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Moments on Catalan numbers

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ABSTRACT

By combining inverse series relations with binomial convolutions and telescoping method, moments of Catalan numbers are evaluated, which resolves a problem recently proposed by Gutiérrez et al. [J.M. Gutiérrez, M.A. Hernández, P.J. Miana, N. Romero, New identities in the Catalan triangle, J. Math. Anal. Appl. 341 (1) (2008) 52–61].

1. Introduction

Catalan numbers can be defined through binomial coefficients (cf. [1,7], [3, §1.15] and [5, §5.4] for example)

$$C_n = \frac{1}{n+1}\binom{2n}{n} \quad \text{for } n \in \mathbb{N}_0$$

which satisfy the recurrence relation

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}$$

as well as the formula due to Touchard [13] (see [4,9,11] also)

$$C_{n+1} = \sum_{0 \leqslant k \leqslant n/2} 2^{n-2k}\binom{n}{2k} C_k. \tag{1}$$

This sequence has many amazing combinatorial properties. In his book [12, Exercise 6.19], Stanley listed 66 enumerative problems which are counted by Catalan numbers. More comprehensive and updated coverage about combinatorial interpretations of Catalan numbers can be found in http://www-math.mit.edu/~rstan/ec/.

Associated with Catalan numbers, Shapiro [10] introduced Catalan triangles with the entries given by

$$B_{n,k} = \frac{k}{n}\binom{2n}{n-k} \quad \text{where } k \leqslant n \text{ and } k, n \in \mathbb{N}.$$

There exist interesting relations for these numbers, for example, the recurrence relation

$$B_{n,k} = B_{n-1,k-1} + 2B_{n-1,k} + B_{n-1,k+1} \quad \text{for } 1 < k \leqslant n$$

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and the expression in terms of Catalan numbers

$$B_{n,k} = \sum_{\ell_1+\ell_2+\cdots+\ell_k=n} C_{\ell_1} C_{\ell_2} \cdots C_{\ell_k}.$$

Recently, Gutiérrez et al. [6] established the following two summation identities

$$\sum_{k=1}^{n} k B_{n,k}^2 = \binom{n+1}{2} C_n C_{n-1}, \tag{2a}$$

$$\sum_{k=1}^{n} k^2 B_{n,k}^2 = (3n-2) C_{2n-2}. \tag{2b}$$

Observing another formula due to Shapiro [10, Corollary 3.5]

$$\sum_{k=1}^{n} B_{n,k}^2 = C_{2n-1} \tag{2c}$$

they proposed as one of the open problems to evaluate the following finite sum

$$\Omega_m(n) := \sum_{k=1}^{n} k^m B_{n,k}^2 \quad \text{for } m \in \mathbb{N}_0. \tag{3a}$$

This can be transformed into the evaluation of the following binomial moments:

$$\Theta_m(n) := \sum_{k=1}^{n} k^m \binom{2n}{n-k}^2 \quad \text{for } m \in \mathbb{N}_0. \tag{3b}$$

The purpose of the present short paper is to resolve this problem in general, which will be accomplished through combining inverse series relations with combinatorial computations. The key step will be expressing monomials as linear combinations of shifted factorial products, which will be given in the second section. Then the moments of even order will be computed in the third section by means of Chu–Vandermonde convolution formula on binomial coefficients. The fourth section will be devoted to the computation of the moments of odd order via telescoping method. Finally, the paper will end with few examples in the fifth section.

2. Symmetric functions and inverse series relations

For a sequence of indeterminate $\{\gamma_k\}_{k\geqslant 0}$, denote the elementary and complete symmetric functions (cf. Macdonald [8, §1.2]) respectively by $e_m(\gamma|\ell)$ and $h_m(\gamma|\ell)$:

$$e_0(\gamma|\ell) = 1 \quad \text{and} \quad e_m(\gamma|\ell) = \sum_{0\leqslant k_1<k_2<\cdots<k_m<\ell} \gamma_{k_1}\gamma_{k_2}\cdots\gamma_{k_m} \quad \text{for } m \in \mathbb{N}, \tag{4a}$$

$$h_0(\gamma|\ell) = 1 \quad \text{and} \quad h_m(\gamma|\ell) = \sum_{0\leqslant k_1\leqslant k_2\leqslant\cdots\leqslant k_m<\ell} \gamma_{k_1}\gamma_{k_2}\cdots\gamma_{k_m} \quad \text{for } m \in \mathbb{N}. \tag{4b}$$

Then there hold the following inverse series relations due to Chu [2]:

$$f_m = \sum_{k=0}^{m} (-1)^k h_{m-k}(\gamma|k+1) g_k, \tag{5a}$$

$$g_m = \sum_{k=0}^{m} (-1)^k e_{m-k}(\gamma|m) f_k. \tag{5b}$$

It is not hard to check that these inversions are equivalent to the following orthogonal relation:

$$\sum_{k=n}^{m} (-1)^{k-n} h_{k-n}(\gamma|n+1) e_{m-k}(\gamma|m) = \begin{cases} 0, & n<m; \\ 1, & n=m. \end{cases}$$

With the same γ-sequence as before, define the formal shifted factorials by

$$(x|\gamma)_0 = 1 \quad \text{and} \quad (x|\gamma)_m = (x+\gamma_0)(x+\gamma_1)\cdots(x+\gamma_{m-1}) \quad \text{for } m \in \mathbb{N}. \tag{6}$$

When $\gamma_k = \pm k$ for $k \in \mathbb{N}_0$, they will reduce to the usual rising and falling factorials

$$(x)_0 = 1 \quad \text{and} \quad (x)_m = x(x+1)\cdots(x+m-1) \quad \text{for } m \in \mathbb{N}, \tag{7a}$$

$$\langle x\rangle_0 = 1 \quad \text{and} \quad \langle x\rangle_m = x(x-1)\cdots(x-m+1) \quad \text{for } m \in \mathbb{N}. \tag{7b}$$

Then we can write explicitly the polynomial

$$(x|\gamma)_m = \sum_{k=0}^{m} x^k e_{m-k}(\gamma|m).$$

Comparing this equation with (5b) specified by $f_k = (-x)^k$ and $g_m = (x|\gamma)_m$, we get from (5a) the following dual relation

$$x^m = \sum_{k=0}^{m} (-1)^{m-k}(x|\gamma)_k h_{m-k}(\gamma|k+1). \tag{8}$$

Let y be another indeterminate. For the sake of brevity, denote by $\sigma_{k,\ell}(y)$ the following special complete symmetric function

$$\begin{aligned}\sigma_{k,\ell}(y) &= h_\ell\big(y^2, (y-1)^2, \ldots, (y-k)^2\big)\\ &= \sum_{0\leqslant k_1\leqslant k_2\leqslant\cdots\leqslant k_\ell\leqslant k} (y-k_1)^2(y-k_2)^2\cdots(y-k_\ell)^2.\end{aligned}$$

Putting $x \to -x^2$ and $\gamma_k = (y-k)^2$ in (8), we may write the resulting equation as

$$x^{2m} = \sum_{k=0}^{m} (-1)^k \langle y+x\rangle_k \langle y-x\rangle_k \sigma_{k,m-k}(y) \tag{9}$$

which will be our starting point to investigate the moments of Catalan numbers.

Furthermore, there is the following explicit expression for the connection coefficients

$$\sigma_{k,\ell}(y) = \frac{2(-1)^k}{\langle 2y\rangle_{2k+1}} \sum_{i=0}^{k} \binom{2y}{i}\binom{2k-2y}{k-i}(y-i)^{2k+2\ell+1}. \tag{10}$$

Let $[x^m]f(x)$ stand for the coefficient of x^m in formal power series $f(x)$. By means of the partial fraction decompositions, we can evaluate $\sigma_{k,\ell}(y)$ as follows:

$$\begin{aligned}\sigma_{k,\ell}(y) &= h_\ell\big(y^2, (y-1)^2, \ldots, (y-k)^2\big) = \big[x^\ell\big]\prod_{i=0}^{k}\frac{1}{1-x(y-i)^2}\\ &= \frac{2}{k!\langle 2y\rangle_{k+1}}\big[x^\ell\big]\sum_{i=0}^{k}(-1)^i\binom{k}{i}\frac{(y-i)^{2k+1}}{1-x(y-i)^2}\frac{\langle 2y\rangle_i}{\langle 2y-k-1\rangle_i}\\ &= \frac{2}{k!\langle 2y\rangle_{k+1}}\sum_{i=0}^{k}(-1)^i\binom{k}{i}\frac{\langle 2y\rangle_i}{\langle 2y-k-1\rangle_i}(y-i)^{2k+2\ell+1}.\end{aligned}$$

Then the formula displayed in (10) follows in view of the binomial relation

$$\frac{\langle 2y\rangle_i}{\langle 2y-k-1\rangle_i} = (-1)^{k-i}\binom{2y}{i}\binom{2k-2y}{k-i}\frac{i!(k-i)!}{\langle 2y-k-1\rangle_k}.$$

3. Moments of even order for Catalan numbers

For the moments of even order, it is trivial to see that

$$\Theta_{2m}(n) = \sum_{k=1}^{n} k^{2m}\binom{2n}{n-k}^2 = \frac{1}{2}\sum_{\ell=-n}^{n}\ell^{2m}\binom{2n}{n-\ell}^2 \quad \text{where } m \neq 0.$$

Recalling the relation (9), we can write

$$\ell^{2m} = \sum_{k=0}^{m}(-1)^k\langle n+\ell\rangle_k\langle n-\ell\rangle_k\sigma_{k,m-k}(n).$$

Substituting this relation into the binomial sum with respect to ℓ, interchanging the summation order and then applying the binomial relation

$$\langle n+\ell\rangle_k\langle n-\ell\rangle_k\binom{2n}{n-\ell}^2=\langle 2n\rangle_k^2\binom{2n-k}{n+\ell}\binom{2n-k}{n-\ell}$$

we can manipulate $\Theta_{2m}(n)$ as follows:

$$\begin{aligned}\Theta_{2m}(n)&=\frac{1}{2}\sum_{\ell=-n}^{n}\binom{2n}{n-\ell}^2\sum_{k=0}^{m}(-1)^k\langle n+\ell\rangle_k\langle n-\ell\rangle_k\sigma_{k,m-k}(n)\\&=\frac{1}{2}\sum_{k=0}^{m}(-1)^k\sigma_{k,m-k}(n)\sum_{\ell=-n}^{n}\langle n-\ell\rangle_k\langle n+\ell\rangle_k\binom{2n}{n+\ell}\binom{2n}{n-\ell}\\&=\frac{1}{2}\sum_{k=0}^{m}(-1)^k\langle 2n\rangle_k^2\sigma_{k,m-k}(n)\sum_{\ell=-n}^{n}\binom{2n-k}{n+\ell}\binom{2n-k}{n-\ell}.\end{aligned}$$

By means of the Chu–Vandermonde convolution formula on binomial coefficients, the inner sum with respect to ℓ can be evaluated as

$$\sum_{\ell=-n}^{n}\binom{2n-k}{n+\ell}\binom{2n-k}{n-\ell}=\binom{4n-2k}{2n-2k}.$$

Keeping in mind (10) and observing further the relation

$$\frac{\langle 2n\rangle_k^2}{\langle 2n\rangle_{2k+1}}\binom{4n-2k}{2n-2k}=\binom{4n-2k}{2n-k}\Big/(2n-2k)$$

we establish finally the following theorem.

Theorem 1 *(Moments of even order for Catalan numbers). There holds the following identity*

$$\Theta_{2m}(n)=\sum_{k=0}^{m}\frac{\lambda_k(m,n)}{2n-2k}\binom{4n-2k}{2n-k}$$

where the λ*-coefficients are explicitly given by the binomial sum*

$$\lambda_k(m,n)=\sum_{i=0}^{k}\binom{2n}{i}\binom{2k-2n}{k-i}(n-i)^{2m+1}.$$

In this theorem, replacing the λ-coefficients by the last line and then interchanging the summation order, we can derive another double sum expression.

Corollary 2 *(Moments of even order for Catalan numbers).*

$$\Theta_{2m}(n)=\sum_{i=0}^{m}\binom{2n}{i}(n-i)^{2m+1}\sum_{k=i}^{m}\frac{1}{2n-2k}\binom{4n-2k}{2n-k}\binom{2k-2n}{k-i}.$$

4. Moments of odd order for Catalan numbers

Instead, for the moments of odd order, we have to take completely different strategy to show the following result.

Theorem 3 *(Moments of odd order for Catalan numbers). Assume the same* λ*-coefficients as in Theorem* 1*. There holds the following identity*

$$\Theta_{2m+1}(n)=n\binom{2n}{n}\sum_{k=0}^{m}\frac{\lambda_k(m,n)}{4n-2k}\binom{2n-2k}{n-k}.$$

Proof. According to (9), reformulate $\Theta_{2m+1}(n)$ similarly as the double sum

$$\begin{aligned}\Theta_{2m+1}(n)&=\sum_{j=1}^{n}j\binom{2n}{n-j}^2\sum_{k=0}^{m}(-1)^k\langle n+j\rangle_k\langle n-j\rangle_k\sigma_{k,m-k}(n)\\&=\sum_{k=0}^{m}(-1)^k\langle 2n\rangle_k^2\sigma_{k,m-k}(n)\sum_{j=1}^{n}j\binom{2n-k}{n+j}\binom{2n-k}{n-j}.\end{aligned}$$

Taking into account the equation

$$j = \frac{1}{2}\{(n+j)-(n-j)\}$$

and the binomial relations

$$\binom{2n-k}{n+j} = \binom{2n-k-1}{n+j} + \binom{2n-k-1}{n+j-1},$$
$$\binom{2n-k}{n-j} = \binom{2n-k-1}{n-j} + \binom{2n-k-1}{n-j-1};$$

we may rewrite the summand in the inner sum as

$$\begin{aligned} j\binom{2n-k}{n+j}\binom{2n-k}{n-j} &= \frac{2n-k}{2}\left\{\binom{2n-k-1}{n+j-1}\binom{2n-k}{n-j} - \binom{2n-k}{n+j}\binom{2n-k-1}{n-j-1}\right\} \\ &= \frac{2n-k}{2}\left\{\binom{2n-k-1}{n+j-1}\binom{2n-k-1}{n-j} - \binom{2n-k-1}{n+j}\binom{2n-k-1}{n-j-1}\right\}. \end{aligned}$$

Now the inner sum with respect to j for the double sum expression of $\Theta_{2m+1}(n)$ can be evaluated, via telescoping method, as follows

$$\sum_{j=1}^{n} j\binom{2n-k}{n+j}\binom{2n-k}{n-j} = \frac{2n-k}{2}\binom{2n-k-1}{n}\binom{2n-k-1}{n-1}.$$

This leads consequently to the expression

$$\Theta_{2m+1}(n) = \frac{1}{2}\sum_{k=0}^{m}(-1)^k\langle 2n\rangle_k\langle 2n\rangle_{k+1}\binom{2n-k-1}{n}\binom{2n-k-1}{n-1}\sigma_{k,m-k}(n).$$

Finally, simplifying the last summand as

$$n\binom{2n}{n}\binom{2n-2k}{n-k}\frac{\lambda_k(m,n)}{2n-k}$$

we derive the formula stated in Theorem 3. □

Similarly, substituting the expression of the λ-coefficients into the equation stated in Theorem 3 and simplifying the result, we get another formula for $\Theta_{2m+1}(n)$.

Corollary 4 *(Moments of odd order for Catalan numbers).*

$$\Theta_{2m+1}(n) = n\binom{2n}{n}\sum_{i=0}^{m}\binom{2n}{i}(n-i)^{2m+1}\sum_{k=i}^{m}\frac{1}{4n-2k}\binom{2n-2k}{n-k}\binom{2k-2n}{k-i}.$$

5. Few examples

According to (3a) and (3b), it is trivial to see

$$\Omega_m(n) = n^{-2}\Theta_{m+2}(n).$$

Then we can reformulate the formulae of Theorems 1 and 3 as the following expressions of $\Omega_m(n)$ in terms of Catalan numbers.

Proposition 5 *(Linear relations in terms of Catalan numbers).*

$$\Omega_{2m}(n) = \sum_{k=0}^{m+1} C_{2n-k}\frac{1+2n-k}{n^2(2n-2k)}\lambda_k(1+m,n), \tag{11a}$$

$$\Omega_{2m+1}(n) = \sum_{k=0}^{m+1} C_nC_{n-k}\frac{(1+n)(1+n-k)}{n(4n-2k)}\lambda_k(1+m,n). \tag{11b}$$

Both formulae together provide a complete solution to the second problem proposed recently by Gutiérrez et al. [6]. In order to facilitate further computations, we display the first few λ-coefficients as follows:

$$\lambda_0(m,n) = n^{1+2m}, \tag{12a}$$

$$\lambda_1(m,n) = 2n(1-n)\{n^{2m} - (n-1)^{2m}\}, \tag{12b}$$

$$\lambda_2(m,n) = n(n-2)\{(2n-3)n^{2m} - 4(n-1)^{2m+1} + (2n-1)(n-2)^{2m}\}, \tag{12c}$$

$$\lambda_3(m,n) = \frac{2n(3-n)}{3}\left\{\begin{array}{l}(n-2)(2n-5)n^{2m} - (n-1)(2n-1)(n-3)^{2m}\\ \quad -3(2n-5)(n-1)^{2m+1} + 3(2n-1)(n-2)^{2m+1}\end{array}\right\}. \tag{12d}$$

Applying these polynomials, we can easily recover the formulae displayed in (2a), (2b) and (2c). Furthermore, we can also derive the following identities:

$$\Omega_3(n) = C_n C_{n-1}\frac{n^2(n+1)}{2}, \tag{13a}$$

$$\Omega_4(n) = C_{2n-3}\frac{15n^3 - 30n^2 + 16n - 2}{n - 1/2}, \tag{13b}$$

$$\Omega_5(n) = C_n C_{n-2}(n+1)n\{3n^2 - 5n + 1\}, \tag{13c}$$

$$\Omega_6(n) = C_{2n-4}\frac{105n^5 - 420n^4 + 588n^3 - 356n^2 + 96n - 10}{(n-1)(n-1/2)}, \tag{13d}$$

$$\Omega_7(n) = C_n C_{n-2}(n+1)n\{6n^3 - 12n^2 + 6n - 1\}, \tag{13e}$$

$$\Omega_8(n) = C_{2n-5}\frac{945n^7 - 6300n^6 + 16380n^5 - 21480n^4 + 15496n^3 - 6306n^2 + 1376n - 126}{(n-1)(n-1/2)(n-3/2)}, \tag{13f}$$

$$\Omega_9(n) = C_n C_{n-3}\frac{2n(n+1)}{n-1}\{30n^5 - 150n^4 + 252n^3 - 185n^2 + 65n - 9\}. \tag{13g}$$

Among these formulae, (13a) has been conjectured numerically by Gutiérrez et al. [6].

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