ELSEVIER

Journal of Computational and Applied Mathematics 54 (1994) 263–272

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

Asymptotics of basic Bessel functions and q-Laguerre polynomials

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Received 4 December 1991; revised 4 November 1992

Abstract

We establish a large n complete asymptotic expansion for q-Laguerre polynomials and a complete asymptotic expansion for a q-Bessel function of large argument. These expansions are needed in our study of an exactly solvable random transfer matrix model for disordered electronic systems. We also give a new derivation of an asymptotic formula due to Littlewood (1907).

Keywords: q-Laguerre polynomials; Complete asymptotic expansions; q-Bessel functions; Stieltjes–Wigert polynomials

1. Introduction

A q-shifted factorial $(a;\, q)_n$ is

$$(a;\, q)_0 := 1, \qquad (a;\, q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}), \quad n>0. \tag{1.1}$$

Sometimes the contracted notation

$$(a_1,\, a_2,\dots,a_r;\, q)_n := (a_1;\, q)_n(a_2;\, q)_n\cdots(a_r;\, q)_n \tag{1.2}$$

is useful. A basic (or q-) hypergeometric series is

$$ {}_{r+s+1}\Phi_s\left(\begin{matrix} a_1,\, a_2,\dots,a_r \\ b_1,\, b_2,\dots,b_s \end{matrix}\middle|\, q,\, z\right) := \sum_{n=0}^{\infty} \frac{(a_1,\, a_2,\dots a_{r+s+1};\, q)_n}{(b_1,\, b_2,\dots b_s;\, q)_n} \frac{(-1)^{nr} q^{rn(n-1)/2} z^n}{(q;\, q)_n}. \tag{1.3}$$

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[1] Research partially supported by NSF grant DMR 8813402.
[2] Research partially supported by NSF grants DMS 8814026 and DMS 8912423.

Elsevier Science B.V.
SSDI 0377-0427(92)00128-L

The little q-Jacobi polynomials

$$\Phi_n^{\alpha,\beta}(x) = {}_2\Phi_1\left(\begin{matrix} q^{-n}, \alpha\beta q^{n+1} \\ \alpha q \end{matrix} \middle| q, qx\right) \tag{1.4}$$

were introduced by Hahn. The terminology is due to [1]. As $q \to 1$, the little q-Jacobi polynomials tend to a multiple of Jacobi polynomials [19]. Complete asymptotic expansions for the little and big q-Jacobi polynomials and the Askey–Wilson polynomials have been established in [10] for $|q| < 1$. The parameter q in (1.4) will be referred to as the *base parameter*. If $|q| > 1$, the little q-Jacobi polynomials with $\alpha\beta \neq 0$ and base parameter q are constant multiples of a little q-Jacobi polynomial with base parameter $1/q$. Thus when $\alpha\beta \neq 0$, $|q| \neq 1$, there is no loss of generality in assuming $|q| < 1$.

Little q-Jacobi polynomials with $\beta = 0$ are q-analogs of Laguerre polynomials and are orthogonal with respect to a discrete measure on a countable set when $|q| < 1$, see [1] or [8, Exercise 1.32]. When $|q| > 1$, the moment problem does not have a unique solution [17]. This aspect was particularly interesting in [2–4] because the vacuum states are not unique. Moak [15] described the spectrum of the extremal measures with respect to which the polynomials in (1.5) are orthogonal. Ismail and Rahman [12] identified the Stieltjes transforms of all measures with respect to which the polynomials in (1.5) are orthogonal. This includes the extremal measures. For solutions of other indeterminate moment problems we refer the interested reader to [6,11]. In particular [11] contains an explicit evaluation of the extremal measures for the q-Hermite polynomials when $q > 1$.

In [3,4] we introduced an exactly solvable random transfer matrix model for disordered systems. This model describes electron transport in disordered systems. The results in the present paper were developed in order to show that our model has correct limiting forms.

The potential of a single particle is required to be linear for small x and behaves as $[\ln x]^2$ for large x. We found it convenient to use the potential $V(x;\, q)$:

$$w(x;\, q) := e^{-V(x;q)} = \frac{x^\alpha}{(-(1-q)x;\, q)_\infty}, \quad 0 < q < 1,\ \alpha > -1.$$

We then needed to study the polynomials orthogonal with respect to $w(x;\, q)$ and estimate the eigenvalue density normalized to N $(= \sigma_N(x;\, q))$. The polynomials orthogonal with respect to $w(x;\, q)$ are the q-Laguerre polynomials. In our model,

$$\sigma_N(x;\, q) = w(x;\, q) \sum_{k=0}^{N-1} \frac{[P_k(x;\, q)]^2}{h_k},$$

where the P's are the orthogonal polynomials and h_k are the squares of the $L_2(w(x;\, q)\, dx)$ norm of $P_k(x)$. To estimate the large N behavior of $\sigma_N(x;\, q)$, we used the Christoffel–Darboux formula and we needed the first two terms in the large n behavior of the q-Laguerre polynomials. We also needed to compare the large x behavior of the limiting function $(N \to \infty)$ of $\sigma_N(x;\, q)$ in order to compare it with the classical model associated with $q = 1$. This brief paragraph was added at the request of a referee but we refer the reader interested in the details of the model to our work [4], whose announcement already appeared in [3].

When we take $\beta=0$, $\alpha\neq 0$ in the little q-Jacobi polynomials, then replacing q by $1/q$, we essentially get the q-Laguerre polynomials

$$L_n^{(\alpha)}(x;q)=\frac{(q^{\alpha+1};q)_n}{(q;q)_n}\sum_{k=0}^{n}\frac{(q^{-n};q)_k q^{k(k-1)/2}}{(q,q^{\alpha+1};q)_k}\left[(1-q)xq^{n+\alpha+1}\right]^k,\quad |q|<1. \tag{1.5}$$

Recall that a basic Bessel function is [8,9]

$$J_\nu^{(2)}(x;q):=\frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{1}{2}x\right)^{\nu+2n}q^{n(n+\nu)}}{(q^{\nu+1},q;q)_n}. \tag{1.6}$$

It is well known that q-Laguerre polynomials converge uniformly on compact subsets of the complex x-plane to $[x(1-q)]^{-\alpha/2}J_\alpha^{(2)}(2\sqrt{x(1-q)};q)$.

The main results of this paper are Theorems 1 and 3 and Lemma 2 stated below.

Theorem 1. *The q-Laguerre polynomials have the representation*

$$L_n^{(\alpha)}(x;q)=\frac{(q^{\alpha+1};q)_\infty}{(q;q)_\infty}\sum_{j=0}^{\infty}\frac{q^{j(\alpha+n+1)}}{(q;q)_j}\sum_{k=0}^{n}\frac{(q^{-k-\alpha};q)_j q^{k(k+\alpha)}}{(q,q^{\alpha+1};q)_k}\left[(q-1)x\right]^k, \tag{1.7}$$

and the complete asymptotic expansion as $n\to\infty$,

$$L_n^{(\alpha)}(x;q)\approx\frac{(q^{\alpha+1};q)_\infty}{(q;q)_\infty}\sum_{j=0}^{\infty}\frac{q^{j(\alpha+n+1)}}{(q;q)_j}\sum_{k=0}^{\infty}\frac{(q^{-k-\alpha};q)_j q^{k(k+\alpha)}}{(q,q^{\alpha+1};q)_k}\left[(q-1)x\right]^k. \tag{1.8}$$

Lemma 2. *Set*

$$f(x,a;q):=\left(\mathrm{i}ax;\sqrt{q}\right)_\infty\,{}_3\phi_2\left(\left.\begin{matrix}a,\,-a,\,0\\-\sqrt{q},\,\mathrm{i}ax\end{matrix}\right|\sqrt{q},\sqrt{q}\right). \tag{1.9}$$

Then,

$$J_\nu^{(2)}(x;q)=\frac{\left(\sqrt{q};q\right)_\infty}{2(q;q)_\infty}\left(\tfrac{1}{2}x\right)^\nu\left[f\left(\tfrac{1}{2}x,q^{(\nu+1/2)/2};q\right)+f\left(-\tfrac{1}{2}x,q^{(\nu+1/2)/2};q\right)\right]. \tag{1.10}$$

Theorem 3. *As $x\to\infty$ with $0\leqslant\arg x<2\pi$, we have*

$$\begin{aligned}J_\nu^{(2)}&(x;q)\\&\approx\left(\tfrac{1}{2}x\right)^\nu\frac{(q^{1/2};q)_\infty}{2(q;q)_\infty}\Bigg[\left(\mathrm{i}q^{(\nu+1/2)/2}\tfrac{1}{2}x;q^{1/2}\right)_\infty\sum_{k=0}^{\infty}\frac{(q^{\nu+1/2};q)_k q^{k/2}}{(q;q)_k\left(\mathrm{i}q^{(\nu+1/2)/2}\tfrac{1}{2}x;q^{1/2}\right)_k}\\&\qquad+\left(-\mathrm{i}q^{(\nu+1/2)/2}\tfrac{1}{2}x;\sqrt{q}\right)_\infty\sum_{k=0}^{\infty}\frac{(q^{\nu+1/2};q)_k q^{k/2}}{(q;q)_k\left(-\mathrm{i}q^{(\nu+1/2)/2}\tfrac{1}{2}x;\sqrt{q}\right)_k}\Bigg].\end{aligned} \tag{1.11}$$

The asymptotic expansion (1.11) is not in terms of the usual asymptotic sequence $\{x^{-n}, n = 0, 1, \ldots\}$ [16], but is a sum of two complete asymptotic expansions in terms of the asymptotic sequences $\{(\pm\frac{1}{2}\mathrm{i}q^{(n+\nu+1/2)/2}x;\, q^{1/2})_\infty\colon n = 0, 1, \ldots\}$. This is to be expected since for real x a q-Bessel function is real and the asymptotic sequences involved are complex. The reader may also recall that Watson [20, Chapter 7] first established complete asymptotic expansions for the Hankel functions and then found a complete asymptotic expansion for $J_\nu(x)$ and $Y_\nu(x)$ from the fact that the Hankel functions are $J_\nu(x) \pm \mathrm{i}Y_\nu(x)$. This process is mirrored in (1.10) and (1.11). It is worth noting that $(x;\, q)_\infty$ is a q-analog of e^{-x}, hence $(\pm\frac{1}{2}\mathrm{i}xq^{(\nu+1/2)/2};\, q^{1/2})_\infty$ are analogs of $\exp(\pm(-\mathrm{i}x))$.

Theorem 4. *Let n be the smallest positive integer such that $q^{(\nu+1/2)/2}|x| < 2q^{-n-1}$ and assume that $0 < \beta < 1$. A basic Bessel function has the asymptotic behavior*

$$J_\nu^{(2)}(x;\, q) \approx \left(\tfrac{1}{2}x\right)^{\nu-1/2}\frac{q^{(\nu-1/2)/4}(q^{1/2};\, q)_\infty}{(q;\, q)_\infty}(-q^{-\beta+1/2}\mathrm{e}^{-\mathrm{i}\theta},\, -q^{\beta+1/2}\mathrm{e}^{\mathrm{i}\theta};\, q^{1/2})_\infty$$
$$\times\cos\left[\frac{2\pi}{\omega}\log\left(\tfrac{1}{2}x\right) - \nu\pi\right]\exp\left\{\frac{1}{4\omega}\left[4\log^2\left(\tfrac{1}{2}xq^{(\nu-1/2)/2}\right) - \pi^2\right]\right\}, \tag{1.12}$$

as $x \to \infty$, $|\arg x| < \frac{1}{2}\pi$, where $\theta = \arg x$ and $\omega = -\ln q$.

In Section 2 we shall provide proofs of Theorems 1 and 3 and Lemma 2. The proofs use mostly the q-binomial theorem and series rearrangements. Lemma 2 is the key to prove Theorem 3 because it provides a convergent asymptotic series representation for a q-Bessel function. We shall also prove Theorem 4 from Theorem 3 and classic results of Littlewood [14]. Littlewood's theorem in the case of q-exponentials is stated as Theorem 5.

Although Lemma 2 follows from general results in q-series stated as [8, Exercise 3.8, p.92 and Excercise 3.15, p.94], we included in Section 2 a simple direct proof to make this work as self-contained as possible.

The Stieltjes–Wigert polynomials [5] are the limiting case $\alpha \to \infty$ of the q-Laguerre polynomials of (1.5) and are orthogonal with respect to a log-normal probability density function. They appeared recently in a study of a q-harmonic oscillator [2]. A complete asymptotic expansion of the Stieltjes–Wigert polynomials follows from (1.8) as the special case $q^\alpha = 0$.

In Section 3 we give a new proof of Littlewood's Theorem 5 when x and q are real. Our proof uses the Jacobi triple product identity and the Poisson summation formula. In Section 4 we included an alternate representation for the q-Laguerre polynomials provided by the referee. We also discuss a complete asymptotic expansion for q-Bessel functions.

2. Proofs

We first give a proof of Theorem 1.

Proof of Theorem 1. Apply

$$(q^{-n};\, q)_k = (-1)^k\frac{q^{-kn+k(k-1)/2}(q;\, q)_n}{(q;\, q)_{n-k}} \tag{2.1}$$

and

$$(s;\,q)_n=\frac{(s;\,q)_\infty}{(sq^n;\,q)_\infty}\tag{2.2}$$

to see that the right-hand side of (1.5) equals

$$(q^{\alpha+1};\,q)_\infty\sum_{k=0}^{n}\frac{(q^{n-k+1};\,q)_\infty q^{k^2+\alpha k}}{(q;\,q)_\infty(q^{\alpha+n+1};\,q)_\infty(q,\,q^{\alpha+1};\,q)_k}[x(q-1)]^k.$$

We then expand $(q^{n-k+1};\,q)_\infty/(q^{\alpha+n+1};\,q)_\infty$ by the q-binomial theorem [8, (1.3.2)]

$$\sum_{n=0}^{\infty}\frac{(a;\,q)_n}{(q;\,q)_n}z^n=\frac{(az;\,q)_\infty}{(z;\,q)_\infty}\tag{2.3}$$

to see that the right-hand side of (1.5) can be expressed in the form

$$\frac{(q^{\alpha+1};\,q)_\infty}{(q;\,q)_\infty}\sum_{k=0}^{n}\sum_{j=0}^{\infty}\frac{q^{k(k+\alpha)}[x(q-1)]^k}{(q,\,q^{\alpha+1};\,q)_k}\,\frac{(q^{-k-\alpha};\,q)_j}{(q;\,q)_j}q^{j(\alpha+n+1)}.\tag{2.4}$$

When we interchange the k- and j-sums in the above expression, we obtain the right-hand side of (1.7). This establishes (1.7) and it easily implies (1.8). □

Proof of Lemma 2. In the series defining $f(x,\,a;\,q)$ we use $(b;\,\sqrt{q})_n(-b;\,\sqrt{q})_n=(b^2;\,q)_n$ and $(w;\,q)_\infty/(w;\,q)_n=(wq^n;\,q)_\infty$ to obtain the series representation

$$f(x,\,a;\,q)=\sum_{n=0}^{\infty}\frac{(a^2;\,q)_n}{(q;\,q)_n}(\mathrm{i}axq^{n/2};\,\sqrt{q})_\infty q^{n/2}.$$

We then expand $(\mathrm{i}axq^{n/2};\,q^{1/2})_\infty$ by Euler's theorem [8]

$$\sum_{n=0}^{\infty}\frac{(-x)^n}{(q;\,q)_n}q^{n(n-1)/2}=(x;\,q)_\infty.\tag{2.5}$$

The result is

$$f(x,\,a;\,q)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(a^2;\,q)_n(-\mathrm{i}ax)^k}{(q;\,q)_n(\sqrt{q};\,\sqrt{q})_k}q^{k(k-1)/4}q^{n(k+1)/2}.$$

Interchange the summations over k and n and evaluate the n-sum by the q-binomial theorem (2.3). The result is

$$f(x,\,a;\,q)=\sum_{k=0}^{\infty}\frac{(a^2q^{(k+1)/2};\,q)_\infty(-\mathrm{i}ax)^k}{(\sqrt{q};\,\sqrt{q})_k(q^{(k+1)/2};\,q)_\infty}q^{k(k-1)/4}.\tag{2.6}$$

The even part of $f(x,\,a;\,q)$ results from summing over even k. Therefore,

$$\tfrac{1}{2}[f(x,\,a;\,q)+f(-x,\,a;\,q)]=\sum_{k=0}^{\infty}\frac{(a^2q^{k+1/2};\,q)_\infty}{(\sqrt{q};\,\sqrt{q})_{2k}(q^{k+1/2};\,q)_\infty}(-\mathrm{i}ax)^{2k}q^{k(k-1/2)},$$

and we obtain

$$\tfrac{1}{2}[f(x, a; q)+f(-x, a; q)]=\frac{(a^2q^{1/2}; q)_\infty}{(q^{1/2}; q)_\infty}\sum_{k=0}^{\infty}\frac{(-1)^k(ax)^{2k}}{(q, a^2q^{1/2}; q)_k}q^{k(k-1/2)}, \tag{2.7}$$

where we used $(\sqrt{q}; \sqrt{q})_{2k}=(\sqrt{q}; q)_k(q; q)_k$ and (2.2). The choice $a=q^{(\nu+1/2)/2}$ in (2.7) readily implies (1.10). □

It is now clear that Theorem 3 is an immediate consequence of Lemma 2.

Theorem 5 (Littlewood [14]). *Let* $z=re^{i\theta}$, $r>1$, *and* $\omega=\omega_1+i\omega_2$. *Assume* $F(z)$ *is defined by*

$$F(z)=\prod_{s=1}^{\infty}(1+ze^{-s\omega}), \quad \operatorname{Re}\omega=\omega_1>0, \tag{2.8}$$

and assume that n *is the smallest positive integer such that* $(n+1)\omega_1>\log r$. *Define* a *and* β *by*

$$(n+1)\omega_1=\log r+\beta\omega_1, \quad 0<\beta<1, \qquad a=\beta+\frac{i[(n+1)\omega_2-\theta]}{\omega_1}. \tag{2.9}$$

Then,

$$\begin{aligned}\log F(z)=&\frac{\omega_1^2+\omega_2^2}{2\omega\omega_1^2}(\log z)^2-\frac{\log z}{2\omega_1}\left\{\omega_1+i\omega_2(2\beta-1)+\frac{2\theta\omega_2}{\omega_1}\right\}\\&+\tfrac{1}{2}\omega\left(\beta-\frac{i\theta}{\omega_1}\right)-\tfrac{1}{2}\omega\left(\beta-\frac{i\theta}{\omega_1}\right)^2+\log F(e^{a\omega_1})+\log F(e^{\omega-a\omega_1})\\&-\sum_{s=1}^{\infty}\frac{(-1)^s z^{-s}}{s(1-e^{-s\omega})}.\end{aligned} \tag{2.10}$$

It is worth mentioning that (2.10) of Theorem 5 is a corrected version of [14, Section 12, (12)]. The original version contained three misprints. The remaining formulas in the same section seem to be accurate.

Proof of Theorem 4. Apply (2.8) and (2.9) with $z=\tfrac{1}{2}ixq^{(\nu-1/2)/2}$, $\omega=-\ln q>0$, so $\omega_2=0$ and $\omega_2=\omega$. □

3. A proof of Theorem 5

The proof of Theorem 5 uses the Poisson summation formula

$$\sum_{n=-\infty}^{\infty}f(n)=\sum_{r=-\infty}^{\infty}\hat{f}(r), \qquad \hat{f}(r)=\int_{-\infty}^{\infty}f(w)e^{-2\pi irw}\,dw, \tag{3.1}$$

which holds if f is defined on $(-\infty, \infty)$; as $|x| \to \infty$, there exist positive constants A and δ such that

$$|f(x)| \leqslant A(1+|x|)^{-1-\delta}, \qquad |\hat{f}(x)| \leqslant A(1+|x|)^{-1-\delta}.$$

For a proof of the Poisson summation formula, see [18, Section 7.2]. Our proof also uses the Jacobi triple product identity

$$\left(q, -qz, -\frac{1}{z}; q\right)_\infty = \sum_{n=-\infty}^{\infty} q^{n(n+1)/2} z^n \tag{3.2}$$

[8, (II.28)]. Next we shall prove an alternate version of Theorem 5, see (3.4) below.

Proof of Theorem 5 (*for positive ω and real x*). Apply (3.1) to the right-hand side of (3.2) to get

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} q^{n(n+1)/2} x^n &= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left[-\tfrac{1}{2}\omega y^2 + y(2\pi i n + \xi - \tfrac{1}{2}\omega)\right] \mathrm{d}y \\
&= \left(\frac{2\pi}{\omega}\right)^{1/2} \exp\left[\frac{1}{2\omega}(\xi - \tfrac{1}{2}\omega)^2\right] \sum_{n=-\infty}^{\infty} (-1)^n \exp\left[\frac{2\pi i n\xi - 2\pi^2 n^2}{\omega}\right] \\
&= \left(\frac{2\pi}{\omega}\right)^{1/2} \exp\left[\frac{1}{2\omega}(\xi - \tfrac{1}{2}\omega)^2\right] \sum_{n=-\infty}^{\infty} (-1)^n p^{n^2} \mathrm{e}^{2\pi i n\xi/\omega},
\end{aligned}$$

where $\xi := \operatorname{Log} x$, $p := \exp(-2\pi^2/\omega)$ and, as we noted earlier, $\ln q = -\omega$. The above expression can be summed by the Jacobi triple product identity (3.1). This proves

$$\left(q, -qx, -\frac{1}{x}; q\right)_\infty = \left(\frac{2\pi}{\omega}\right)^{1/2} \exp\left[\frac{1}{2\omega}(\xi - \tfrac{1}{2}\omega)^2\right] (p^2, p\mathrm{e}^{2\pi i\xi/\omega}, p\mathrm{e}^{-2\pi i\xi/\omega}; p^2)_\infty. \tag{3.3}$$

Observe that when $|x| > 1$, then

$$\log\left(-\frac{1}{x}; q\right)_\infty = \sum_{n=0}^{\infty} \log\left(1 + \frac{q^n}{x}\right) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^m}{m} q^{nm} x^{-m} = \sum_{m=1}^{\infty} \frac{(-1)^m x^{-m}}{m(1-q^m)}.$$

Furthermore we have

$$\begin{aligned}
&\log(p\mathrm{e}^{2\pi i\xi/\omega}, p\mathrm{e}^{-2\pi i\xi/\omega}; p^2)_\infty \\
&= \sum_{s=\pm 1} \sum_{n=0}^{\infty} \log(1 - p^{2n+1}\mathrm{e}^{2\pi i s\xi/\omega}) = -\sum_{s=\pm 1} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m} p^{(2n+1)m} \mathrm{e}^{2\pi i m s\xi/\omega} \\
&= -\sum_{s=\pm 1} \sum_{m=1}^{\infty} \frac{1}{m} \mathrm{e}^{2\pi i m s\xi/\omega} p^m (1-p^{2m})^{-1} = -\sum_{m=1}^{\infty} \frac{1}{m} \frac{\mathrm{e}^{2\pi i m\xi/\omega} + \mathrm{e}^{-2\pi i m\xi/\omega}}{\mathrm{e}^{2\pi^2 m/\omega} - \mathrm{e}^{-2\pi^2 m/\omega}} \\
&= -\sum_{m=1}^{\infty} \frac{\cos(2\pi m\xi/\omega)}{m \sinh(2\pi^2 m/\omega)}.
\end{aligned}$$

In the above steps we used the fact that $|p \exp(\pm 2\pi i\xi/\omega)| = p < 1$. The above calculations yield

$$\log F_q(x) = \frac{(\log x)^2}{2\omega} - \tfrac{1}{2}\log x + \tfrac{1}{2}\log\left(\frac{2\pi}{\omega}\right) + \tfrac{1}{2}\omega + \log(p^2; p^2)_\infty$$
$$- \log(q; q)_\infty - \sum_{m=1}^{\infty} \frac{\cos(2\pi m\xi/\omega)}{m \sinh(2\pi^2 m/\omega)} - \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \frac{x^{-m}}{1 - e^{-m\omega}}. \quad \square \tag{3.4}$$

4. Remarks

The interested reader will observe that by combining Littlewood's Theorem (Theorem 5) with our Theorem 3, one will establish a complete asymptotic expansion for q-Bessel functions. We have not been able to find a closed form for the coefficients in this asymptotic expansion but it is obvious that one can find as many terms in the above-mentioned expansion of q-Bessel functions as desired.

The referee kindly pointed out that we can deduce the identity

$$\sum_{k=0}^{n} \frac{(q^n; q)_k q^{k(k-1)/2}}{(q, q^{\alpha+1}; q)_k} \left[(1-q)xq^{n+\alpha+1}\right]^k$$
$$= (-(1-q)xq^{n+\alpha-1}; q)_\infty \sum_{k=0}^{\infty} \frac{(q^{\alpha+n+1}; q)_k q^{k(k+\alpha)}[(1-q)x]^k}{(q, q^{\alpha+1}\ (q-1)xq^{\alpha+n+1}; q)_k} \tag{4.1}$$

from Jackson's formula [8, (1.5.4)] in a straightforward manner. The referee then observed that this gives the following representation of the q-Laguerre polynomials:

$$L_n^{(\alpha)}(x; q) = \frac{(q^{\alpha+1}; q)_n}{(q; q)_n}((q-1)xq^{\alpha+n+1}; q)_\infty \sum_{k=0}^{\infty} \frac{(q^{\alpha+n+1}; q)_k q^{k(\alpha+k)}[(q-1)x]^k}{(q, q^{\alpha+1}, (q-1)xq^{\alpha+n+1}; q)_k}. \tag{4.2}$$

The referee pointed out that the $n \to \infty$ limit of the expression on the right-hand side of (4.2) is a q-Bessel function. It is clear that (4.2) is a single sum, so it may seem that (4.2) has an advantage over our (1.8). However, a closer look at (4.2) easily shows that it is not an asymptotic expansion because the terms in (4.2) do not form an asymptotic sequence. Thus our (1.8) is indeed a complete asymptotic expansion, while the formulas like (4.2) can be thought of as alternate representations for q-Laguerre polynomials, which may say something about the large n behavior of q-Laguerre polynomials.

The reader undoubtedly noticed the power of Littlewood's result (Theorem 5). It is clear that Theorem 5 will be useful in any asymptotic analysis involving q-functions. Somehow Theorem 5 is not well known to workers in the area of q-series. Littlewood's paper [14] is cited in the excellent book [8] and a special case of Theorem 5 is used in [8, Section 4.3] but the theorem itself is not stated explicitly.

It is worth reminding the reader that Littlewood obtained Theorem 5 as a special case of his more general results and we only give a new and simple proof of Theorem 5. We feel the interest in the special case (Theorem 5) is justified for two reasons. Firstly, Theorem 5 is just what is needed to study asymptotic properties of q-infinite products. Secondly, the logarithm of the function $F(z)$ of (2.8) arises in the problem of the motion of a spinless electron gas under the influence of a strong external homogeneous and stationary magnetic field. The free energy is proportional to

$$\sum_{n=0}^{\infty} \log\left[1 - 2p^{2n+1} \cos\left(\frac{2\pi\xi}{\omega}\right) + p^{4n+2}\right]$$
$$= \sum_{n=0}^{\infty} \log\left[\left(1 - p^{n+1/2} e^{-2\pi i\xi/\omega}\right)\left(1 - p^{n+1/2} e^{-2\pi i\xi/\omega}\right)\right].$$

It can be shown that the oscillatory terms in the free energy as a function of the external field gives rise to oscillatory magnetic susceptibility; an experimentally observed phenomenon called the de Haas–van Alphen effect [13].

Acknowledgements

The authors thank George E. Andrews of Pennsylvania State University for bringing them together and for encouragement. We also thank a referee for his helpful comments and constructive criticism.

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