

A GENERALIZATION OF LAGUERRE POLYNOMIALS

By

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1. INTRODUCTION. Recently F. J. PALAS [1] has introduced a k -set of polynomials $\{T_{kn}(x)\}$ as sequence of polynomials such that $T_{kn}(x)$ is of exactly degree kn , $n = 0, 1, 2, \dots$; k is a natural number. He has established that the k -set of polynomials defined by the generating function

$$(1.1) \quad (1-t)^{-1} \exp [x^k u(t)] = \sum_{n=0}^{\infty} T_{kn}(x) t^n,$$

where

$$u(t) = 1 - (1-t)^{-k}$$

satisfies the RODRIGUES formula

$$(1.2) \quad T_{kn}(x) = \exp(x^k) D^n [x^n \exp(-x^k)] / n!.$$

His explicit formula for such polynomials is

$$(1.3) \quad T_{kn}(x) = \frac{1}{n!} \sum_{p=0}^n \frac{x^{kp}}{p!} \sum_{r=0}^p (-1)^r \binom{p}{r} (kr+1)_n$$

where

$$(kr+1)_n = (kr+1)(kr+2)\dots(kr+n).$$

His differentiation formula is

$$(1.4) \quad (r+1) T_{k(r+1)}(x) = x T'_{kr}(x) + (r+1 - kx^k) T_{kr}(x).$$

This work of PALAS generalizes the results of a paper by P. HUMBERT [2] concerning a set of polynomials of even degree. It may be noted that the polynomials set of PALAS reduces to the LAGUERRE type of order zero, when $k = 1$.

The object of the present paper is to develop certain new properties which arise from a generalization of the LAGUERRE polynomials. It is interesting to note that our generalized polynomials include both the polynomials set of PALAS and the general LAGUERRE polynomials as special cases.

2. DEFINITION. We start with the definition

$$(2.1) \quad T_{kn}^{(\alpha)}(x) = \frac{1}{n!} x^{-\alpha} e^{x^k} D^n (x^{\alpha+n} e^{-x^k})$$

where k is a natural number.

We first show that the polynomial $T_{kn}^{(\alpha)}(x)$ is of exactly degree kn ($n = 0, 1, 2, \dots$). For this we notice the result [3]:

$$(2.2) \quad D_x^s f(z) = \sum_{k=0}^s \frac{(-1)^k}{k!} D_z^k f(z) \sum_{j=0}^k (-1)^j \binom{k}{j} z^{k-j} D_x^s z^j$$

Thus from (2.1) we derive

$$\begin{aligned} & T_{kn}^{(\alpha)}(x) \\ &= \frac{1}{n!} x^{-\alpha} e^{x^k} \sum_{s=0}^n \binom{n}{s} (D^{n-s} x^{\alpha+n}) (D^s e^{-x^k}) \\ &= \frac{1}{n!} e^{x^k} \sum_{s=0}^n \binom{n}{s} \binom{\alpha+n}{n-s} (n-s)! x^s (D^s e^{-x^k}) \\ &= \sum_{s=0}^n \binom{\alpha+n}{n-s} \sum_{i=0}^s \frac{x^{ki}}{i!} \sum_{j=0}^i (-1)^j \binom{i}{j} \binom{kj}{s} \\ &= \sum_{i=0}^n \frac{x^{ki}}{i!} \sum_{j=0}^i (-1)^j \binom{i}{j} \sum_{s=i}^n \binom{\alpha+n}{n-s} \binom{kj}{s} \end{aligned}$$

Next we know that

$$\sum_{j=0}^i (-1)^j \binom{i}{j} \sum_{s=0}^{i-1} \binom{\alpha+n}{n-s} \binom{kj}{s} = 0$$

Thus we finally obtain

$$\begin{aligned} & T_{kn}^{(\alpha)}(x) \\ &= \sum_{i=0}^n \frac{x^{ki}}{i!} \sum_{j=0}^i (-1)^j \binom{i}{j} \sum_{s=0}^n \binom{\alpha+n}{n-s} \binom{kj}{s} \end{aligned}$$

$$(2.3) \quad = \sum_{i=0}^n \frac{x^{ki}}{i!} \sum_{j=0}^i (-1)^j \binom{i}{j} \binom{\alpha + n + kj}{n},$$

which is the explicit formula for $T_{kn}^{(\alpha)}(x)$.

It may be noted that if $\alpha = 0$, we have

$$(2.4) \quad T_{kn}^{(0)}(x) = \sum_{i=0}^n \frac{x^{ki}}{i!} \sum_{j=0}^i (-1)^j \binom{i}{j} \binom{n + kj}{n},$$

which is (1.3). Thus $T_{kn}^{(0)}(x) \equiv T_{kn}(x)$.

Again if $k = 1$, we have from (2.3)

$$(2.5) \quad \begin{aligned} & T_n^{(\alpha)}(x) \\ &= \sum_{i=0}^n \frac{x^i}{i!} \sum_{j=0}^i (-1)^j \binom{i}{j} \binom{\alpha + n + j}{n} \\ &= \sum_{i=0}^n \frac{x^i}{i!} \cdot (-1)^i \binom{\alpha + n}{n - i}, \end{aligned}$$

which is the explicit formula for the general LAGUERRE polynomials $L_n^{(\alpha)}(x)$. Thus $T_n^{(\alpha)}(x) \equiv L_n^{(\alpha)}(x)$.

3. OPERATIONAL FORMULAE. Recently we [4] have derived the general operational formula

$$(3.1) \quad x^{-\alpha} D^n (x^{kn+\alpha} Y) = \prod_{j=1}^n \{x^{k-1} (\delta + \alpha + kj)\} Y, \\ (k = 1, 2, 3, \dots)$$

where $\delta \equiv x \frac{d}{dx}$ and Y is any sufficiently differentiable function of x .

The operators on the right of (3.1) commute only when $k = 1$.

Thus we derive

$$\begin{aligned} & x^{-\alpha} D^n (x^{\alpha+n} e^{-x^k} Y) \\ &= e^{-x^k} \prod_{j=1}^n (x D - kx^k + \alpha + j) Y \end{aligned}$$

In other words, we have

$$(3.2) \quad x^{-\alpha} e^{x^k} D^n (x^{\alpha+n} e^{-x^k} Y) = \prod_{j=1}^n (xD - kx^k + \alpha + j) Y$$

Again we note that

$$\begin{aligned} & D^n (x^{\alpha+n} e^{-x^k} Y) \\ &= \sum_{r=0}^n \binom{n}{r} D^{n-r} (x^{\alpha+n} e^{-x^k}) D^r Y \\ &= n! x^\alpha e^{-x^k} \sum_{r=0}^n \frac{x^r}{r!} T_{k(n-r)}^{(\alpha+r)}(x) D^r Y, \end{aligned}$$

whence we obtain

$$(3.3) \quad \frac{1}{n!} x^{-\alpha} e^{x^k} D^n (x^{\alpha+n} e^{-x^k} Y) = \sum_{r=0}^n \frac{x^r}{r!} T_{k(n-r)}^{(\alpha+r)}(x) D^r Y$$

It therefore follows from (3.2) and (3.3) that

$$\begin{aligned} & \prod_{j=1}^n (xD - kx^k + \alpha + j) Y \\ (3.4) \quad &= (n!) \sum_{r=0}^n \frac{x^r}{r!} T_{k(n-r)}^{(\alpha+r)}(x) D^r Y \end{aligned}$$

If we set $Y = 1$, we derive from (3.4)

$$(3.5) \quad (n!) T_{kn}^{(\alpha)}(x) = \prod_{j=1}^n (xD - kx^k + \alpha + j). 1$$

Again if $\alpha = 0$, we obtain from (3.5) the operational representation for the polynomials of PALAS :

$$(3.6) \quad (n!) T_{kn}^{(0)}(x) = \prod_{j=1}^n (xD - kx^k + j). 1$$

Lastly if $k = i = 1$ we get from (3.4) the operational formula for the general LAGUERRE polynomials :

$$\begin{aligned} & \prod_{j=1}^n (xD - x + \alpha + j) Y \\ (3.7) \quad &= (n!) \sum_{r=0}^n \frac{x^r}{r!} T_{n-r}^{(\alpha+r)}(x) D^r Y, \end{aligned}$$

which was previously obtained by L. CARLITZ [5].

4. SOME CONSEQUENCES OF OUR OPERATIONAL FORMULA :

From (3.5) we note that

$$(4.1) \quad n T_{kn}^{(\alpha)}(x) = (x D - kx^k + \alpha + n) T_{k(n-1)}^{(\alpha)}(x),$$

which is the differentiation formula satisfied by $T_{kn}^{(\alpha)}(x)$.

In particular, when $\alpha = 0$, we derive

$$(4.2) \quad n T_{kn}^{(0)}(x) = (x D - kx^k + n) T_{k(n-1)}^{(0)}(x)$$

which is (1.4).

On the other hand, if $k = 1$, we obtain from (4.1)

$$(4.3) \quad n T_n^{(\alpha)}(x) = (x D - x + \alpha + n) T_{n-1}^{(\alpha)}(x),$$

which is also implied by the familiar recurrences satisfied by $L_n^{(\alpha)}(x)$.

Next we consider

$$\begin{aligned} & (m+n)! T_{k(m+n)}^{(\alpha)}(x) \\ &= \prod_{j=1}^m (x D - kx^k + \alpha + n + j) \cdot \prod_{i=1}^n (x D - kx^k + \alpha + i) \cdot 1 \\ &= n! \prod_{j=1}^m (x D - kx^k + \alpha + n + j) \cdot T_{kn}^{(\alpha)}(x) \\ &= m! n! \prod_{r=0}^m \frac{x^r}{r!} T_{k(m-r)}^{(\alpha+n+r)}(x) D^r T_{kn}^{(\alpha)}(x), \end{aligned}$$

which implies that

$$(4.4) \quad \binom{m+n}{m} T_{k(m+n)}^{(\alpha)}(x) = \sum_{r=0}^{\min(m,n)} \frac{x^r}{r!} T_{k(m-r)}^{(\alpha+n+r)}(x) D^r T_{kn}^{(\alpha)}(x).$$

If we set $k = 1$, we derive from (4.4)

$$(4.5) \quad \binom{m+n}{m} T_{m+n}^{(\alpha)}(x) = \sum_{r=0}^{\min(m,n)} \frac{x^r}{r!} T_{m-r}^{(\alpha+n+r)}(x) D^r T_n^{(\alpha)}(x),$$

which is a well-known formula satisfied by the general LAGUERRE polynomials.

5. GENERATING FUNCTION. We shall now show that the polynomials $T_{kn}^{(\alpha)}(x)$ are generated by

$$(5.1) \quad g(x, t) \equiv (1-t)^{-\alpha-1} \exp [x^k u(t)] = \sum_{n=0}^{\infty} T_{kn}^{(\alpha)}(x) t^n,$$

where

$$u(t) = 1 - (1-t)^{-k}.$$

From the definition (2.1) we observe

$$(5.2) \quad T_{kn}^{(\alpha)}(x) = e^{x^k} \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} x^{kr} \binom{kr + \alpha + n}{n}$$

It is to be noted here that (2.3) is a consequence of (5.2).

Now we notice

$$g(x, t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \left[\frac{\partial^n}{\partial t^n} g(x, 0) \right] = \sum_{n=0}^{\infty} T_{kn}^{(\alpha)}(x) t^n,$$

whence we obtain

$$(5.3) \quad T_{kn}^{(\alpha)}(x) = \frac{1}{n!} \left[\frac{\partial^n}{\partial t^n} g(x, 0) \right]$$

Again we note that

$$\begin{aligned} & \left[\frac{\partial^n}{\partial t^n} \left\{ (1-t)^{-\alpha-1} \exp(x^k u(t)) \right\} \right]_{t=0} \\ &= e^{x^k} \left[\frac{\partial^n}{\partial t^n} \left\{ (1-t)^{-\alpha-1} \exp\left(\left(\frac{-x}{1-t}\right)^k\right) \right\} \right]_{t=0} \\ (5.4) \quad &= n! e^{x^k} \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} x^{kr} \binom{kr + \alpha + n}{n} \end{aligned}$$

Thus a comparison of (5.3) and (5.4) with (5.2) confirms (5.1).

Now from (5.1) we notice that

$$(1-t)^{k+1} \frac{\partial g(x, t)}{\partial t} = [\alpha + 1] (1-t)^k - kx^k g(x, t)$$

whence we obtain

$$(1-t)^{k+1} \sum_{n=1}^{\infty} n t^{n-1} T_{kn}^{(\alpha)}(x) = [\alpha + 1] (1-t)^k - kx^k \sum_{n=0}^{\infty} T_{kn}^{(\alpha)}(x) t^n$$

Performing the indicated multiplications on both sides and comparing coefficients of t^n on both sides, we derive

$$\begin{aligned}
 (5.5) \quad & \sum_{r=0}^{k+1} (-1)^r \binom{k+1}{r} (n+1-r) T_{k(n+1-r)}^{(\alpha)}(x) \\
 & = (\alpha+1) \sum_{r=0}^k (-1)^r \binom{k}{r} T_{k(n-r)}^{(\alpha)}(x) - kx^k T_{kn}^{(\alpha)}(x)
 \end{aligned}$$

Now if in particular, $k = 1$, we obtain from (5.5)

$$(5.6) \quad (n+1) T_{n+1}^{(\alpha)}(x) - (2n+\alpha+1-x) T_n^{(\alpha)}(x) + (n+\alpha) T_{n-1}^{(\alpha)}(x) = 0,$$

which is the well-known recursion relation satisfied by the general LAGUERRE polynomials.

Again returning to (5.1) we deduce in like manner

$$(5.7) \quad \sum_{r=0}^k (-1)^r \binom{k}{r} D T_{k(n-r)}^{(\alpha)}(x) = kx^{k-1} \sum_{r=1}^k (-1)^r \binom{k}{r} T_{k(n-r)}^{(\alpha)}(x)$$

where $D \equiv d/dx$.

As a special case, when $k = 1$, we notice from (5.7)

$$(5.8) \quad D [T_{n-1}^{(\alpha)}(x) - T_n^{(\alpha)}(x)] = T_{n-1}^{(\alpha)}(x)$$

which is another well-known recurrence relation satisfied by the general LAGUERRE polynomials.

Lastly we remark that the generating function (5.1) leads to the following finite summation formula for the polynomials $T_{kn}^{(\alpha)}(x)$.

For we observe

$$\begin{aligned}
 (1-t)^{-\alpha-1} \exp [x^k \{1 - (1-t)^{-k}\}] & = (1-t)^{-(\alpha-\beta)} (1-t)^{-\beta-1} \\
 & \quad \exp [x^k \{1 - (1-t)^{-k}\}]
 \end{aligned}$$

whence we obtain

$$\sum_{n=0}^{\infty} T_{kn}^{(\alpha)}(x) t^n = (1-t)^{-(\alpha-\beta)} \sum_{n=0}^{\infty} T_{kn}^{(\beta)}(x) t^n.$$

Now comparing the coefficient of t^n on both sides we get

$$(5.9) \quad T_{kn}^{(\alpha)}(x) = \sum_{r=0}^n \frac{(\alpha-\beta)_r}{r!} T_{k(n-r)}^{(\beta)}(x)$$

where α and β are arbitrary real numbers.

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