a pyramid over Q, is also empty.

One has the following relations between the special value and the q-umbra.

Proposition 5.3 *One has*

$$v_{B_+(Q),B_+(\lambda)} = \Psi(L_{Q,\lambda}). \tag{48}$$

Proof. The right hand-side is the action of a $\mathbb{Q}(q)$-linear operator on a element of $\mathbb{Q}(q)[x]$.

By definition of the special value, mapping a multiple of $1 + qx$ to its special value is a $\mathbb{Q}(q)$-linear operator.

By proposition 5.2, the left-hand side is obtained from $L_{Q,\lambda}$ by first applying the inverse of Δ, then taking the special value. This is also a $\mathbb{Q}(q)$-linear operator.

It is therefore enough to check that this equality holds for enough polytopes, such that the $L_{Q,\lambda}$ span a basis of the space of polynomials. This has been done in the companion article [5, §4] for the order polytopes of all tree posets. ∎

Proposition 5.4 *One has*

$$v_{B_+B_+(Q),B_+B_+(\lambda)} = \Psi(-xL_{Q,\lambda}). \tag{49}$$

Proof. For the same reasons as before, both sides are $\mathbb{Q}(q)$-linear operators acting on $L_{Q,\lambda}$. It is therefore enough to check it on enough polytopes.

This has been done in the companion article [5, §4] for the order polytopes of all tree posets. ∎

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