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An extension of generalized Apostol-Euler polynomials

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Abstract

Recently, Tremblay, Gaboury and Fugère introduced a class of the generalized Bernoulli polynomials (see Tremblay in *Appl. Math. Let.* 24:1888-1893, 2011). In this paper, we introduce and investigate an extension of the generalized Apostol-Euler polynomials. We state some properties for these polynomials and obtain some relationships between the polynomials and Apostol-Bernoulli polynomials, Stirling numbers of the second kind, Jacobi polynomials, Laguerre polynomials, Hermite polynomials and generalized Bernoulli polynomials.

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1 Introduction, definitions and motivation

The generalized Bernoulli polynomials $B_n^{(\alpha)}(x)$ of order $\alpha \in \mathbb{Z}$ and the generalized Euler polynomials $E_n^{(\alpha)}(x)$ of order $\alpha \in \mathbb{Z}$ are defined by the following generating functions (see, [1–3, Vol. 3, p.253 et seq.] and [4, Section 2.8]):

$$\left(\frac{t}{e^t - 1}\right)^\alpha \cdot e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x) \frac{t^n}{n!} \quad (|t| < 2\pi) \quad (1.1)$$

and

$$\left(\frac{2}{e^t + 1}\right)^\alpha \cdot e^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x) \frac{t^n}{n!} \quad (|t| < \pi). \quad (1.2)$$

Recently, Luo and Srivastava introduced the generalized Apostol-Bernoulli polynomials $\mathfrak{B}_n^{(\alpha)}(x; \lambda)$ and the generalized Apostol-Euler polynomials $\mathfrak{E}_n^{(\alpha)}(x; \lambda)$ as follows.

Definition 1.1 (Luo and Srivastava [5]) The generalized Apostol-Bernoulli polynomials $\mathfrak{B}_n^{(\alpha)}(x; \lambda)$ of order $\alpha \in \mathbb{N}$ are defined by means of the following generating function:

$$\left(\frac{t}{\lambda e^t - 1}\right)^\alpha \cdot e^{xt} = \sum_{n=0}^{\infty} \mathfrak{B}_n^{(\alpha)}(x; \lambda) \frac{t^n}{n!} \quad (|t| < 2\pi \text{ when } \lambda = 1; |t| < |\log(\lambda)| \text{ when } \lambda \neq 1). \quad (1.3)$$

Clearly, the Apostol-Bernoulli polynomials $\mathfrak{B}_n(x; \lambda) := \mathfrak{B}_n^{(1)}(x; \lambda)$ and generalized Bernoulli polynomials $B_n^\alpha(x) = \mathfrak{B}_n^{(\alpha)}(x; 1)$.

Definition 1.2 (Luo [6]) The generalized Apostol-Euler polynomials $\mathfrak{E}_n^{(\alpha)}(x; \lambda)$ of order $\alpha \in \mathbb{C}$ are defined by means of the following generating function:

$$\left(\frac{2}{\lambda e^t + 1}\right)^\alpha \cdot e^{xt} = \sum_{n=0}^{\infty} \mathfrak{E}_n^{(\alpha)}(x; \lambda) \frac{t^n}{n!} \quad (|t| < |\log(-\lambda)|). \tag{1.4}$$

Clearly, the Apostol-Euler polynomials $\mathfrak{E}_n(x; \lambda) := \mathfrak{E}_n^{(1)}(x; \lambda)$ and generalized Euler polynomials $E_n^\alpha(x) = \mathfrak{E}_n^{(\alpha)}(x; 1)$.

Recently, Kurt [7] gave the following generalization of the Bernoulli polynomials of order α .

Definition 1.3 The generalized Bernoulli polynomials $B_n^{[m-1, \alpha]}(x)$, $m \in \mathbb{N}$, are defined, in a suitable neighborhood of $t = 0$, by means of the generating function

$$\left(\frac{t^m}{e^t - \sum_{l=0}^{m-1} \frac{t^l}{l!}}\right)^\alpha \cdot e^{xt} = \sum_{n=0}^{\infty} B_n^{[m-1, \alpha]}(x) \frac{t^n}{n!}. \tag{1.5}$$

Clearly, if we take $m = 1$ in (1.5), then the definition (1.5) becomes the definition (1.1).

More recently, Tremblay, Gaboury and Fugère [8] further gave the following generalization of Kurt's definition (1.5) in the following form.

Definition 1.4 For arbitrary real or complex parameters λ and the natural numbers $m, \alpha \in \mathbb{N}$, the generalized Bernoulli polynomials $\mathfrak{B}_n^{[m-1, \alpha]}(x)$ are defined, in a suitable neighborhood of $t = 0$, by means of the generating function

$$\left(\frac{t^m}{\lambda e^t - \sum_{l=0}^{m-1} \frac{t^l}{l!}}\right)^\alpha \cdot e^{xt} = \sum_{n=0}^{\infty} \mathfrak{B}_n^{[m-1, \alpha]}(x; \lambda) \frac{t^n}{n!}. \tag{1.6}$$

Clearly, if we take $m = 1$ in (1.6), then the definition (1.6) becomes the definition (1.3).

In view of (1.6) in Definition 1.4 and (1.5) in Definition 1.3, we give the following analogous definitions, (1.7) of which is a natural generalization for the generalized Euler polynomials $E_n^{(\alpha)}(x)$.

Definition 1.5 For complex numbers $\alpha \in \mathbb{C}$, natural numbers $m \in \mathbb{N}$, the generalized Euler polynomials $E_n^{[m-1, \alpha]}(x)$ are defined, in a suitable neighborhood of $t = 0$, by means of the generating function

$$\left(\frac{2^m}{e^t + \sum_{l=0}^{m-1} \frac{t^l}{l!}}\right)^\alpha \cdot e^{xt} = \sum_{n=0}^{\infty} E_n^{[m-1, \alpha]}(x) \frac{t^n}{n!}. \tag{1.7}$$

Obviously, setting $m = 1$ in (1.7), we have $\mathfrak{E}_n^{[0, \alpha]}(x; 1) = E_n^\alpha(x)$.

Definition 1.6 For arbitrary real or complex parameters λ, α and natural numbers $m \in \mathbb{N}$, the generalized Euler polynomials $\mathfrak{E}_n^{[m-1,\alpha]}(x)$ are defined, in a suitable neighborhood of $t = 0$, by means of the generating function

$$\left(\frac{2^m}{\lambda e^t + \sum_{l=0}^{m-1} \frac{t^l}{l!}} \right)^\alpha \cdot e^{xt} = \sum_{n=0}^{\infty} \mathfrak{E}_n^{[m-1,\alpha]}(x; \lambda) \frac{t^n}{n!}. \tag{1.8}$$

It is easy to see that setting $m = 1$ in (1.8), we have $\mathfrak{E}_n^{[0,\alpha]}(x; \lambda) = \mathfrak{E}_n^{(\alpha)}(x; \lambda)$. From (1.8) we readily get

$$\mathfrak{E}_0^{[m-1,\alpha]}(x; \lambda) = \left(\frac{2^m}{\lambda + 1} \right)^\alpha. \tag{1.9}$$

In the present paper, we give some properties of the polynomials $\mathfrak{E}_n^{[m-1,\alpha]}(x; \lambda)$ and obtain some relationships between the polynomials $\mathfrak{E}_n^{[m-1,\alpha]}(x; \lambda)$ and other polynomials and numbers, for example, the Stirling numbers of the second kind, Jacobi polynomials, Laguerre polynomials, Hermite polynomials and generalized Bernoulli polynomials.

2 Some basic properties for the polynomials $\mathfrak{E}_n^{[m-1,\alpha]}(x; \lambda)$

In this section, we state some basic properties for the generalized Apostol-Euler polynomials $\mathfrak{E}_n^{[m-1,\alpha]}(x; \lambda)$ defined by (1.8).

Proposition 2.1 *The generalized Apostol-Euler polynomials $\mathfrak{E}_n^{[m-1,\alpha]}(x; \lambda)$ satisfy the following relations:*

$$\mathfrak{E}_n^{[m-1,\alpha+\beta]}(x + y; \lambda) = \sum_{k=0}^n \binom{n}{k} \mathfrak{E}_k^{[m-1,\alpha]}(x; \lambda) \mathfrak{E}_{n-k}^{[m-1,\beta]}(y; \lambda), \tag{2.1}$$

$$\mathfrak{E}_n^{[m-1,\alpha]}(x + y; \lambda) = \sum_{k=0}^n \binom{n}{k} \mathfrak{E}_k^{[m-1,\alpha]}(y; \lambda) x^{n-k}. \tag{2.2}$$

Proof By (1.8), we have

$$\begin{aligned} \sum_{n=0}^{\infty} \mathfrak{E}_n^{[m-1,\alpha+\beta]}(x + y; \lambda) \frac{t^n}{n!} &= \left(\frac{2^m}{\lambda e^t + \sum_{l=0}^{m-1} \frac{t^l}{l!}} \right)^{\alpha+\beta} e^{(x+y)t} \\ &= \left(\frac{2^m}{\lambda e^t + \sum_{l=0}^{m-1} \frac{t^l}{l!}} \right)^\alpha e^{xt} \left(\frac{2^m}{\lambda e^t + \sum_{l=0}^{m-1} \frac{t^l}{l!}} \right)^\beta e^{yt} \\ &= \sum_{n=0}^{\infty} \mathfrak{E}_n^{[m-1,\alpha]}(x; \lambda) \frac{t^n}{n!} \sum_{n=0}^{\infty} \mathfrak{E}_n^{[m-1,\beta]}(y; \lambda) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^n \binom{n}{k} \mathfrak{E}_k^{[m-1,\alpha]}(x; \lambda) \mathfrak{E}_{n-k}^{[m-1,\beta]}(y; \lambda) \frac{t^n}{n!}. \end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on the both sides of the above equation, we arrive at (2.1) immediately. In the same way, we may get (2.2), so it is omitted. □

Proposition 2.2 *The generalized Apostol-Euler polynomials $\mathfrak{E}_n^{[m-1,\alpha]}(x; \lambda)$ satisfy the following relation:*

$$\lambda \mathfrak{E}_n^{[m-1,\alpha]}(x+1; \lambda) + \mathfrak{E}_n^{[m-1,\alpha]}(x; \lambda) = 2 \sum_{k=0}^n \binom{n}{k} \mathfrak{E}_k^{[m-1,\alpha]}(x; \lambda) \mathfrak{E}_{n-k}^{(-1)}(0; \lambda). \quad (2.3)$$

Proof By (1.4) and (1.8), we have

$$\begin{aligned} & \sum_{n=0}^{\infty} [\lambda \mathfrak{E}_n^{[m-1,\alpha]}(x+1; \lambda) + \mathfrak{E}_n^{[m-1,\alpha]}(x; \lambda)] \frac{t^n}{n!} \\ &= \lambda \sum_{n=0}^{\infty} \mathfrak{E}_n^{[m-1,\alpha]}(x+1; \lambda) \frac{t^n}{n!} + \sum_{n=0}^{\infty} \mathfrak{E}_n^{[m-1,\alpha]}(x; \lambda) \frac{t^n}{n!} = \left(\frac{2^m}{\lambda e^t + \sum_{l=0}^{m-1} \frac{t^l}{l!}} \right)^\alpha e^{xt} (\lambda e^t + 1) \\ &= 2 \left(\frac{2^m}{\lambda e^t + \sum_{l=0}^{m-1} \frac{t^l}{l!}} \right)^\alpha e^{xt} \left(\frac{2}{\lambda e^t + 1} \right)^{(-1)} = 2 \sum_{n=0}^{\infty} \mathfrak{E}_n^{[m-1,\alpha]}(x; \lambda) \frac{t^n}{n!} \sum_{n=0}^{\infty} \mathfrak{E}_n^{(-1)}(0; \lambda) \frac{t^n}{n!} \\ &= 2 \sum_{n=0}^{\infty} \sum_{k=0}^n \binom{n}{k} \mathfrak{E}_k^{[m-1,\alpha]}(x; \lambda) \mathfrak{E}_{n-k}^{(-1)}(0; \lambda) \frac{t^n}{n!}. \end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on the both sides of the above equation, we obtain the identity (2.3) at once. \square

Remark 2.3 Setting $m = 1$ in (2.3), we obtain the following familiar relations for the generalized Apostol-Euler polynomials (see [6]):

$$\lambda \mathfrak{E}_n^{(\alpha)}(x+1; \lambda) + \mathfrak{E}_n^{(\alpha)}(x; \lambda) = 2 \mathfrak{E}_n^{(\alpha-1)}(x; \lambda). \quad (2.4)$$

3 Some generalizations of the analogues of the Luo-Srivastava addition theorem

In this section, we give a generalization of the Luo-Srivastava addition theorem and an analogue.

Theorem 3.1 *The relationship*

$$\begin{aligned} & \mathfrak{E}_n^{[m-1,\alpha]}(x+y; \lambda) \\ &= \sum_{j=0}^n \frac{1}{n+1} \binom{n+1}{j} \left[\lambda \sum_{k=0}^{n-j+1} \binom{n-j+1}{k} \mathfrak{E}_k^{[m-1,\alpha-1]}(y; \lambda) - \mathfrak{E}_{n+1-j}^{[m-1,\alpha]}(y; \lambda) \right] \mathfrak{B}_j(x; \lambda) \\ & \quad + \frac{\lambda-1}{n+1} \left(\frac{2^m}{\lambda+1} \right)^\alpha \mathfrak{B}_{n+1}(x; \lambda) \end{aligned} \quad (3.1)$$

holds between the polynomials $\mathfrak{E}_n^{[m-1,\alpha]}(x; \lambda)$ and the Apostol-Bernoulli polynomials $\mathfrak{B}_n(x; \lambda)$ defined by (1.3) (with $\alpha = 1$).

Proof First of all, according to the equation [9, p.634, Eq. (29)], we substitute

$$x^n = \frac{1}{n+1} \left[\lambda \sum_{j=0}^{n+1} \binom{n+1}{j} \mathfrak{B}_j(x; \lambda) - \mathfrak{B}_{n+1}(x; \lambda) \right] \quad (3.2)$$

into the right-hand side of (2.2) and we get

$$\begin{aligned}
 & \mathfrak{E}_n^{[m-1,\alpha]}(x+y;\lambda) \\
 &= \sum_{k=0}^n \binom{n}{k} \mathfrak{E}_k^{[m-1,\alpha]}(y;\lambda) x^{n-k} \\
 &= \sum_{k=0}^n \binom{n}{k} \mathfrak{E}_k^{[m-1,\alpha]}(y;\lambda) \frac{1}{n+1-k} \left[\lambda \sum_{j=0}^{n+1-k} \binom{n+1-k}{j} \mathfrak{B}_j(x;\lambda) - \mathfrak{B}_{n+1-k}(x;\lambda) \right] \\
 &= \sum_{k=0}^n \binom{n}{k} \mathfrak{E}_k^{[m-1,\alpha]}(y;\lambda) \frac{1}{n+1-k} \left[\lambda \sum_{j=0}^{n-k} \binom{n-k+l}{j} \mathfrak{B}_j(x;\lambda) + (\lambda-1) \mathfrak{B}_{n-k+1}(x;\lambda) \right] \\
 &= \sum_{k=0}^n \binom{n}{k} \mathfrak{E}_k^{[m-1,\alpha]}(y;\lambda) \frac{1}{n+1-k} \lambda \sum_{j=0}^{n-k} \binom{n-k+l}{j} \mathfrak{B}_j(x;\lambda) \\
 &\quad + \sum_{k=0}^n \binom{n}{k} \mathfrak{E}_k^{[m-1,\alpha]}(y;\lambda) \frac{\lambda-1}{n+1-k} \mathfrak{B}_{n-k+1}(x;\lambda). \tag{3.3}
 \end{aligned}$$

The first sum in (3.3) is equal to

$$\begin{aligned}
 & \sum_{k=0}^n \binom{n}{k} \mathfrak{E}_k^{[m-1,\alpha]}(y;\lambda) \frac{1}{n+1-k} \lambda \sum_{j=0}^{n-k} \binom{n-k+l}{j} \mathfrak{B}_j(x;\lambda) \\
 &= \sum_{j=0}^n \sum_{k=0}^{n-j} \frac{\lambda}{n+1} \binom{n+l}{n-k+1} \binom{n-k+l}{j} \mathfrak{E}_k^{[m-1,\alpha]}(y;\lambda) \mathfrak{B}_j(x;\lambda) \\
 &= \sum_{j=0}^n \frac{\lambda}{n+1} \binom{n+1}{j} \mathfrak{B}_j(x;\lambda) \sum_{k=0}^{n-j} \binom{n+l-j}{k} \mathfrak{E}_k^{[m-1,\alpha]}(y;\lambda) \\
 &= \sum_{j=0}^n \frac{\lambda}{n+1} \binom{n+1}{j} \mathfrak{B}_j(x;\lambda) [\mathfrak{E}_{n+1-j}^{[m-1,\alpha]}(y+1;\lambda) - \mathfrak{E}_{n+1-j}^{[m-1,\alpha]}(y;\lambda)]. \tag{3.4}
 \end{aligned}$$

Upon inverting the order of summation and using the following elementary combinational identity:

$$\binom{m}{l} \binom{l}{n} = \binom{m}{n} \binom{m-n}{n-l} \quad (m \geq l \geq n; l, m, n \in \mathbb{N}_0), \tag{3.5}$$

the second sum in (3.3) is equal to (noting that $\mathfrak{B}_0(x;\lambda) = 0$)

$$\begin{aligned}
 & \sum_{k=0}^n \binom{n}{k} \mathfrak{E}_k^{[m-1,\alpha]}(y;\lambda) \frac{\lambda-1}{n+1-k} \mathfrak{B}_{n-k+1}(x;\lambda) \\
 &= \sum_{k=0}^n \binom{n+1}{k} \frac{\lambda-1}{n+1} \mathfrak{B}_{n-k+1}(x;\lambda) \mathfrak{E}_k^{[m-1,\alpha]}(y;\lambda) \\
 &= \sum_{k=0}^{n+1} \binom{n+1}{k} \frac{\lambda-1}{n+1} \mathfrak{B}_{n-k+1}(x;\lambda) \mathfrak{E}_k^{[m-1,\alpha]}(y;\lambda) - \frac{\lambda-1}{n+1} \mathfrak{B}_0(x;\lambda) \mathfrak{E}_{n+1-k}^{[m-1,\alpha]}(y;\lambda)
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{j=0}^{n+1} \binom{n+1}{j} \frac{\lambda-1}{n+1} \mathfrak{B}_j(x; \lambda) \mathfrak{E}_{n-j+1}^{[m-1, \alpha]}(y; \lambda) \\
 &= \sum_{j=0}^n \binom{n+1}{j} \frac{\lambda-1}{n+1} \mathfrak{B}_j(x; \lambda) \mathfrak{E}_{n-j+1}^{[m-1, \alpha]}(y; \lambda) + \frac{\lambda-1}{n+1} \mathfrak{B}_{n+1}(x; \lambda) \mathfrak{E}_0^{[m-1, \alpha]}(y; \lambda). \tag{3.6}
 \end{aligned}$$

Combining (3.4) and (3.6), and noting that (2.2) with $y = 1$ and (1.9), we obtain that

$$\begin{aligned}
 &\mathfrak{E}_n^{[m-1, \alpha]}(x+y; \lambda) \\
 &= \sum_{k=0}^n \binom{n}{k} \mathfrak{E}_k^{[m-1, \alpha]}(y; \lambda) \frac{1}{n+1-k} \lambda \sum_{j=0}^{n-k} \binom{n-k+l}{j} \mathfrak{B}_j(x; \lambda) \\
 &\quad + \sum_{k=0}^n \binom{n}{k} \mathfrak{E}_k^{[m-1, \alpha]}(y; \lambda) \frac{\lambda-1}{n+1-k} \mathfrak{B}_{n-k+1}(x; \lambda) \\
 &= \sum_{j=0}^n \frac{\lambda}{n+1} \binom{n+1}{j} \left[\mathfrak{E}_{n+1-j}^{[m-1, \alpha]}(y+1; \lambda) - \mathfrak{E}_{n+1-j}^{[m-1, \alpha]}(y; \lambda) \right] \mathfrak{B}_j(x; \lambda) \\
 &\quad + \sum_{j=0}^n \frac{\lambda-1}{n+1} \binom{n+1}{j} \mathfrak{B}_j(x; \lambda) \mathfrak{E}_{n-j+1}^{[m-1, \alpha]}(y; \lambda) + \frac{\lambda-1}{n+1} \mathfrak{B}_{n+1}(x; \lambda) \mathfrak{E}_0^{[m-1, \alpha]}(y; \lambda) \\
 &= \sum_{j=0}^n \frac{1}{n+1} \binom{n+1}{j} \\
 &\quad \times \left[\lambda \mathfrak{E}_{n+1-j}^{[m-1, \alpha]}(y+1; \lambda) - \lambda \mathfrak{E}_{n+1-j}^{[m-1, \alpha]}(y; \lambda) + (\lambda-1) \mathfrak{E}_{n+1-j}^{[m-1, \alpha]}(y; \lambda) \right] \mathfrak{B}_j(x; \lambda) \\
 &\quad + \frac{\lambda-1}{n+1} \mathfrak{E}_0^{[m-1, \alpha]}(y; \lambda) \mathfrak{B}_{n+1}(x; \lambda) \\
 &= \sum_{j=0}^n \frac{1}{n+1} \binom{n+1}{j} \left[\lambda \sum_{k=0}^{n-j+1} \binom{n-j+1}{k} \mathfrak{E}_k^{[m-1, \alpha-1]}(y; \lambda) - \mathfrak{E}_{n+1-j}^{[m-1, \alpha]}(y; \lambda) \right] \mathfrak{B}_j(x; \lambda) \\
 &\quad + \frac{\lambda-1}{n+1} \left(\frac{2^m}{\lambda+1} \right)^\alpha \mathfrak{B}_{n+1}(x; \lambda).
 \end{aligned}$$

This completes the proof. □

Remark 3.2 Letting $m = 1$ in (3.1) and noting (2.4), we get the Luo-Srivastava addition theorem (see [10, p.5711, Theorem B]):

$$\begin{aligned}
 &\mathfrak{E}_n^{(\alpha)}(x+y; \lambda) \\
 &= \sum_{k=0}^n \binom{n}{k} \frac{2}{k+1} (\mathfrak{E}_{k+1}^{(\alpha-1)}(y; \lambda) - \mathfrak{E}_{k+1}^{(\alpha)}(y; \lambda)) \mathfrak{B}_{n-k}(x; \lambda) \\
 &\quad + \frac{\lambda-1}{n+1} \left(\frac{2}{\lambda+1} \right)^\alpha \mathfrak{B}_{n+1}(x; \lambda). \tag{3.7}
 \end{aligned}$$

4 Some relationships between the polynomials $\mathfrak{E}_n^{[m-1, \alpha]}(x; \lambda)$ and other polynomials and numbers

In this section, by applying the same method as in the proof of (3.1) of Theorem 3.1, we give and display some relationships between the polynomials $\mathfrak{E}_n^{[m-1, \alpha]}(x; \lambda)$ and other polyno-

mials and numbers, for example, the Genocchi polynomials, Stirling numbers of the second kind, Laguerre polynomials, Jacobi polynomials, Hermite polynomials, generalized Bernoulli polynomials $B_n^{[m-1]}(x)$ and generalized Bernoulli polynomials $B_n^{(\alpha)}(x)$.

Theorem 4.1 *The relationship*

$$\begin{aligned} & \mathfrak{E}_n^{[m-1,\alpha]}(x+y;\lambda) \\ &= \frac{1}{2} \sum_{k=0}^n \frac{1}{k+1} \left[\binom{n}{k} \mathfrak{E}_{n-k}^{[m-1,\alpha]}(y;\lambda) + \sum_{j=k}^n \binom{n}{j} \binom{j}{k} \mathfrak{E}_{n-j}^{[m-1,\alpha]}(y;\lambda) \right] G_{k+1}(x) \end{aligned} \tag{4.1}$$

holds between the polynomials $\mathfrak{E}_n^{[m-1,\alpha]}(x;\lambda)$ and the Genocchi polynomials $G_n(x;\lambda)$ defined by [11, p.291, Eq. (1.1)].

Proof By substituting (see [11])

$$x^n = \frac{1}{2(n+1)} \left[\sum_{k=0}^n \binom{n+l}{k+1} G_{k+1}(x) + G_{n+1}(x) \right] \tag{4.2}$$

into the right-hand side of (2.2), we can get (4.1). □

Theorem 4.2 *The relationship*

$$\mathfrak{E}_n^{[m-1,\alpha]}(x+y;\lambda) = \sum_{k=0}^n k! \binom{x}{k} \sum_{j=k}^n \binom{n}{j} \mathfrak{E}_{n-j}^{[m-1,\alpha]}(y;\lambda) S(j,k) \tag{4.3}$$

holds between the polynomials $\mathfrak{E}_n^{[m-1,\alpha]}(x;\lambda)$ and the Stirling numbers $S(n,k)$ of the second kind defined by [12, p.58, Eq. (15)].

Proof By substituting (see [12, p.58, Eq. (14)])

$$x^n = \sum_{k=0}^n \binom{x}{k} k! S(n,k) \tag{4.4}$$

into the right-hand side of (2.2), we can get (4.3). □

Theorem 4.3 *The relationship*

$$\mathfrak{E}_n^{[m-1,\mu]}(x+y;\lambda) = \sum_{k=0}^n (-1)^k \sum_{j=k}^n j! \binom{n}{j} \binom{j+\alpha}{j-k} \mathfrak{E}_{n-j}^{[m-1,\mu]}(y;\lambda) L_k^{(\alpha)}(x) \tag{4.5}$$

holds between the polynomials $\mathfrak{E}_n^{[m-1,\alpha]}(x;\lambda)$ and the Laguerre polynomials defined by [12, p.55, Eq. (71)].

Proof By substituting (see [13, p.207, Eq. (2)])

$$x^n = n! \sum_{k=0}^n (-1)^k \binom{n+\alpha}{n-k} L_k^{(\alpha)}(x) \tag{4.6}$$

into the right-hand side of (2.2), we can obtain (4.5). □

Theorem 4.4 *The relationship*

$$\begin{aligned} & \mathfrak{E}_n^{[m-1, \mu]}(x+y; \lambda) \\ &= \sum_{k=0}^n (-1)^k \sum_{j=k}^n j! \binom{j+\alpha}{j-k} \binom{n}{j} \frac{\alpha+\beta+2k+1}{(\alpha+\beta+k+1)_{j+1}} \mathfrak{E}_{n-j}^{[m-1, \mu]}(y; \lambda) P_k^{(\alpha, \beta)}(1-2x) \end{aligned} \quad (4.7)$$

holds between the polynomials $\mathfrak{E}_n^{[m-1, \alpha]}(x; \lambda)$ and the Jacobi polynomials defined by [12, p.49, Eq. (35)].

Proof By substituting (see [13, p.262, Eq. (2)])

$$x^n = n! \sum_{k=0}^n (-1)^k \binom{n+\alpha}{n-k} \frac{\alpha+\beta+2k+1}{(\alpha+\beta+k+1)_{n+1}} P_k^{(\alpha, \beta)}(1-2x) \quad (4.8)$$

into the right-hand side of (2.2), we can get (4.7). □

Theorem 4.5 *The relationship*

$$\mathfrak{E}_n^{[m-1, \mu]}(x+y; \lambda) = \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{j=2k}^n 2^{\binom{n}{j}} \binom{j}{2k} \frac{(2k)!}{k!} \mathfrak{E}_{n-j}^{[m-1, \mu]}(y; \lambda) H_{j-2k}(x) \quad (4.9)$$

holds between the polynomials $\mathfrak{E}_n^{[m-1, \alpha]}(x; \lambda)$ and the Hermite polynomials defined by [12, p.55, Eq. (70)].

Proof By substituting (see [13, p.194, Eq. (4)])

$$(2x)^n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \frac{(2k)!}{k!} H_{n-2k}(x) \quad (4.10)$$

into the right-hand side of (2.2), we can get (4.9). □

Theorem 4.6 *The relationship*

$$\mathfrak{E}_n^{[m-1, \alpha]}(x+y; \lambda) = \sum_{k=0}^n \sum_{j=k}^n \frac{k!}{(j+m)!} \binom{n}{j} \binom{j}{k} \mathfrak{E}_{n-j}^{[m-1, \alpha]}(y; \lambda) B_{j-k}^{[m-1]}(x) \quad (4.11)$$

holds between the polynomials $\mathfrak{E}_n^{[m-1, \alpha]}(x; \lambda)$ and the generalized Bernoulli polynomials $B_n^{[m-1]}(x)$ defined by (1.5) (with $\alpha = 1$).

Proof According to the equation [14, p.158, (2.6)], we substitute

$$x^n = \sum_{k=0}^n \binom{n}{k} \frac{k!}{(k+m)!} B_{n-k}^{[m-1]}(x) \quad (m \in \mathbb{N}) \quad (4.12)$$

into the right-hand side of (2.2) and we can obtain (4.11). □

Theorem 4.7 *The relationship*

$$\mathfrak{E}_n^{[m-1,\alpha]}(x+y;\lambda) = \sum_{k=0}^n \sum_{l=k}^n \binom{n}{l} \binom{l}{k} \binom{k+j}{j}^{-1} \mathfrak{E}_{n-l}^{[m-1,\alpha]}(y;\lambda) S(k+j,j) B_{l-k}^{(j)}(x) \quad (4.13)$$

holds between the polynomials $\mathfrak{E}_n^{[m-1,\alpha]}(x;\lambda)$ and the generalized Bernoulli polynomials defined by (1.1).

Proof By substituting (see [15, p.1329, (2.15)])

$$x^n = \sum_{l=0}^n \binom{n}{l} \binom{l+j}{j}^{-1} S(l+j,j) B_{n-l}^{(j)}(x) \quad (j \in \mathbb{N}_0) \quad (4.14)$$

into the right-hand side of (2.2), we can get (4.13). □

Remark 4.8 If we set $m = 1$ in (4.1), (4.3), (4.5), (4.7), (4.9) and (4.11), then we obtain the corresponding results of the Apostol-Euler polynomials $\mathfrak{E}_n^{(\alpha)}(x;\lambda)$.

If we set $m = 1, \lambda = 1$ in (4.1), (4.3), (4.5), (4.7), (4.9) and (4.11), then we obtain the corresponding results of the generalized Euler polynomials $E_n^{(\alpha)}(x)$.

5 A remark for the Apostol-type polynomials

In [16, pp.939-940], Srivastava, Kurt and Simsek gave the following remarks about the Apostol-type polynomials:

‘... It should be reiterated in passing that the investigations of the corresponding generalizations of the Apostol-Euler polynomials, which are associated with any admissible (real or complex) order f , are not at all affected by the observations made here.

For the sake of the interested readers, we list below the following additional sequels, some relevant parts of which are believed to be similarly affected by the works of Luo and Srivastava (see [5, 7–10, 14, 15, 17–34]).

In each of the main results in most of the aforementioned works, which involve the generalized Apostol-Bernoulli polynomials and/or the generalized Apostol-Genocchi polynomials, only the nonnegative integer orders of these polynomials are considered and used correctly. Finally, it should be mentioned here that a suitable research-cum-expository article which would deal in detail, both analytically and rigorously, with each and every aspect of this situation is under preparation.’

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally in writing this paper, and read and approved the final manuscript.

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