Available online at www.sciencedirect.com
ScienceDirect
European Journal of Combinatorics 28 (2007) 1331–1344
European Journal of Combinatorics
www.elsevier.com/locate/ejc

Area of Catalan paths on a checkerboard

Szu-En Cheng[a], Sen-Peng Eu[a], Tung-Shan Fu[b]

[a] *Department of Applied Mathematics, National University of Kaohsiung, Kaohsiung 811, Taiwan, ROC*
[b] *Mathematics Faculty, National Pingtung Institute of Commerce, Pingtung 900, Taiwan, ROC*

Received 13 August 2005; accepted 17 January 2006
Available online 15 March 2006

Abstract

It is known that the area of all Catalan paths of length n is equal to $4^n - \binom{2n+1}{n}$, which coincides with the number of inversions of all 321-avoiding permutations of length $n+1$. In this paper, a bijection between the two sets is established. Meanwhile, a number of interesting bijective results that pave the way to the required bijection are presented.

1. Introduction

Among many other combinatorial structures, the nth Catalan number $c_n = \frac{1}{n+1}\binom{2n}{n}$ enumerates the number of lattice paths, called *Catalan paths of length n*, in the plane $\mathbb{Z} \times \mathbb{Z}$ from $(0, 0)$ to (n, n) using north steps $(0, 1)$ and east steps $(1, 0)$ that never pass below the line $y = x$. Let $\mathcal{C}_n$ denote the set of Catalan paths of length n. A Catalan path is said to be *elevated* if it remains strictly above the line $y = x$ except at the start and end points. The *area* of a Catalan path is defined to be the number of triangles of the region enclosed by the path and the line $y = x$. For example, the area of the path shown in Fig. 1 is 13. In [8], Merlini et al. derived that the area a_n of all Catalan paths of length n is $a_n = 4^n - \binom{2n+1}{n}$, which is also equal to $\sum_{k=1}^{n} 4^{n-k} c_k$ as shown in [15]. Shapiro et al. proved that the area of all elevated Catalan paths of length n is 4^{n-1} [11]. There is other literature concerning the area and moments of Catalan paths (e.g., [3,6, 9]).

A permutation $\sigma = \sigma_1 \cdots \sigma_n$ of $\{1, \ldots, n\}$, where $\sigma_i = \sigma(i)$, is called a 321-*avoiding permutation of length n* if there are no integers $i < j < k$ such that $\sigma_i > \sigma_j > \sigma_k$ (i.e.,

E-mail addresses: chengszu@nuk.edu.tw (S.-E. Cheng), speu@nuk.edu.tw (S.-P. Eu), tsfu@npic.edu.tw (T.-S. Fu).

0195-6698/$ - see front matter © 2006 Elsevier Ltd. All rights reserved.
doi:10.1016/j.ejc.2006.01.006

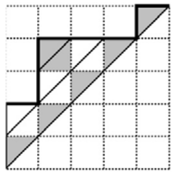

Fig. 1. A Catalan path of length 5.

every decreasing subsequence is of length at most two). Let $S_n(321)$ denote the set of 321-avoiding permutations of length n. A pair (σ_i, σ_j) is called an *inversion* of σ if $i < j$ and $\sigma_i > \sigma_j$. What catches our attention is that, as reported by Deutsch in [13, A008549], the number sequence $\{a_n\}_{n\geq 0} = \{0, 1, 6, 29, 130, 562, \ldots\}$ counts the number of inversions of all 321-avoiding permutations of length $n+1$. The main purpose of this paper is to establish a bijection Π_n between the set of triangles under all Catalan paths of length n and the set of inversions of all 321-avoiding permutations of length $n+1$. The bijection is composed of two major stages (see Theorems 1.1 and 1.2).

To resolve this problem, we color the unit squares in the plane $\mathbb{Z} \times \mathbb{Z}$ in black and white like a checkerboard. A unit square B is colored black if the upper left corner (i, j) of B satisfies the condition that $i + j$ is odd, and white otherwise. For example, there are 1 black square and 3 white squares under the path shown in Fig. 1. An intriguing observation is that the number of white squares under all Catalan paths of length $n+1$ is also equal to a_n (see Theorem 2.1). As the first stage of Π_n, the following bijection is one of the major results in this paper.

Theorem 1.1. *There is a bijection between the set of triangles under all Catalan paths of length n and the set of white squares under all Catalan paths of length* $n+1$.

For the second stage of Π_n, we employ a variant of parallelogram polyominoes to establish the following bijection $\Psi_n : \mathcal{C}_n \to S_n(321)$, which is different from the one given by Billy et al. [2, p. 361].

Theorem 1.2. *There is a bijection Ψ_n between the set $\mathcal{C}_n$ of Catalan paths of length n and the set $S_n(321)$ of 321-avoiding permutations of length n such that there is a one-to-one correspondence between the white squares under a path $\pi \in \mathcal{C}_n$ and the inversions of $\Psi_n(\pi) \in S_n(321)$.*

We organize this paper as follows. Regarding the plane as a checkerboard, we enumerate the black and white squares under Catalan paths in Section 2. The proofs of Theorems 1.1 and 1.2 are given in Sections 3 and 4, respectively. Finally, some enumerative results for variants of parallelogram polyominoes are given in Section 5.

2. Area of Catalan paths on a checkerboard

In this section, we shall enumerate the black and white squares under all Catalan paths of length n by the method of generating functions. The generating function $C = C(z) = \sum_{n\geq 0} c_n z^n$ for Catalan numbers $\{c_n\}_{n\geq 0}$ satisfies the equation $C = 1 + zC^2$. Another useful fact is $[z^n]C^t = \frac{t}{2n+t}\binom{2n+t}{n}$, which is known as the *ballot number* [4, p. 21]. Let N and E denote a north step and an east step, respectively. A *block* of a Catalan path is a section of the form NμE, where N is a north step leaving the line $y = x$, E is the first east step returning to the line

$y = x$ afterward, and μ is a Catalan path of certain length (possibly empty). A *peak* (resp. *valley*) of a path is formed by a consecutive NE (resp. EN) pair.

Theorem 2.1. *For $n \geq 2$, the following results hold.*

(i) *The number of white squares under all Catalan paths of length n is $4^{n-1} - \binom{2n-1}{n-1}$.*

(ii) *The number of black squares under all Catalan paths of length n is $4^{n-1} - \binom{2n}{n-1}$.*

(iii) *The number of white squares under all elevated Catalan paths of length n is 4^{n-2}.*

Proof. Let $f_{n,k}$ (resp. $g_{n,k}$) denote the number of paths $\pi \in \mathcal{C}_n$ with k white squares (resp. black squares) under π. Define the generating functions $F(t,z) = \sum_{n,k\geq 0} f_{n,k}t^k z^n$, and $G(t,z) = \sum_{n,k\geq 0} g_{n,k}t^k z^n$. Taking the partial derivative with respect to t and then setting $t = 1$, we have $\left(\frac{\partial F(t,z)}{\partial t}\right)_{t=1} = \sum_{n\geq 0}\left(\sum_{k\geq 0} kf_{n,k}\right) z^n$ and $\left(\frac{\partial G(t,z)}{\partial t}\right)_{t=1} = \sum_{n\geq 0}\left(\sum_{k\geq 0} kg_{n,k}\right) z^n$, which are the generating functions for the numbers in (i) and (ii), respectively.

A non-trivial path $\pi \in \mathcal{C}_n$ has a factorization $\pi = \mathsf{N}\mu\mathsf{E}\nu$, where E is the first east step that returns to the line $y = x$, and μ and ν are Catalan paths of certain lengths (possibly empty). Since, in the elevated path $\mathsf{N}\mu\mathsf{E}$, the black squares under μ become white and vice versa, we observe that the number of white squares under the first block $\mathsf{N}\mu\mathsf{E}$ of π is equal to the sum of the number of black squares under μ and the length of μ. Moreover, the number of black squares under the first block $\mathsf{N}\mu\mathsf{E}$ of π is equal to the number of white squares under μ. Hence $F(t,z)$ and $G(t,z)$ satisfy the following equations:

$$\begin{cases} F(t,z) = 1 + zG(t,tz)F(t,z), \\ G(t,z) = 1 + zF(t,z)G(t,z). \end{cases} \tag{1}$$

Let $F' = \left(\frac{\partial F(t,z)}{\partial t}\right)_{t=1}$ and $G' = \left(\frac{\partial G(t,z)}{\partial t}\right)_{t=1}$. Taking the partial derivative with respect to t, setting $t = 1$, and taking into account that $F(1,z) = G(1,z) = C(z)$, we have

$$\begin{cases} F' = z((G' + C'z)C + F'C), \\ G' = z(F'C + G'C). \end{cases} \tag{2}$$

Since $C = 1 + zC^2$, $1 - zC = \frac{1}{C}$ and $C' = C^2 + 2zCC'$. Solving (2) with $C = \frac{1-\sqrt{1-4z}}{2z}$, we have

$$F' = \frac{z^2C'}{1-2zC} = \frac{1-2z-\sqrt{1-4z}}{2(1-4z)}, \quad \text{and} \quad G' = F' - z^2CC' = F' - \frac{z}{2}(C' - C^2).$$

It follows that

$$[z^n]F' = \frac{1}{2}[z^n]\frac{1}{1-4z} - [z^{n-1}]\frac{1}{1-4z} - \frac{1}{2}[z^n]\frac{1}{\sqrt{1-4z}} = 4^{n-1} - \binom{2n-1}{n-1},$$

and

$$[z^n]G' = [z^n]F' - \frac{1}{2}[z^{n-1}]C' + \frac{1}{2}[z^{n-1}]C^2 = 4^{n-1} - \binom{2n}{n-1}.$$

Hence (i) and (ii) follow.

Let $h_{n,k}$ denote the number of elevated Catalan paths τ of length n with k white squares under τ, and let $H(t,z) = \sum_{n,k\geq 0} h_{n,k}t^k z^n$. We observe that $H(t,z)$ satisfies the equation $H(t,z) =$

$zG(t, tz)$. Let $H' = \left(\frac{\partial H(t,z)}{\partial t}\right)_{t=1}$. By the same method as above, we have $H' = z(G' + C'z)$. Hence $[z^n]H' = [z^{n-1}]G' + [z^{n-2}]C' = 4^{n-2}$, and (iii) follows. □

Similarly, the area of a Catalan path is partitioned into regions of the four types: white up-triangles, white down-triangles, black up-triangles, and black down-triangles. For example, the area of the path in Fig. 1 consists of 3 white up-triangles, 3 white down-triangles, 6 black up-triangles, and 1 black down-triangle. The following corollary is an immediate consequence of Theorem 2.1.

Corollary 2.2. *Among the area of all Catalan paths of length n, there are*

(i) $4^{n-1} - \binom{2n-1}{n-1}$ *white up-triangles,*
(ii) $4^{n-1} - \binom{2n-1}{n-1}$ *white down-triangles,*
(iii) 4^{n-1} *black up-triangles, and*
(iv) $4^{n-1} - \binom{2n}{n-1}$ *black down-triangles.*

Proof. It is clear that (i) and (ii) are equivalent to Theorem 2.1(i), and that (iv) is equivalent to Theorem 2.1(ii). Note that the number of black up-triangles under a path $\pi \in \mathcal{C}_n$ is equal to the number of white squares under the elevated path $\mathsf{N}\pi\mathsf{E} \in \mathcal{C}_{n+1}$. Hence (iii) follows from Theorem 2.1(iii). □

Remarks. In [1, p. 6], Barcucci et al. derived that the generating function for the number of inversions of all 321-avoiding permutations of length n is $\frac{1-2z-\sqrt{1-4z}}{2(1-4z)}$. Corollary 2.2(iii) has appeared in [15, Theorem A], which is obtained by making use of an enumerative result on parallelogram polyominoes in [11].

3. Proof of Theorem 1.1

Let $\mathcal{T}_n$ denote the set of ordered pairs (A, π), where $\pi \in \mathcal{C}_n$ and A is a triangle under π, and let $\mathcal{W}_{n+1}$ denote the set of ordered pairs (B, τ), where $\tau \in \mathcal{C}_{n+1}$ and B is a white square under τ. In this section, we shall establish a bijection $\Phi_n : \mathcal{T}_n \to \mathcal{W}_{n+1}$. Let $\mathcal{T}_n$ be partitioned into the following four subsets:

$$T_1(n) = \{(A, \pi) \in \mathcal{T}_n \mid A \text{ is a black up-triangle under } \pi\},$$
$$T_2(n) = \{(A, \pi) \in \mathcal{T}_n \mid A \text{ is a white up-triangle under } \pi\},$$
$$T_3(n) = \{(A, \pi) \in \mathcal{T}_n \mid A \text{ is a white down-triangle under } \pi\},$$
$$T_4(n) = \{(A, \pi) \in \mathcal{T}_n \mid A \text{ is a black down-triangle under } \pi\}.$$

For any $(A, \pi) \in T_1(n) \cup T_2(n)$ (i.e., A is an up-triangle), A is said to be at position (i, j) if the upper left corner of A is (i, j), and A is said to be on the line $L : x + y = i + j$. For each up-triangle A, the *top triangle* of A is the up-triangle $\widehat{A}$ to the northwest of A at the intersection of π and L.

On the other hand, for any $(B, \tau) \in \mathcal{W}_{n+1}$, B is said to be at position (i, j) if the upper left corner of B is (i, j), and B is said to be on the line $L : x + y = i + j$ (note that $i + j$ is even). For each white square B, the *top box* of B is the white square $\widehat{B}$ to the northwest of B at the intersection of τ and L. Moreover, we say that $\widehat{B}$ is *falling* if the top edge of $\widehat{B}$ coincides with an east step of τ, and *rising* otherwise. For any $(B, \tau) \in \mathcal{W}_{n+1}$, B is called a *downhill square* (resp.

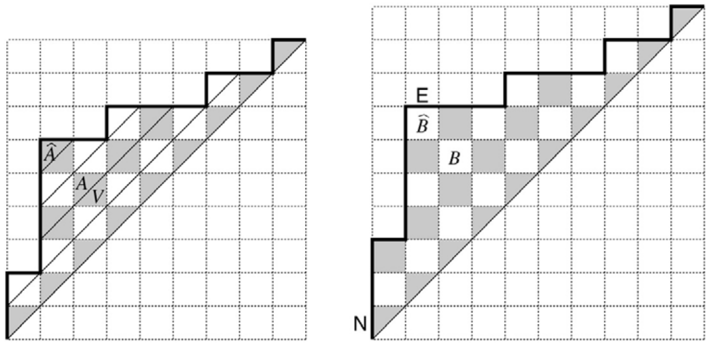

Fig. 2. A pair $(A, \pi) \in T_1(9)$ and the corresponding pair $\Phi_{9,1}((A, \pi)) = (B, \tau) \in W_1(10)$.

uphill square) of τ if the top box of B is falling (resp. rising). Let $\mathcal{W}_{n+1}$ be partitioned into the following four subsets:

$W_1(n+1) = \{(B, \tau) \in \mathcal{W}_{n+1} \mid B$ is a downhill square in the first block of $\tau\}$,
$W_2(n+1) = \{(B, \tau) \in \mathcal{W}_{n+1} \mid$ the first block β of τ is of length 1, i.e., $\beta = \mathsf{NE}\}$,
$W_3(n+1) = \{(B, \tau) \in \mathcal{W}_{n+1} \mid B$ is an uphill square in the first block of $\tau\}$,
$W_4(n+1) = \{(B, \tau) \in \mathcal{W}_{n+1} \mid$ the first block β of τ is of length > 1, and B is not in $\beta\}$.

For each i $(1 \le i \le 4)$, we shall establish a bijection $\Phi_{n,i} : T_i(n) \to W_i(n+1)$ (see Propositions 3.1–3.4). Then Φ_n is established by the refinement $\Phi_n|_{T_i(n)} = \Phi_{n,i}$, for $1 \le i \le 4$, and hence Theorem 1.1 is proved.

Proposition 3.1. *There is a bijection $\Phi_{n,1}$ between $T_1(n)$ and $W_1(n+1)$.*

Proof. Given a pair $(A, \pi) \in T_1(n)$, say A is at (i, j), we have $i + j = 2h - 1$, for some h $(h \ge 1)$. Let $\widehat{A}$ be the top triangle of A. We factorize π as $\pi = \mu\nu$, where μ goes from the origin to the upper left corner of $\widehat{A}$, and ν is the remaining part of π. Define a mapping $\Phi_{n,1}$ that carries (A, π) into $\Phi_{n,1}((A, \pi)) = (B, \tau)$, where $\tau = \mathsf{N}\mu\mathsf{E}\nu \in \mathcal{C}_{n+1}$ (i.e., with a north step N attached to the beginning and an east step E inserted between μ and ν) and B is the white square at $(i, j+1)$. Note that the top box $\widehat{B}$ of B is at the end point of μ, and that E is the top edge of $\widehat{B}$. Hence $\widehat{B}$ is a falling box and B is downhill. Hence $\Phi_{n,1}((A, \pi)) \in W_1(n+1)$.

To find $\Phi_{n,1}^{-1}$, given a pair $(B, \tau) \in W_1(n+1)$, say B is at (i, j), we have $i + j = 2h'$, for some h'. Since B is a downhill square, the top box $\widehat{B}$ of B is a falling box. We factorize τ as $\tau = \mathsf{N}\mu\mathsf{E}\nu$, where N is the first step of τ, E is the top edge of $\widehat{B}$, μ is the section between N and E, and ν is the remaining part of τ. Since B is in the first block of τ, μ remains above the line $y = x + 1$ and hence $\mu\nu \in \mathcal{C}_n$. Hence $\Phi_{n,1}^{-1}((B, \tau)) = (A, \pi) \in T_1(n)$, where $\pi = \mu\nu$ and A is the black up-triangle at $(i, j-1)$. $\square$

For example, on the left of Fig. 2 is a pair $(A, \pi) \in T_1(9)$, where A is at $(2, 5)$. The top triangle $\widehat{A}$ of A in π is at $(1, 6)$. Note that A is the second up-triangle on the line $x + y = 7$ from $\widehat{A}$. The corresponding pair $\Phi_{9,1}((A, \pi)) = (B, \tau) \in W_1(10)$ is shown on the right of Fig. 2, where B is at $(2, 6)$ and $\widehat{B}$ is at $(1, 7)$. Note that B is the second square on the line $x + y = 8$ from $\widehat{B}$.

Proposition 3.2. *There is a bijection $\Phi_{n,2}$ between $T_2(n)$ and $W_2(n+1)$.*

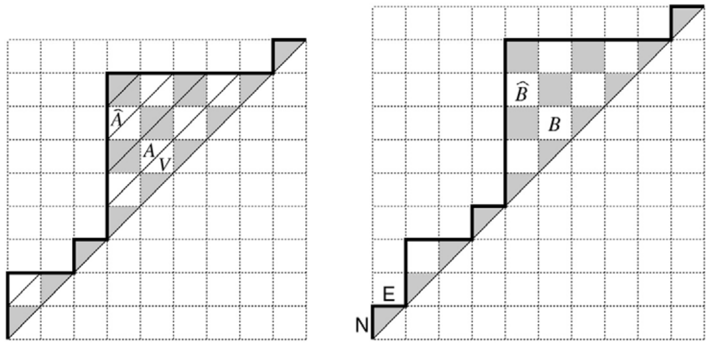

Fig. 3. A pair $(A, \pi) \in T_2(9)$ and the corresponding pair $\Phi_{9,2}((A, \pi)) = (B, \tau) \in W_2(10)$.

Proof. Given a pair $(A, \pi) \in T_2(n)$, say A is at (i, j), we have $i + j = 2h$, for some h ($h \geq 1$). Define a mapping $\Phi_{n,2} : T_2(n) \to W_2(n + 1)$ that carries (A, π) into $\Phi_{n,2}((A, \pi)) = (B, \tau) \in W_2(n + 1)$, where $\tau = \mathsf{NE}\pi \in \mathcal{C}_{n+1}$ and B is the white square at $(i + 1, j + 1)$. It is easy to find $\Phi^{-1}_{n,2}$ by a reverse process. □

For example, on the left of Fig. 3 is a pair $(A, \pi) \in T_2(9)$, where A is at $(4, 6)$. The corresponding pair $\Phi_{9,2}((A, \pi)) = (B, \tau) \in W_2(10)$ is shown on the right of Fig. 3, where B is at $(5, 7)$.

Proposition 3.3. *There is a bijection $\Phi_{n,3}$ between $T_3(n)$ and $W_3(n + 1)$.*

Proof. Given a pair $(V, \pi) \in T_3(n)$, say the lower right corner of V is (i, j), we have $i + j = 2h$, for some h ($h \geq 1$). Let A be the white up-triangle at $(i - 1, j + 1)$. Clearly, $(A, \pi) \in T_2(n)$. We shall use the mapping $\Phi_{n,2}$ given in Proposition 3.2 as an intermediate stage for establishing $\Phi_{n,3}$.

Let $\Phi_{n,2}((A, \pi)) = (B, \tau) \in W_2(n + 1)$. Then B is at $(i, j + 2)$. Let $\widehat{B}$ be the top box of B in τ, and let B be the kth square on the line $L : x + y = i + j + 2$ from $\widehat{B}$, for some k. We factorize τ as $\tau = \mathsf{NE}\mu\beta\nu$, where NE is the first block of τ, β is the block containing B, μ is the section between the first block and β, and ν is the remaining part of τ. Moreover, β is further factorized as $\beta = \alpha\gamma$, where α goes from the beginning of β to the upper left corner of $\widehat{B}$, and γ is the remaining part of β. Let p_α denote the end point of α. Define a mapping $\Phi_{n,3}$ that carries (V, π) into $\Phi_{n,3}((V, \pi)) = (C, \omega)$, where $\omega = \alpha\mathsf{N}\mu\mathsf{E}\gamma\nu$, $\widehat{C}$ is the top box at p_α in ω, and C is the kth square from $\widehat{C}$. Since α is followed by a north step, $\widehat{C}$ is a rising box and C is uphill. Moreover, C is in the first block $\alpha\mathsf{N}\mu\mathsf{E}\gamma$ of ω. Hence $\Phi_{n,3}((V, \pi)) \in W_3(n + 1)$.

To find $\Phi^{-1}_{n,3}$, given a pair $(C, \omega) \in W_3(n + 1)$, say C is at (i, j), we have $i + j = 2h'$, for some h'. Let $\widehat{C}$ be the top box of C in ω, say $\widehat{C}$ is at (i', j'), and let C be the k'th square on the line $x + y = 2h'$ from $\widehat{C}$. First, we factorize ω as $\omega = \beta\nu$, where β is the first block of ω, and ν is the remaining part of ω. Since C is an uphill square in β, $\widehat{C}$ is a rising box and β has a factorization $\beta = \alpha\mathsf{N}\mu\mathsf{E}\gamma$, where α goes from the origin to the upper left corner of $\widehat{C}$, E is the first step after $\widehat{C}$ that returns to the line $y = x + j' - i'$, and γ is the remaining part of β. Let p_α denote the end point of α. Locate the pair (B, τ), where $\tau = \mathsf{NE}\mu\alpha\gamma\nu$, $\widehat{B}$ is the top box at p_α in τ, and B is the k'th square from $\widehat{B}$. Since the first block of τ is of length 1, $(B, \tau) \in W_2(n + 1)$. Let $\Phi^{-1}_{n,2}((B, \tau)) = (A, \pi) \in T_2(n)$. Then we retrieve the required pair

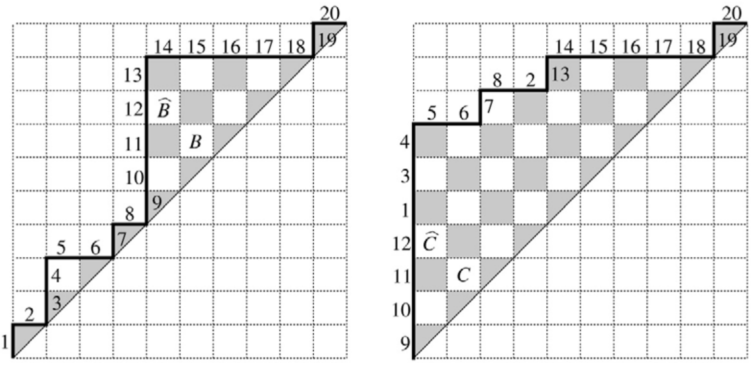

Fig. 4. The pairs $\Phi_{9,2}((A,\pi)) = (B,\tau) \in W_2(10)$ and $\Phi_{9,3}((V,\pi)) = (C,\omega) \in W_3(10)$ that are associated with the pairs $(A,\pi) \in T_2(9)$ and $(V,\pi) \in T_3(9)$ shown on the left of Fig. 3.

$\Phi_{n,3}^{-1}((C,\omega)) = (V,\pi) \in T_3(n)$ from (A,π), where V is the white down-triangle that shares an edge with A. □

For example, given the pair $(V,\pi) \in T_3(9)$ shown on the left of Fig. 3, where the lower right corner of V is $(5,5)$, let A be the white up-triangle at $(4,6)$. The intermediate pair $\Phi_{9,2}((A,\pi)) = (B,\tau)$ is shown on the left of Fig. 4. Factorize τ as $\tau = \mathsf{NE}\mu\beta\nu$, where $\mathsf{N} = 1$, $\mathsf{E} = 2$, $\mu = (3,\ldots,8)$, $\beta = (9,\ldots,18)$, and $\nu = (19,20)$. Moreover, β is further factorized as $\beta = \alpha\gamma$, where $\alpha = (9,10,11,12)$ and $\gamma = (13,\ldots,18)$. The corresponding pair $\Phi_{9,3}((V,\pi)) = (C,\omega) \in W_3(10)$ is shown on the right of Fig. 4, where $\omega = \alpha\mathsf{N}\mu\mathsf{E}\gamma\nu$, and C is at $(1,3)$.

Proposition 3.4. *There is a bijection $\Phi_{n,4}$ between $T_4(n)$ and $W_4(n+1)$.*

Proof. Given a pair $(V,\pi) \in T_4(n)$, say the lower right corner of V is (i,j), we have $i + j = 2h + 1$, for some h ($h \geq 1$). Let A be the up-triangle at $(i-1, j+1)$. Clearly, $(A,\pi) \in T_1(n)$. We shall use the mapping $\Phi_{n,1}$ given in Proposition 3.1 as an intermediate stage for establishing $\Phi_{n,4}$. Let $\Phi_{n,1}((A,\pi)) = (B,\tau) \in W_1(n+1)$. Then B is at $(i-1, j+2)$. Let $\widehat{B}$ be the top box of B in τ, and let B be the kth square on the line $L : x + y = i + j + 1$ from $\widehat{B}$, for some k. Since B is at $(i-1, j+2)$ and $j > i$, B is above the line $y = x + 2$. First, we factorize τ as $\tau = \beta\nu$, where β is the first block of τ and ν is the remaining part of τ. Next, β is further factorized as $\beta = \mathsf{NN}\mu_1\mu_2$, where μ_1 goes from $(0,2)$ to the first step after $\widehat{B}$ that returns to the line $L_2 : y = x + 2$, and μ_2 is the remaining part of β. Form a new path $\beta' = \mathsf{NN}\mu_2\mu_1$ from β by switching μ_1 and μ_2. Note that $\mathsf{NN}\mu_2$ is the first block of β', and that B is in μ_1. Moreover, the section μ_1 of β' might have a valley on the line $L_1 : y = x - 1$ (in front of $\widehat{B}$). There are two cases.

Case I. μ_1 has no valley on the line L_1. We define a mapping $\Phi_{n,4}$ that carries (V,π) into $\Phi_{n,4}((V,\pi)) = (C,\omega)$, where $\omega = \beta'\nu = \mathsf{NN}\mu_2\mu_1\nu$, and C is the white square B in μ_1. Since the first block $\mathsf{NN}\mu_2$ is of length at least 2, $\Phi_{n,4}((V,\pi)) \in W_4(n+1)$. It is worth mentioning that C is a downhill square since B is downhill in μ_1.

Case II. μ_1 has at least one valley on the line L_1. Then we factorize μ_1 as $\mu_1 = \lambda\mathsf{EN}\alpha\gamma$, where EN is the last valley on the line L_1, α goes from the end point of N to the upper left corner of $\widehat{B}$, and γ is the remaining part of μ_1. Let p_α be the end point of α. The mapping $\Phi_{n,4}$ is then defined by carrying (V,π) into $\Phi_{n,4}((V,\pi)) = (C,\omega)$, where $\omega = \mathsf{NN}\mu_2\alpha\mathsf{N}\lambda\mathsf{E}\gamma\nu$, $\widehat{C}$ is the top box at

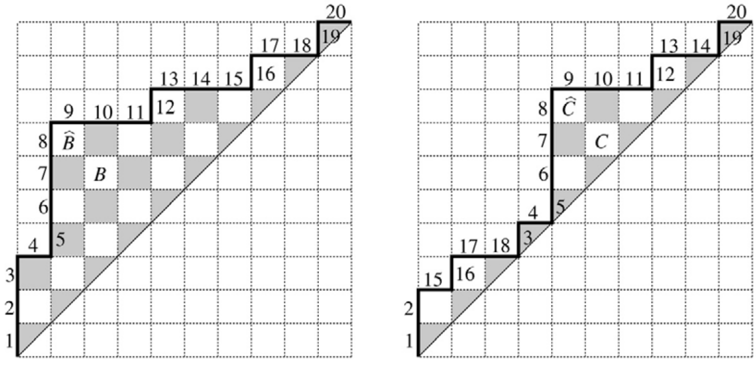

Fig. 5. The pairs $\Phi_{9,1}((A, \pi)) = (B, \tau) \in W_1(10)$ and $\Phi_{9,4}((V, \pi)) = (C, \omega) \in W_4(10)$ that are associated with the pairs $(A, \pi) \in T_1(9)$ and $(V, \pi) \in T_4(9)$ shown on the left of Fig. 2.

p_α in ω, and C is the kth square from $\widehat{C}$. Since the first block $\mathsf{NN}\mu_2$ of ω is of length at least 2 and since C is not in the first block, $\Phi_{n,4}((V, \pi)) \in W_4(n+1)$. Note that, since α is followed by a north step, $\widehat{C}$ is a rising box and C is uphill.

To find $\Phi_{n,4}^{-1}$, given a pair $(C, \omega) \in W_4(n+1)$, say C is at (i, j), for some $i \geq 2$, $j \geq 4$, first, we factorize ω as $\omega = \mathsf{NN}\mu_2\beta\nu$, where $\mathsf{NN}\mu_2$ is the first block of ω, β is the section that ends with the block containing C, and ν is the remaining part of ω. There are two cases.

Case i. C is a downhill square. We locate the pair (B, τ), where $\tau = \mathsf{NN}\beta\mu_2\nu$, and B is the square C in β. We observe that B is a downhill square in the first block $\mathsf{NN}\beta\mu_2$ of ω. Hence $(B, \tau) \in W_1(n+1)$.

Case ii. C is an uphill square. The top box $\widehat{C}$ of C in β is a rising box, say $\widehat{C}$ is at (i', j'). Let C be the k'th square on the line $x + y = i + j$ from $\widehat{C}$. We further factorize β as $\beta = \alpha\mu_1\mathsf{E}\gamma$, where α goes from the beginning of β to the upper left corner of $\widehat{C}$, E is the first east step that goes from the line $y = x + j' - i'$ to the line $y = x + j' - i' - 1$, and γ is the remaining part of β. Let p_α denote the end point of α. Since $\widehat{C}$ is a rising box, μ_1 starts with a north step. Factorize μ_1 as $\mu_1 = \mathsf{N}\lambda\mathsf{E}$, and let $\mu_1' = \lambda\mathsf{EN}$. We locate the pair (B, τ), where $\tau = \mathsf{NN}\mu_1'\alpha\mathsf{E}\gamma\mu_2\nu$, $\widehat{B}$ is the top box at p_α in τ, and B is the k'th square from $\widehat{B}$. Since α is followed by an east step, $\widehat{B}$ is a falling box and B is a downhill square in the first block $\mathsf{NN}\mu_1'\alpha\mathsf{E}\gamma\mu_2$ of τ. Hence $(B, \tau) \in W_1(n+1)$.

For both cases, let $\Phi_{n,1}^{-1}((B, \tau)) = (A, \pi) \in T_1(n)$. Then we retrieve the required pair $\Phi_{n,4}^{-1}((C, \omega)) = (V, \pi) \in T_4(n)$ from (A, π), where V is the black down-triangle that shares an edge with A. $\square$

For example, given the pair $(V, \pi) \in T_4(9)$ shown on the left of Fig. 2, where the lower right corner of V is $(3, 4)$, let A be the up-triangle at $(2, 5)$. The intermediate pair $\Phi_{9,1}((A, \pi)) = (B, \tau) \in W_1(10)$ is shown on the left of Fig. 5. First, factorize $\tau = \beta\nu$, where $\beta = (1, \ldots, 18)$ and $\nu = (19, 20)$. Next, β is further factorized as $\beta = \mathsf{N}_1\mathsf{N}_2\mu_1\mu_2$, where $\mathsf{N}_1 = 1$, $\mathsf{N}_2 = 2$, $\mu_1 = (3, \ldots, 14)$ and $\mu_2 = (15, 16, 17, 18)$. Let $\beta' = \mathsf{N}_1\mathsf{N}_2\mu_2\mu_1$. On the right of Fig. 5 is the path $\beta'\nu$. We observe that $\mathsf{N}_1\mathsf{N}_2\mu_2$ is the first block of β', and that μ_1 has no valley on the line $L_1 : y = x - 1$. Hence we have the corresponding pair $\Phi_{9,4}((V, \pi)) = (C, \omega) \in W_4(10)$, where $\omega = \beta'\nu = \mathsf{N}_1\mathsf{N}_2\mu_2\mu_1\nu$ and C is at $(5, 7)$.

For the latter case, consider the pair $(V, \pi) \in T_4(11)$ shown on the left of Fig. 6, where the lower right corner of V is $(7, 8)$. Let A be the up-triangle at $(6, 9)$. The intermediate pair

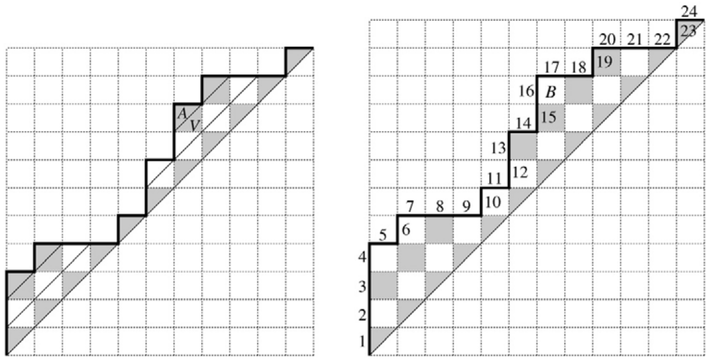

Fig. 6. A pair $(V, \pi) \in T_4(11)$ and the corresponding pair $\Phi_{11,1}((A, \pi)) = (B, \tau) \in W_1(12)$.

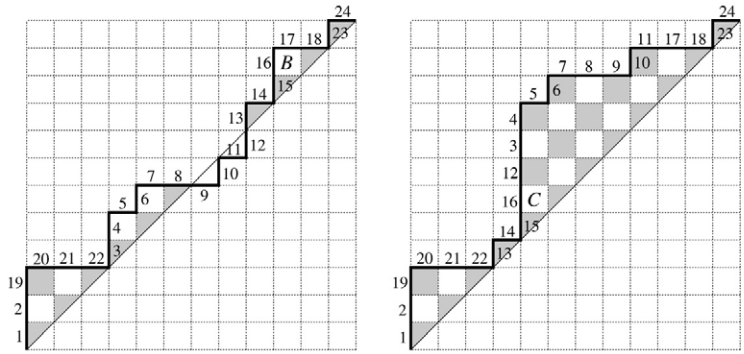

Fig. 7. The intermediate path $\beta'\nu$ and the corresponding pair $\Phi_{11,4}((V, \pi)) = (C, \omega) \in W_4(12)$.

$\Phi_{11,1}((A, \pi)) = (B, \tau) \in W_1(12)$ is shown on the right of Fig. 6. First, τ is factorized as $\tau = \beta\nu$, where $\beta = (1, \ldots, 22)$ and $\nu = (23, 24)$. Next, β is factorized as $\beta = \mathsf{N}_1\mathsf{N}_2\mu_1\mu_2$, where $\mu_1 = (3, \ldots, 18)$ and $\mu_2 = (19, 20, 21, 22)$. Let $\beta' = \mathsf{N}_1\mathsf{N}_2\mu_2\mu_1$. On the left of Fig. 7 is the path $\beta'\nu$. We observe that $\mathsf{N}_1\mathsf{N}_2\mu_2$ is the first block of β', and that μ_1 has two valleys on the line $L_1 : y = x - 1$. Hence μ_1 is further factorized as $\mu_1 = \lambda\mathsf{E}_3\mathsf{N}_3\alpha\gamma$, where $\mathsf{E}_3 = 11$ and $\mathsf{N}_3 = 12$ form the last valley on the line L_1 of μ_1, $\lambda = (3, \ldots, 10)$, $\alpha = (13, 14, 15, 16)$, and $\gamma = (17, 18)$. With N_3 moved in front of λ, we have $\mathsf{N}_3\lambda\mathsf{E}_3 = (12, 3, 4, \ldots, 11)$. The corresponding pair $\Phi_{11,4}((V, \pi)) = (C, \omega) \in W_4(12)$ is shown on the right of Fig. 7, where $\omega = \mathsf{N}_1\mathsf{N}_2\mu_2\alpha\mathsf{N}_3\lambda\mathsf{E}_3\gamma\nu$ and C is at $(4, 6)$.

4. Proof of Theorem 1.2

In this section, making use of a variant of parallelogram polyominoes, we shall prove Theorem 1.2 in two stages (see Propositions 4.1 and 4.3).

A *shortened polyomino* is formed by a pair (P, Q) of paths using north steps $(0, 1)$ and east steps $(1, 0)$ that start from the origin, end in a common point, and satisfy the following conditions

(H1) P never goes below Q, and

(H2) there are no north steps of P and Q overlapped.

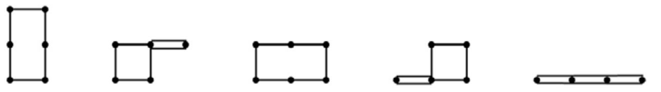

Fig. 8. The shortened polyominoes with perimeter 6.

The *perimeter* of a polyomino is twice the length of its paths, and its *area* is the number of unit squares enclosed. As another occurrence of Catalan numbers, it is known that the number of shortened polyominoes of perimeter $2n$ is c_n (see [7, Section 5]). The shortened polyominoes of perimeter 6 are shown in Fig. 8. Making use of an argument similar to the one in [15, Theorem A], we prove the following proposition. Here, the end point of a step is said to be at *level* h if it is on the line $y = x + h$, for some integer h.

Proposition 4.1. *There is a bijection Ω_n between the set $\mathcal{C}_n$ of Catalan paths of length n and the set $\mathcal{H}_n$ of shortened polyominoes of perimeter $2n$ such that there is a one-to-one correspondence between the white squares under a path $\omega \in \mathcal{C}_n$ and the squares in $\Omega_n(\omega) \in \mathcal{H}_n$.*

Proof. Given a path $\omega \in \mathcal{C}_n$, let P (resp. Q) be the path formed by the even steps (resp. odd steps) of ω, and let Q^* be the path obtained from Q by interchanging north steps and east steps. Define a mapping Ω_n by carrying ω into $\Omega_n(\omega) = (P, Q^*)$. Let $P = p_1 \cdots p_n$ and $Q^* = q_1 \cdots q_n$. Clearly, P and Q^* have the same number of north steps (as well as east steps), and P always remains above Q^* since the distance between the end points of p_i and q_i $(1 \le i \le n)$ is one half of the level of the end point of p_i in ω. Moreover, whenever two steps in (P, Q^*) overlap, they are east steps since their corresponding steps in ω form a peak at level 1. Hence $\Omega_n(\omega) \in \mathcal{H}_n$. To find Ω_n^{-1}, one simply reverses the procedure.

We observe that each white square under ω is on the line $x + y = 2h$, for some h $(1 \le h \le n - 1)$, and that the number of white squares under ω on the line $x + y = 2h$ is equal to the number of squares on the line $x + y = h$ in $\Omega_n(\omega)$. Hence there is a one-to-one correspondence between the set of white squares under ω and the set of squares in $\Omega_n(\omega)$ such that the kth square on the line $x + y = 2h$ from its top box under ω corresponds to the kth square on the line $x + y = h$ (from upper left to lower right) in $\Omega_n(\omega)$. □

We remark that the actual distance between the end points of p_i and q_i in (P, Q^*) has a factor $\sqrt{2}$, but we omit it.

For example, given the pair $(C, \omega) \in \mathcal{W}_{10}$ shown on the right of Fig. 5, the shortened polyomino $\Omega_{10}(\omega) = (P, Q^*)$ is shown on the left of Fig. 9, where $P =$ NNEENNENEE consists of the even steps of ω and $Q^* =$ ENNEEENNNE is obtained from the odd steps $Q =$ NEENNNEEEN of ω by interchanging north steps and east steps. The white square C under ω is carried into the square D in $\Omega_{10}(\omega)$.

Let us turn to the second half of the proof of Theorem 1.2. Let S_n be the set of permutations of $[n] := \{1, \ldots, n\}$. We write $\sigma = \sigma_1 \cdots \sigma_n \in S_n$, where $\sigma_i = \sigma(i)$. For a $\sigma \in S_n$, an *excedance* (resp. *weak excedance*) of σ is an integer $i \in [n - 1]$ such that $\sigma_i > i$ (resp. $\sigma_i \ge i$). Here the element σ_i is called an *excedance letter* (resp. *weak excedance letter*). *Non-weak excedances* and *non-weak excedance letters* are defined in the obvious way, in terms of i and σ_i, such that $\sigma_i < i$. Let $E(\sigma)$ be the set of excedances of σ, and let $\mathsf{inv}(\sigma)$ be the number of inversions of σ. The following characterization of 321-avoiding permutations was given by Simion [12, Lemma 5.6] (see also [10, Proposition 2.3]).

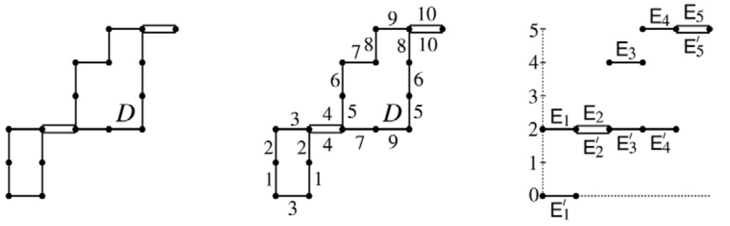

Fig. 9. The shortened polyomino $\Omega_{10}(\omega)$ associated with the path $\omega \in \mathcal{C}_{10}$ in Fig. 5, and its labeling.

Lemma 4.2. *A permutation σ is* 321*-avoiding if and only if*

$$\mathsf{inv}(\sigma) = \sum_{k \in E(\sigma)} (\sigma_k - k).$$

Proposition 4.3. *There is a bijection Υ_n between the set $\mathcal{H}_n$ of shortened polyominoes of perimeter $2n$ and the set $S_n(321)$ of* 321*-avoiding permutations of length n such that there is a one-to-one correspondence between the squares in a polyomino $(P, Q) \in \mathcal{H}_n$ and the inversions of $\Upsilon_n((P, Q)) \in S_n(321)$.*

Proof. Given a shortened polyomino $(P, Q) \in \mathcal{H}_n$, let $P = p_1 \cdots p_n$ and $Q = q_1 \cdots q_n$. Let the steps $p_1, \ldots, p_n$ of P be labeled from 1 to n. For each i $(1 \le i \le n)$, we assign the ith step q_i of Q the label z_i of the opposite step across the polyomino. The mapping Υ_n is defined by carrying (P, Q) into $\Upsilon_n((P, Q)) = z_1 \cdots z_n$. Since the labels of the north steps (resp. east steps) of Q are increasing, every decreasing subsequence of $\Upsilon_n((P, Q))$ is of length at most two. Hence $\Upsilon_n((P, Q)) \in S_n(321)$.

To find Υ_n^{-1}, we shall retrieve a shortened polyomino $\Upsilon_n^{-1}(\sigma)$ for any $\sigma = \sigma_1 \cdots \sigma_n \in S_n(321)$. Let $\{j_1, \ldots, j_t\}$ be the set of weak excedances of σ (i.e., $\sigma(j_i) \ge j_i$, for $1 \le i \le t$). For each i $(1 \le i \le t)$, put an east step E_i at height $y = \sigma(j_i) - i$ as the top of the ith column of $\Upsilon_n^{-1}(\sigma)$. The upper path of $\Upsilon_n^{-1}(\sigma)$ goes from $(0, 0)$ to the end point of E_t containing $\mathsf{E}_1, \ldots, \mathsf{E}_t$. On the other hand, for each i $(1 \le i \le t)$, put an east step E'_i at height $y = j_i - i$ as the bottom of the ith column of $\Upsilon_n^{-1}(\sigma)$. The lower path of $\Upsilon_n^{-1}(\sigma)$ goes from $(0, 0)$ to the end point of E_t containing $\mathsf{E}'_1, \ldots, \mathsf{E}'_t$. Since $\sigma(j_i) \ge j_i \ge i$ $(1 \le i \le t)$, $\Upsilon_n^{-1}(\sigma) \in \mathcal{H}_n$ is well defined.

Note that there are $\sigma(j_i) - j_i$ squares in the ith column of $\Upsilon_n^{-1}(\sigma)$, and that, by Lemma 4.2, $\mathsf{inv}(\sigma) = \sum_{i=1}^{t} (\sigma(j_i) - j_i)$. Hence the number of inversions of σ is equal to the number of squares in $\Upsilon_n^{-1}(\sigma)$. Moreover, the columns (resp. rows) of $\Upsilon_n^{-1}(\sigma)$ are labeled with weak excedance letters (resp. non-weak excedance letters) increasingly. Since each square D in $\Upsilon_n^{-1}(\sigma)$ is the intersection of the column with label σ_i and the row with label σ_j, for some excedance i and non-weak excedance j, there is one-to-one correspondence between the squares in $\Upsilon_n^{-1}(\sigma)$ and the inversions of σ such that D is carried into the inversion (σ_i, σ_j). □

For example, in Fig. 9, the labeling of the shortened polyomino (P, Q^*) on the left is shown in the center. The corresponding permutation $\sigma = \Upsilon_{10}((P, Q^*)) = 312479568a$ $(a = 10)$ can be obtained from the labeling of the lower path Q^*. Note that the square D in (P, Q^*) is carried into the inversion $(\sigma_6, \sigma_7) = (9, 5)$ of $\Upsilon_{10}((P, Q^*))$. To show $\Upsilon_{10}^{-1}(\sigma)$, note that the weak excedances of σ are $\{1, 4, 5, 6, 10\}$, i.e., $\sigma_1 = 3$, $\sigma_4 = 4$, $\sigma_5 = 7$, $\sigma_6 = 9$, and $\sigma_{10} = 10$. The east steps on the upper path and lower path of $\Upsilon_{10}^{-1}(\sigma)$ are shown on the right of Fig. 9.

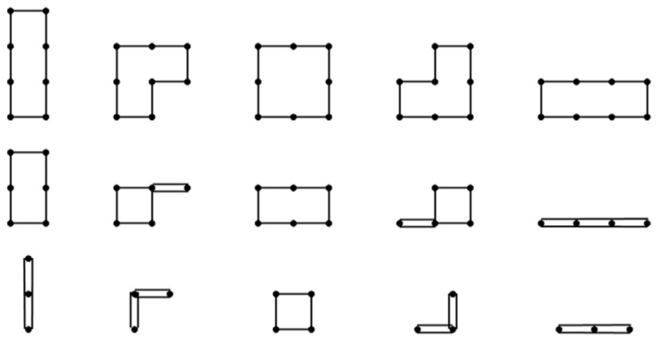

Fig. 10. The polyominoes of three kinds for the case $n = 3$.

By the composition $\Psi_n = \Upsilon_n \circ \Omega_n$, Theorem 1.2 is proved. Hence, by Theorems 1.1 and 1.2, we establish the required bijection between the area of all Catalan paths of length n and the inversions of all 321-avoiding permutations of length $n + 1$.

5. Some enumerative results for parallelogram polyominoes

In the previous section, we introduced a variant of parallelogram polyominoes, called shortened polyominoes. A *parallelogram polyomino* is a pair of non-intersecting paths that start from the origin and end in a common point. A *shrunk polyomino* is a pair of paths that starts from the origin and ends in a common point such that one path never goes below the other. In fact, a shortened polyomino of perimeter $2n$ can be obtained from a parallelogram polyomino (P, Q) of perimeter $2n + 2$ by deleting the initial (north) step of the upper path P and deleting the final (north) step of the lower path Q. Moreover, a shrunk polyomino of perimeter $2n - 2$ can be obtained from a shortened polyomino (P', Q') of perimeter $2n$ by further deleting the final (east) step of the upper path P' and deleting the first (east) step of the lower path Q'. Fig. 10 shows polyominoes of the three types for the case of $n = 3$. Refer also to [14, Exercise 6.19(l)(m)].

A bijection Ω'_n between Catalan paths of length n and parallelogram polyominoes of perimeter $2n + 2$ can be obtained from the bijection Ω_n in Proposition 4.1 as follows. Given a path $\omega \in \mathcal{C}_n$, let $(P, Q^*) = \Omega_n(\omega) \in \mathcal{H}_n$ be the corresponding shortened polyomino. The bijection Ω'_n is defined by $\Omega'_n(\omega) = (\mathsf{N}P, Q^*\mathsf{N})$, which is obtained from $\Omega_n(\omega)$ with a north step attached to the beginning of the upper path and a north step attached to the end of the lower path. We remark that this bijection is different from the one given by Delest and Viennot in [5, Section 4] and the one given by Reifegerste in [10, Theorem 3.10]. The following proposition is also an immediate consequence of the bijection Ω_n.

Proposition 5.1. *There is a bijection Θ_n between the set $\mathcal{C}_n$ of Catalan paths of length n and the set $\mathcal{R}_n$ of shrunk polyominoes of perimeter $2n-2$ such that there is a one-to-one correspondence between the black squares under a path $\pi \in \mathcal{C}_n$ and the squares in $\Theta_n(\pi) \in \mathcal{R}_n$.*

Proof. Given a path $\pi \in \mathcal{C}_n$, consider the shortened polyomino $\Omega_n(\pi) = (P, Q^*)$ under the mapping Ω_n in Proposition 4.1. Let $P = p_1 \cdots p_n$ and $Q^* = q_1 \cdots q_n$. There is an immediate bijection $\Theta_n : \mathcal{C}_n \to \mathcal{R}_n$ that carries π into $\Theta_n(\pi) = (P', Q^{*\prime}) \in \mathcal{R}_n$, where $P' = p_1 \cdots p_{n-1}$ and $Q^{*\prime} = q_2 \cdots q_n$. Moreover, the number of black squares under π on the line $x + y = 2h + 1$, $(1 \le h \le n - 2)$ is equal to the distance between the end points of p_h and q_{h+1} in $(P', Q^{*\prime})$.

Hence there is a one-to-one correspondence between the black squares under π and the squares in $\Theta_n(\pi)$. $\square$

The following bijective result can be obtained by the same argument as in the proof of Proposition 4.1, which appeared implicitly in [15, Theorem A].

Proposition 5.2. *There is a bijection Λ_n between the set $\mathcal{E}_n$ of elevated Catalan paths of length $n+1$ and the set $\mathcal{P}_n$ of parallelogram polyominoes of perimeter $2n+2$ such that there is a one-to-one correspondence between the white squares under a path $\pi \in \mathcal{E}_n$ and the squares in $\Lambda_n(\pi) \in \mathcal{P}_n$.*

By Theorem 2.1 and Propositions 4.1, 5.1 and 5.2, we deduce the enumerative results on the areas of the various polyominoes.

Theorem 5.3. *For $n \geq 2$, the following results hold.*

(i) *The area of all shortened polyominoes of perimeter $2n$ is $4^{n-1} - \binom{2n-1}{n-1}$.*
(ii) *The area of all shrunk polyominoes of perimeter $2n-2$ is $4^{n-1} - \binom{2n}{n-1}$.*
(iii) *The area of all parallelogram polyominoes of perimeter $2n+2$ is 4^{n-1}.*

A *2-Motzkin path of length n* is a lattice path from $(0, 0)$ to $(n, 0)$ that never goes below the x-axis, using up steps $(1, 1)$, down steps $(1, -1)$, and level steps $(1, 0)$, where the level steps can be of either of two kinds: straight and wavy. The *area* of a 2-Motzkin path is defined to be the sum of the heights of the end points of all steps. By a simple substitution, there is a bijection between the set $\mathcal{M}_n$ of 2-Motzkin paths of length n and the set $\mathcal{R}_{n+1}$ of shrunk polyominoes of perimeter $2n$. Given a $\tau \in \mathcal{M}_n$, for each i $(1 \leq i \leq n)$, we associate the ith step t_i of τ with a pair (p_i, q_i) of steps, where

$$(p_i, q_i) = \begin{cases} (\mathsf{N}, \mathsf{E}) & \text{if } t_i \text{ is an up step} \\ (\mathsf{E}, \mathsf{N}) & \text{if } t_i \text{ is a down step} \\ (\mathsf{N}, \mathsf{N}) & \text{if } t_i \text{ is a straight level step} \\ (\mathsf{E}, \mathsf{E}) & \text{if } t_i \text{ is a wavy level step.} \end{cases}$$

The corresponding shrunk polyomino of τ is the pair (P, Q) of paths, where $P = p_1 \cdots p_n$ and $Q = q_1 \cdots q_n$. It is straightforward to verify that the height of the end point of t_i in τ is equal to the distance between p_i and q_i in (P, Q). By Theorem 5.3(ii), we have the following result.

Corollary 5.4. *The area of all 2-Motzkin paths of length n is $4^n - \binom{2n+2}{n}$.*

Acknowledgments

The authors would like to thank Professor Louis Shapiro for useful suggestions during his visit to Academia Sinica, Taiwan, and thank the referees for helpful remarks. The authors are partially supported by the National Science Council, Taiwan, under grants NSC 93-2115-M-390-006 (S.-E. Cheng), NSC 94-2115-M-390-005 (S.-P. Eu), and NSC 94-2115-M-251-001 (T.-S. Fu).

References

[1] E. Barcucci, A. Del Lungo, E. Pergola, R. Pinzani, Some permutations with forbidden subsequences and their inversion numbers, Discrete Math. 234 (2001) 1–15.

[2] S.C. Billey, W. Jockusch, R.P. Stanley, Some combinatorial properties of Schubert polynomials, J. Algebraic Combin. 2 (1993) 345–374.
[3] E. Chapman, Moments of Dyck paths, Discrete Math. 204 (1999) 113–117.
[4] L. Comtet, Advanced Combinatorics, Reidel, Dordrecht, Boston, 1974.
[5] M.P. Delest, X.G. Viennot, Algebraic languages and polyomino enumeration, Theoret. Comput. Sci. 34 (1984) 169–206.
[6] E. Deutsch, Dyck paths enumeration, Discrete Math. 204 (1999) 167–202.
[7] E. Deutsch, L.W. Shapiro, A survey of the Fine numbers, Discrete Math. 241 (2001) 241–265.
[8] M. Merlini, R. Sprugnoli, M.C. Verri, The area determined by under-diagonal lattice path, in: Proc. CAAP, in: Lecture Notes in Computer Science, vol. 1095, 1996, pp. 59–71.
[9] E. Pergola, Two bijections for the area of Dyck paths, Discrete Math. 241 (2001) 435–447.
[10] A. Reifegerste, Excedances and descents of bi-increasing permutations, arxiv:math.CO/0212247 (preprint).
[11] L.W. Shapiro, D. Rogers, W.-J. Woan, The Catalan numbers, the Lebesgue integral, and 4^{n-2}, Amer. Math. Monthly 104 (1997) 926–931.
[12] R. Simion, Combinatorial statistics on non-crossing partitions, J. Combin. Theory Ser. A 66 (1994) 270–301.
[13] N.J.A. Sloane, The On-Line Encyclopedia of Integer Sequences, published electronically at http://www.research.att.com/~njas/sequences/.
[14] R.P. Stanley, Enumerative Combinatorics, vol. 2, Cambridge University Press, Cambridge, 1999.
[15] W.-J. Woan, Area of Catalan paths, Discrete Math. 226 (2001) 439–444.