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**On a generalization of Laguerre polynomials**

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# ON A GENERALIZATION OF LAGUERRE POLYNOMIALS

*Nota* *) *di* S. K. CHATTERJEA (*a Calcutta*)

1. – In a recent paper [1], the writer has defined the polynomials $T_{kn}^{(\alpha)}(x)$ by the Rodrigues' formula

$$(1.1) \qquad T_{kn}^{(\alpha)}(x) = \frac{1}{n\,!}\, x^{-\alpha} e^{x^k} D^n (x^{\alpha+n} e^{-x^k}) \,,$$

where $k$ is a natural number. The polynomials $T_{kn}^{(\alpha)}(x)$ are of exactly degree $kn$ ($n = 0, 1, 2, \ldots$). They satisfy the operational formula

$$(1.2) \qquad \prod_{j=1}^{n} (xD - kx^k + \alpha + j) = n\,! \sum_{r=0}^{n} \frac{x^r}{r\,!}\, T_{k(n-r)}^{(\alpha+r)}(x) D^r \,.$$

The following are the consequences of the operational formula (1.2):

$$(1.3) \qquad nT_{kn}^{(\alpha)}(x) = (xD - kx^k + \alpha + n) T_{k(n-1)}^{(\alpha)}(x)$$

$$(1.4) \qquad \binom{m+n}{m} T_{k(m+n)}^{(\alpha)}(x) = \sum_{r=0}^{\min(m,n)} \frac{x^r}{r\,!}\, T_{k(m-r)}^{(\alpha+n+r)}(x) D^r T_{kn}^{(\alpha)}(x) \,.$$

The polynomials $T_{kn}^{(\alpha)}(x)$ are generated by the function

$$(1.5) \qquad (1 - t)^{-\alpha-1} \exp\, [x^k u(t)] = \sum_{n=0}^{\infty} T_{kn}^{(\alpha)}(x) t^n$$

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where

$$u(t) = 1 - (1 - t)^{-k}.$$

In the same paper the writer has also proved the following properties

$$\sum_{r=0}^{k+1} (-1)^r \binom{k+1}{r} (n + 1 - r) T^{(\alpha)}_{k(n+1-r)}(x)$$

$$(1.6) \qquad = (\alpha + 1) \sum_{r=0}^{k} (-1)^r \binom{k}{r} T^{(\alpha)}_{k(n-r)}(x) - kx^k T^{(\alpha)}_{kn}(x)$$

$$(1.7) \quad \sum_{r=0}^{k} (-1)^r \binom{k}{r} DT^{(\alpha)}_{k(n-r)}(x) = kx^{k-1} \sum_{r=1}^{k} (-1)^r \binom{k}{r} T^{(\alpha)}_{k(n-r)}(x)$$

$$(1.8) \qquad T^{(\alpha)}_{kn}(x) = \sum_{r=0}^{n} \frac{(\alpha - \beta)_r}{r!} T^{(\beta)}_{k(n-r)}(x)$$

This work of the writer generalizes some properties of the Laguerre polynomials $L^{(\alpha)}_n(x)$. Indeed, when $k = 1$, $T^{(\alpha)}_{kn}(x) \equiv L^{(\alpha)}_n(x)$. A similar generalization viz., $T^{(0)}_{kn}(x)$, has been previously studied by Palas [2]. The purpose of this paper is to discuss a more general class of Laguerre polynomials.

2. Definition: We first make the definition

$$(2.1) \qquad T^{(\alpha)}_{kn}(x, p) = \frac{1}{n!} x^{-\alpha} e^{px^k} D^n (x^{\alpha+n} e^{-px^k})$$

where $k$ is a natural number.

We now show that the polynomial $T^{(\alpha)}_{kn}(x, p)$ is of exactly degree $kn$ $(n = 0, 1, 2, \ldots)$. In this connection we know the result [3], for which I must thank Prof. H. W. Gould:

$$(2.2) \qquad D^s_z(z) = \sum_{k=0}^{s} \frac{(-1)^k}{k!} D^k_s f(z) \sum_{j=0}^{k} (-1)^j \binom{k}{j} z^{k-j} D^s_z z^j$$

Thus we obtain from (2.1)

$$T^{(\alpha)}_{kn}(x, p) =$$

$$= \frac{1}{n!} x^{-\alpha} e^{px^k} \sum_{s=0}^{n} \binom{n}{s} (D^{n-s} x^{\alpha+n})(D^s e^{-px^k}) =$$

$$= \frac{1}{n!} e^{px^k} \sum_{s=0}^{n} \binom{n}{s} \binom{\alpha+n}{n-s} (n-s)!\, x^s (D^s e^{-px^k})$$

$$= \sum_{s=0}^{n} \binom{\alpha+n}{n-s} \sum_{i=0}^{s} \frac{p^i}{i!} x^{ki} \sum_{j=0}^{i} (-1)^j \binom{i}{j} \binom{kj}{s}$$

$$= \sum_{i=0}^{n} \frac{p^i}{i!} x^{ki} \sum_{j=0}^{i} (-1)^j \binom{i}{j} \sum_{s=i}^{n} \binom{\alpha+n}{n-s} \binom{kj}{s}$$

Now we know that

$$\sum_{j=0}^{i} (-1)^j \binom{i}{j} \sum_{s=0}^{i-1} \binom{\alpha+n}{n-s} \binom{kj}{s} = 0 .$$

Thus we finally obtain

$$T_{kn}^{(\alpha)}(x, p) = \sum_{i=0}^{n} \frac{p^i}{i!} x^{ki} \sum_{j=0}^{i} (-1)^j \binom{i}{j} \binom{\alpha+n+kj}{n} , \tag{2.3}$$

which is the explicit formula for $T_{kn}^{(\alpha)}(x, p)$.

In particular, when $k = 1$, and $p = 1$, we derive

$$T_n^{(\alpha)}(x, 1) = \sum_{i=0}^{n} \frac{x^i}{i!} \sum_{j=0}^{i} (-1)^i \binom{i}{j} \binom{\alpha+n+j}{n} = \tag{2.4}$$

$$= \sum_{i=0}^{n} \frac{x^i}{i!} \cdot (-1)^i \binom{\alpha+n}{n-i}$$

which is the explicit formula for the general Laguerre polynomials $L_n^{(\alpha)}(x)$. Thus $T_n^{(\alpha)}(x, 1) \equiv L_n^{(\alpha)}(x)$.

3. – Operational formulae: Recently we [4] have derived the general operational formula

$$x^{-\alpha} D^n (x^{kn+\alpha} Y) = \prod_{j=1}^{n} \{x^{k-1}(z + \alpha + kj)\} Y , \qquad (k = 1, 2, 3, \ldots), \tag{3.1}$$

where $z \equiv x D$ and $Y$ is any sufficiently differentiable function of $x$. The operators on the right of (3.1) commute only when $k = 1$.

Thus we derive

$$(3.2)\qquad x^{-\alpha}e^{px^k}D^n(x^{\alpha+n}e^{-px^k}Y)=\prod_{j=1}^{n}(xD-pkx^k+\alpha+j)Y$$

Again we observe

$$D^n(x^{\alpha+n}e^{-px^k}Y)=\sum_{r=0}^{n}\binom{n}{r}D^{n-r}(x^{\alpha+n}e^{-px^k})D^rY=$$

$$=n\,!\,x^{\alpha}e^{-px^k}\sum_{r=0}^{n}\frac{x^r}{r\,!}T_{k(n-r)}^{(\alpha+r)}(x,p)D^rY,$$

whence we obtain

$$(3.3)\qquad \frac{1}{n\,!}x^{-\alpha}e^{px^k}D^n(x^{\alpha+n}e^{-px^k}Y)=\sum_{r=0}^{n}\frac{x^r}{r\,!}T_{k(n-r)}^{(\alpha+r)}(x,p)D^rY$$

It therefore follows from (3.2) and (3.3) that

$$(3.4)\qquad \prod_{j=1}^{n}(xD-pkx^k+\alpha+j)Y=n\,!\sum_{r=0}^{n}\frac{x^r}{r\,!}T_{k(n-r)}^{(\alpha+r)}(x,p)D^rY$$

If we set $Y=1$, we derive from (3.4)

$$(3.5)\qquad n\,!\,T_{kn}^{(\alpha)}(x,p)=\prod_{j=1}^{n}(xD-pkx^k+\alpha+j)\cdot 1$$

Further if $k=1$, and $p=1$, we obtain from (3.4)

$$(3.6)\qquad \prod_{j=1}^{n}(xD-x+\alpha+j)Y=n\,!\sum_{r=0}^{n}\frac{x^r}{r\,!}T_{n-r}^{(\alpha+r)}(x,1)D^rY;$$

which may be compared with the operational formula for the general Laguerre polynomials, derived by Carlitz [5].

In a recent paper [6], Gould and Hopper have generalized the Hermite polynomials by the definition

$$(3.7)\qquad H_n^k(x,\alpha,p)=(-1)^nx^{-\alpha}e^{px^k}D^n(x^{\alpha}e^{-px^k})$$

We remark that $x^n H_n^k(x, \alpha, p)$ yeilds a generalized class of polynomials of exactly degree $kn$ $(n = 0, 1, 2, \ldots)$, provided $k$ is a natural number. Consequently if we write

$$x^n H_n^k(x, \alpha, p) = H_{kn}^{(\alpha)}(x, p)$$

then

$$H_{kn}^{(\alpha)}(x, p) = (-1)^n x^{n-\alpha} e^{px^k} D^n (x^\alpha e^{-px^k}) \tag{3.8}$$

Thus the polynomials $H_{kn}^{(\alpha)}(x, p)$ are related to our polynomials by

$$H_{kn}^{(\alpha)}(x, p) = (-1)^n n!\, T_{kn}^{(\alpha-n)}(x, p). \tag{3.9}$$

Now returning to the operational formula (3.5) we obtain

$$(-1)^n H_{kn}^{(\alpha)}(x, p) = \prod_{j=1}^{n} (xD - pkx^k + \alpha - n + j) \cdot 1 \tag{3.10}$$

More generally we have from (3.4)

$$\prod_{j=1}^{n} (xD - pkx^k + \alpha - n + j) Y = \tag{3.11}$$
$$= \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} x^r H_{k(n-r)}^{(\alpha)}(x, p) D^r Y$$

This operational formula viz., (3.11) seems to be of particular interest. Indeed, using $k = 2$, $p = 1$, and $\alpha = 0$, we have

$$\prod_{j=1}^{n} (xD - 2x^2 - n + j) Y =$$
$$= \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} x^r H_{2(n-r)}^{(0)}(x, 1) D^r Y$$

Now noticing that

$$H_{2(n-r)}^{(0)}(x, 1) = x^{n-r} H_{n-r}(x),$$

where $H_n(x)$ denotes the ordinary Hermite polynomials defined by

$$H_n(x) = (-1)^n e^{x^2} D^n e^{-x^2},$$

we obtain

$$\prod_{j=1}^{n}(xD - 2x^2 - n + j)Y = x^n \sum_{r=0}^{n}(-1)^{n-r}\binom{n}{r} H_{n-r}(x) D^r Y$$

Now we note that

$$x^{-n}\prod_{j=1}^{n}(xD - 2x^2 - n + j) \equiv (D - 2x)^n. \tag{3.12}$$

For, (3.12) is evidently true for $n = 1$. Next assume that (3.12) is true for $n = m$. Then we have

$$\begin{aligned} x^{-(m+1)}\prod_{j=1}^{m+1}(xD - 2x^2 - m - 1 + j) &= \\ &= x^{-(m+1)}(xD - 2x^2 - m)\prod_{j=1}^{m}(xD - 2x^2 - m + j) = \\ &= x^{-(m+1)}(xD - 2x^2 - m)x^m(D - 2x)^m = \\ &= x^{-(m+1)} \cdot x^m(xD - 2x^2)(D - 2x)^m = \\ &= (D - 2x)^{m+1}. \end{aligned}$$

Hence by induction (3.12) is true for all positive integers n. Thus we finally derive

$$(D - 2x)^n = \sum_{r=0}^{n}(-1)^{n-r}\binom{n}{r} H_{n-r}(x) D^r, \tag{3.13}$$

a formula which Burchnall [7] derived some years ago.

## 4. - Some applications of the operational formula:

From (3.5) we note that

$$nT_{kn}^{(\alpha)}(x, p) = (xD - pkx^k + \alpha + n)T_{k(n-1)}^{(\alpha)}(x, p). \tag{4.1}$$

In particular, when $k = 1$, and $p = 1$, we derive

$$nT_n^{(\alpha)}(x, 1) = (xD - x + \alpha + n)T_{n-1}^{(\alpha)}(x, 1) \tag{4.2}$$

which is well-known for the Laguerre polynomials $L_n^{(\alpha)}(x)$.

Again in terms of the polynomials of Gould and Hopper, (4.1) stands thus

$$(4.3) \qquad H_{kn}^{(\alpha)}(x, p) + (xD - pkx^k + \alpha)H_{k(n-1)}^{(\alpha-1)}(x, p) = 0.$$

Next we consider

$$(m + n)!\, T_{k(m+n)}^{(\alpha)}(x, p) =$$

$$= \prod_{j=1}^{m} (xD - pkx^k + \alpha + n + j) \prod_{i=1}^{n} (xD - pkx^k + \alpha + i) \cdot 1$$

$$= n! \prod_{j=1}^{m} (xD - pkx^k + \alpha + n + j) \cdot T_{kn}^{(\alpha)}(x, p)$$

$$= m!\, n! \sum_{r=0}^{m} \frac{x^r}{r!} T_{k(m-r)}^{(\alpha+n+r)}(x, p) D^r T_{kn}^{(\alpha)}(x, p);$$

which implies that

$$(4.4) \qquad \binom{m+n}{m} T_{k(m+n)}^{(\alpha)}(x, p) = \\ = \sum_{r=0}^{\min(m,n)} \frac{x^r}{r!} T_{k(m-r)}^{(\alpha+n+r)}(x, p) D^r T_{kn}^{(\alpha)}(x, p).$$

The formula (4.4) readily yields the corresponding formula for the polynomials of Gould and Hopper:

$$(4.5) \quad H_{k(m+n)}^{(\alpha+m+n)}(x, p) \\ = \sum_{r=0}^{\min(m,n)} (-1)^r \binom{m}{r} x^r H_{k(m-r)}^{(\alpha+m+n)}(x, p) D^r H_{kn}^{(\alpha+n)}(x, p).$$

5. – Generating function: We shall now show that the polynomials $T_{kn}^{(\alpha)}(x, p)$ are generated by

$$(5.1) \qquad g(x, t) = (1 - t)^{-\alpha-1} \exp\,[px^k u(t)] = \sum_{n=0}^{\infty} T_{kn}^{(\alpha)}(x, p) t^n,$$

where

$$u(t) = 1 - (1 - t)^{-k}.$$

From the definition (2.1) we observe

$$(5.2) \qquad T_{kn}^{(\alpha)}(x, p) = e^{px^k} \sum_{r=0}^{\infty} \frac{(-p)^r}{r!} \binom{kr + \alpha + n}{n} x^{kr}.$$

It may be noted that (2.3) is a consequence of (5.2).

Now we notice that

$$(5.3) \qquad T_{kn}^{(\alpha)}(x, p) = \frac{1}{n!} \left[\frac{\partial^n}{\partial t^n} g(x, 0)\right]$$

Also

$$(5.4) \quad \left[\frac{\partial^n}{\partial t^n} \{(1 - t)^{-\alpha-1} \exp(px^k u(t))\}\right]_{t=0}$$

$$= e^{px^k} \left[\frac{\partial^n}{\partial t^n} \left\{(1 - t)^{-\alpha-1} \exp\left(- p \left(\frac{x}{1-t}\right)^k\right)\right\}\right]_{t=0}$$

$$= n!\, e^{px^k} \sum_{r=0}^{\infty} \frac{(-p)^r}{r!} \binom{kr + \alpha + n}{n} x^{kr}$$

Thus a comparison of (5.3) and (5.4) with (5.2) confirms (5.1).

Now from the generating function (5.1) we easily derive the following multiplication formula:

$$(5.5) \qquad T_{kn}^{(\alpha)}(xm^{1/k}, p) = T_{kn}^{(\alpha)}(x, mp),$$

which, in terms of the polynomials of Gould and Hopper, shapes into

$$(5.6) \qquad H_{kn}^{(\alpha)}(xm^{1/k}, p) = H_{kn}^{(\alpha)}(x, mp),$$

which may well be compared with (3.9) of [6, p. 54].

It is also interesting to note from (5.5) that

$$(5.7) \qquad T_n^{(\alpha)}(x, m) = L_n^{(\alpha)}(mx).$$

Again we observe

$$(1 - t)^{-\alpha-1} \exp[px^k\{1 - (1 - t)^{-k}\}]$$

$$= (1 - t)^{(\alpha-\beta)}(1 - t)^{-\beta-1} \exp[px^k\{1 - (1 - t)^{-k}\}]$$

whence we obtain

$$\sum_{n=0}^{\infty} T_{kn}^{(\alpha)}(x, p)t^n = (1 - t)^{-(\alpha-\beta)} \sum_{n=0}^{\infty} T_{kn}^{(\beta)}(x, p)t^n .$$

Now comparing the coefficients of $t^n$ on both sides we get

$$T_{kn}^{(\alpha)}(x, p) = \sum_{r=0}^{n} \frac{(\alpha - \beta)_r}{r!} T_{k(n-r)}^{(\beta)}(x, p) , \tag{5.8}$$

where $\alpha$ and $\beta$ are arbitrary real numbers.

Next we notice that

$$\sum_{n=0}^{\infty} T_{kn}^{(\alpha+\beta+1)}(x, p + q)t^n$$

$$= (1 - t)^{-\alpha-1}e^{pxk\{1-(1-t)^{-k}\}} \cdot (1 - t)^{-\beta-1}e^{qxk\{1-(1-t)^{-k}\}}$$

$$= \sum_{m=0}^{\infty} T_{km}^{(\alpha)}(x, p)t^m \cdot \sum_{n=0}^{\infty} T_{kn}^{(\beta)}(x, q)t^n$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{n} T_{km}^{(\alpha)}(x, p)T_{k(n-m)}^{(\beta)}(x, q)t^n .$$

Thus we obtain the following 'doubly-additive' addition formula

$$T_{kn}^{(\alpha+\beta+1)}(x, p + q) = \sum_{m=0}^{n} T_{km}^{(\alpha)}(x, p)T_{k(n-m)}^{(\beta)}(x, q) . \tag{5.9}$$

In particular, when $p = q = 1$, and $k = 1$, we derive

$$T_{n}^{(\alpha+\beta+1)}(x, 2) = \sum_{m=0}^{n} L_{m}^{(\alpha)}(x)L_{n-m}^{(\beta)}(x) . \tag{5.10}$$

It follows therefore from (5.7) and (5.10) that

$$L_{n}^{(\alpha+\beta+1)}(2x) = \sum_{m=0}^{n} L_{m}^{(\alpha)}(x)L_{n-m}^{(\beta)}(x) , \tag{5.11}$$

which is implied by the well-known formula of the Laguerre polynomials

$$(5.12) \qquad L_n^{(\alpha+\beta+1)}(x+y) = \sum_{m=0}^{n} L_m^{(\alpha)}(x) L_{n-m}^{(\beta)}(y) .$$

Again returning to (5.1) we obtain

$$(1-t)^{k+1} \frac{\partial g(x,t)}{\partial t} = [(\alpha+1)(1-t)^k - pkx^k] g(x,t)$$

whence we notice

$$(1-t)^{k+1} \sum_{n=1}^{\infty} n t^{n-1} T_{kn}^{(\alpha)}(x,p) = [(\alpha+1)(1-t)^k - pkx^k] \cdot \sum_{n=0}^{\infty} T_{kn}^{(\alpha)}(x,p) t^n .$$

Performing the indicated multiplication on both sides and comparing coefficients of $t^n$ on both sides, we derive

$$(5.13) \qquad \sum_{r=0}^{k+1} (-1)^r \binom{k+1}{r} (n+1-r) T_{k(n+1-r)}^{(\alpha)}(x,p) .$$
$$= (\alpha+1) \sum_{r=0}^{k} (-1)^r \binom{k}{r} T_{k(n-r)}^{(\alpha)}(x,p) - pkx^k T_{kn}^{(\alpha)}(x,p) .$$

Lastly we observe

$$(1-t)^k \frac{\partial g(x,t)}{\partial x} = pkx^{k-1}\{(1-t)^k - 1\} g(x,t) ,$$

whence we obtain in like manner

$$(5.14) \qquad \sum_{r=0}^{k} (-1)^r \binom{k}{r} D T_{(n-r)}^{(\alpha)}(x,p)$$
$$= pkx^{k-1} \sum_{r=1}^{k} (-1)^r \binom{k}{r} T_{(n-r)}^{(\alpha)}(x,p) .$$

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