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Simple proofs of open problems about the structure of involutions in the Riordan group ☆

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Abstract

We prove that if $D = (g(x), f(x))$ is an element of order 2 in the Riordan group then $g(x) = \pm\exp[\Phi(x, xf(x))]$ for some antisymmetric function $\Phi(x, z)$. Also we prove that every element of order 2 in the Riordan group can be written as BMB^{-1} for some element B and $M = (1, -1)$ in the Riordan group. These proofs provide solutions to two open problems presented by Shapiro [L.W. Shapiro, Some open questions about random walks, involutions, limiting distributions and generating functions, Adv. Appl. Math. 27 (2001) 585–596].

AMS classification: 05A15

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1. Introduction

We begin with a brief description of the Riordan group developed by Shapiro et al. [7] in 1991. A *Riordan matrix* [7,9] is an infinite lower triangular array $D = \{d_{n,k}\}_{n\geqslant k\geqslant 0}$ generated by a pair of analytic functions or generating functions $g(x) = \sum_{n\geqslant 0} g_n x^n$ and $f(x) = \sum_{n\geqslant 0} f_n x^n$ with $g(0) \neq 0$ and $f(0) \neq 0$ such that

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$$d_{n,k} = [x^n]g(x)(xf(x))^k,$$

where the notation $[x^n]$ stands for the 'coefficient of' operator. We often denote a Riordan matrix by $D = (g(x), f(x))$.

A well known example of a Riordan matrix is the Pascal matrix:

$$P = \left(\frac{1}{1-x}, \frac{1}{1-x}\right) = \left[\binom{n}{k}\right]_{n,k\geqslant 0}.$$

The set of all Riordan matrices forms a group denoted $(\mathscr{R}, *)$ with the operation being matrix multiplication $*$. In terms of the generating functions this works out as

$$(g(x), f(x)) * (h(x), \ell(x)) = (g(x)h(xf(x)), f(x)\ell(xf(x))). \quad (1)$$

We call $\mathscr{R}$ the *Riordan group*. It is easy to see that $I = (1, 1)$ is the identity element of $\mathscr{R}$ and the inverse of $(g(x), f(x)) \in \mathscr{R}$ is $(\bar{g}(x), \bar{f}(x))$ where $\bar{g}(y) = \{1/g(x)|y = xf(x)\}$ and $\bar{f}(y) = \{1/f(x)|y = xf(x)\}$.

As noted in [6], for the case of the Riordan group any element with integer entries having finite order must have order 1 or 2. In this paper, we are interested in the structure of elements of order 2 in the Riordan group $\mathscr{R}$. We call an element of order 2 in $\mathscr{R}$ a *Riordan involution*.

In combinatorial situations, a Riordan matrix will often have all nonnegative entries on and below the main diagonal and hence it cannot itself have order 2. We define an element D in the Riordan group to have *pseudo-order 2* if DM has order 2 where

$$M = (1, -1) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ & & \cdots & \end{bmatrix}. \quad (2)$$

An element of pseudo-order 2 in the Riordan group will be called a *pseudo-Riordan involution* or briefly a *pseudo-involution* [1,6]. It is obvious that the main diagonal entries of pseudo-involutions are equal to 1. The Pascal matrix is an example of pseudo-involution.

Clearly, BMB^{-1} is a Riordan involution for any element B in the Riordan group. How about the converse? In 2001, Shapiro [6] presented some open questions (Q8, Q8.1, Q9.1) concerning involutions in the Riordan group:

Q8: Can every Riordan involution be written as BMB^{-1} for some element B in the Riordan group?

Q8.1: If the Riordan involution D has a combinatorial significance, can we find a B such that $D = BMB^{-1}$ and B has a related combinatorial significance?

If $D = (g(x), f(x))$ is a Riordan involution then it might be natural to ask whether there is a relationship between $g(x)$ and $f(x)$. In fact, it was briefly conjectured (Q9, [6]) that if $D = (g(x), f(x))$ is Riordan involution then $g(x) = (f(x))^m$ for some constant m. However, it is known [6] that there are several counterexamples. The question now for involutions in the Riordan group is:

Q9.1: If $D = (g(x), f(x))$ is Riordan involution, is there a simple condition for $g(x)$ in terms of $f(x)$?

In [1], Cameron and Nkwanta studied classes of combinatorial matrices having pseudo-order 2 in the Riordan group and obtained some partial results on the problem Q8.

In this paper, we will focus our attention on the open questions, Q8, Q8.1, Q9.1 for Riordan involutions. More specifically, in Section 2, we give simple proofs for two questions Q8 and Q9.1.

In Section 3, we explore the question Q8.1 and we obtain a class of Riordan matrices for which Q8.1 is affirmative.

2. Simple proofs for two open questions

In this section, we address answers for the questions Q8 and Q9.1. Given a Riordan involution $D = (g(x), f(x))$, if we know the relationship between $g(x)$ and $f(x)$ it might make question Q8 more tractable. So, first we consider the question Q9.1.

We begin with the following lemma.

Lemma 2.1. *If $D = (g(x), f(x))$ is a Riordan involution then $g(x)$ is the solution of the functional equation*:

$$g(x)g(xf(x)) = 1. \tag{3}$$

Conversely, if $g(x)$ satisfies (3) for any analytic function $f(x)$ such that $f(x)f(xf(x)) = 1$, then $D = (g(x), f(x))$ is a Riordan involution.

Proof. A Riordan matrix $D = (g(x), f(x))$ is involution if and only if

$$(g(x), f(x))^2 = (g(x)g(xf(x)), f(x)f(xf(x))) = (1, 1).$$

Equivalently, we have

$$g(x)g(xf(x)) = 1 \quad \text{and} \quad f(x)f(xf(x)) = 1, \tag{4}$$

which completes the proof. □

In order to give an answer for the question Q9.1, it suffices to solve the functional equation with quadratic nonlinearity given in (3). The following lemma is easy to apply and very useful.

Lemma 2.2 [3]. *Let $y(x)y(\omega(x)) = b^2$ be a nonlinear functional equation in one variable where $\omega(\omega(x)) = x$. Then the solutions are*

$$y(x) = \pm b \exp[\Phi(x, \omega(x))], \tag{5}$$

where $\Phi(x, z) = -\Phi(z, x)$ is a suitable antisymmetric function of two arguments.

Now we are ready to solve the question Q9.1.

Theorem 2.3 (Q9.1). *If $D = (g(x), f(x))$ is a Riordan involution, then*

$$g(x) = \pm\exp[\Phi(x, xf(x)] \tag{6}$$

for some antisymmetric function $\Phi(x, z)$. Conversely, if $f(x)$ is an analytic function such that $f(x)f(xf(x)) = 1$ and $g(x)$ satisfies (6) for any antisymmetric function $\Phi(x, z)$, then $D = (g(x), f(x))$ is a Riordan involution.

Proof. Let $D = (g(x), f(x))$ be a Riordan involution. By setting $\omega(x) := xf(x)$, we see that $f(x)f(xf(x)) = 1$ if and only if $\omega(\omega(x)) = x$. Hence by Lemma 2.2, there exists an antisymmetric function $\Phi(x, z)$ such that the solution of $g(x)g(xf(x)) = 1$ is $g(x) = \pm\exp[\Phi(x, xf(x)]$.

The converse is an immediate consequence of Lemmas 2.1 and 2.2. Hence the proof is complete. □

The second statement of Theorem 2.3 tells us that given an analytic function $f(x)$ such that $f(x)f(xf(x)) = 1$, we can obtain infinitely many Riordan involutions $(g(x), f(x))$ where $g(x)$ is obtained from any antisymmetric function $\Phi(x, z)$ via (6).

A few of particular solutions of the functional equation $\omega(\omega(x)) = x$ (called the *Babbage equation* in [4]) are

$$\omega_1(x) = x, \quad \omega_2(x) = C - x, \quad \omega_3(x) = \frac{C}{x}, \quad \omega_4(x) = \frac{C_1 - x}{1 + C_2 x},$$

where C, C_1 and C_2 are arbitrary constants. Since we want $f(x) = \omega(x)/x$ to be analytic, we pick constants carefully and end up with

$$f(x) = \pm 1 \quad \text{or} \quad f(x) = \frac{-1}{1 + Cx},$$

where C is arbitrary constant. Examples of antisymmetric function $\Phi(x, z)$ are

$$\ln\left|\frac{z}{x}\right|, \quad \ln\left|\frac{x}{z}\right|, \quad \text{and} \quad \rho(x - z), \quad \text{etc.},$$

where $\rho(x)$ is an odd function.

For example, let $f(x) = \frac{-1}{1-x}$. Taking $\Phi(x, z) = \ln\left|\frac{z}{x}\right|$, $\Phi(x, z) = \ln\left|\frac{x}{z}\right|$, and $\Phi(x, z) = x - z$, respectively, we obtain Riordan involutions, respectively:

$$\left(\frac{1}{1-x}, \frac{-1}{1-x}\right), \quad \left(1 - x, \frac{-1}{1-x}\right), \quad \text{and} \quad \left(\mathrm{e}^{\frac{x(2-x)}{1-x}}, \frac{-1}{1-x}\right).$$

Here is a list of some important subgroups [5] of the Riordan group considered in this paper.

1. the *Appell subgroup* $\mathscr{A} = \{(g(x), 1)\}$,
2. the *Bell subgroup* $\mathscr{B} = \{(g(x), g(x))\}$,
3. the *checkerboard subgroup* $\mathscr{C} = \{(g(x), f(x)) | g, f$ both even functions$\}$.

Corollary 2.4. *If* $D = (g(x), 1)$ *is a pseudo-involution in the Appell subgroup then* $g(x) = \pm \mathrm{e}^{\rho(x)}$ *for some odd function* $\rho(x)$. *Conversely, for any odd function* $\rho(x)$, $D = (\mathrm{e}^{\rho(x)}, 1)$ *is a pseudo-involution.*

Proof. Let $D = (g(x), 1)$ be a pseudo-involution. Then $(g(x), -1)$ is the corresponding Riordan involution. By Theorem 2.3, we have $g(x) = \pm\exp[\Phi(x, -x)]$ for some antisymmetric function $\Phi(x, z)$. Let $\rho(x) = \Phi(x, -x)$. Since $\rho(-x) = \Phi(-x, x) = -\Phi(x, -x) = -\rho(x)$, we see that $\rho(x)$ is odd function. Thus $g(x)$ has the form $\pm\mathrm{e}^{\rho(x)}$ for some odd function $\rho(x)$. The converse is obvious. Hence the proof is complete. □

Now, we solve the question Q8 with the same antisymmetric function satisfying (6).

Theorem 2.5 (Q8). *Let* $D = (g(x), f(x))$ *be a Riordan involution. Then there exists a Riordan matrix* B *such that* $D = BMB^{-1}$, *where*

$$B = \left(\exp\left[\frac{\Phi(x, xf(x))}{2}\right], \frac{\Phi(x, xf(x))}{x}\right) \tag{7}$$

for the same antisymmetric function $\Phi(x, z)$ *given by* (6).

Proof. Let $D = (g(x), f(x))$ be a Riordan involution. By Theorem 2.3, there exists some antisymmetric function $\Phi(x, z)$ such that $g(x) = \pm\exp[\Phi(x, xf(x))]$. Since $(-g(x), f(x)) = -(g(x), f(x))$ for some Riordan matrix B we have

$$(g(x), f(x)) = BMB^{-1} \quad \text{if and only if } (-g(x), f(x)) = B(-M)B^{-1}. \tag{8}$$

Hence we may assume that

$$g(x) = \exp[\Phi(x, xf(x))]. \tag{9}$$

With the same antisymmetric function $\Phi(x, z)$ in (9), let

$$h(x) = \exp\left[\frac{\Phi(x, xf(x))}{2}\right] \quad \text{and} \quad \ell(x) = \frac{\Phi(x, xf(x))}{x}. \tag{10}$$

We claim that $B = (h(x), \ell(x))$ satisfies (8). By using $f(x)f(xf(x)) = 1$, we obtain

$$\begin{aligned} h(xf(x)) &= \exp\left[\frac{\Phi(xf(x), xf(x)f(xf(x)))}{2}\right] = \exp\left[\frac{\Phi(xf(x), x)}{2}\right] \\ &= \exp\left[\frac{-\Phi(x, xf(x))}{2}\right] \end{aligned} \tag{11}$$

and

$$\begin{aligned} \ell(xf(x)) &= \frac{\Phi(xf(x), xf(x)f(xf(x)))}{xf(x)} = \frac{\Phi(xf(x), x)}{xf(x)} \\ &= -\frac{\Phi(x, xf(x))}{xf(x)}. \end{aligned} \tag{12}$$

Applying (9), (11) and (12) shows

$$g(x)h(xf(x)) = h(x) \quad \text{and} \quad f(x)\ell(xf(x)) = -\ell(x). \tag{13}$$

Hence it follows from (1) that

$$(g(x), f(x)) * (h(x), \ell(x)) = (h(x), \ell(x)) * (1, -1).$$

Equivalently, we have $D = BMB^{-1}$ as we wanted, which completes the proof. □

Corollary 2.6. *Let $D = (g(x), f(x))$ be a pseudo-involution with nonnegative entries. If $f(x) = 1$ or $f(x) = g(x)$ then there exists the Riordan matrix B such that $DM = BMB^{-1}$, where*

$$B = \left(\sqrt{g(x)}, \frac{1}{x}\ln g(x)\right). \tag{14}$$

Proof. Let $f(x) = 1$. Then D is an element of the Appell subgroup. By Corollary 2.4, there exists an odd function $\rho(x)$ such that $g(x) = \mathrm{e}^{\rho(x)}$. Since $\Phi(x, -x) = \rho(x)$, we obtain (14) immediately from (7). Now let $f(x) = g(x)$. Then D is an element of the Bell subgroup. Taking $\Phi(x, z) = \ln\left|\frac{z}{x}\right|$ yields also (14) from (7). □

We note that if $D = (g(x), 1)$ is a pseudo-involution with nonnegative entries in the Appell subgroup $\mathscr{A}$, then there exists also $B = (\sqrt{g(x)}, 1) \in \mathscr{A}$ such that $DM = BMB^{-1}$. It is known (e.g. see [1]) that important pseudo-involutions in the Bell subgroup are the Pascal matrix, Nkwanta's RNA matrix $(g(x), g(x))$ and Aigner's directed animal matrix $(1 + xm(x), 1 + xm(x))$ where

$$g(x) = \frac{1 - x + x^2 - \sqrt{(1 - x + x^2)^2 - 4x^2}}{2x^2}$$
$$= 1 + x + x^2 + 2x^3 + 4x^4 + 8x^5 + 17x^6 + O(x^7) \tag{15}$$

and for the Motzkin numbers given by $m(x) = (1 - x - \sqrt{1 - 2x - 3x^2})/2x^2$

$$1 + xm(x) = 1 + x + x^2 + 2x^3 + 4x^4 + 9x^5 + 21x^6 + O(x^7). \tag{16}$$

3. Some remarks on the question Q8.1

We now turn to the question Q8.1 concerning the existence of a certain Riordan matrix B having the related combinatorial significance such that $D = BMB^{-1}$ for which Riordan involution D has some combinatorial significance.

In this section, we discuss the possibility for the question Q8.1 and we obtain an affirmative answer for a pseudo-involution in the Bell subgroup.

In many cases, the Riordan matrix B determined by (7) does not have integer entries. So, it does not seem easy to find directly the related significance in B for which $D = BMB^{-1}$ has some combinatorial significance. In this case, we will look at the checkerboard subgroup. Because, the checkerboard subgroup is the centralizer of M and thus this allows many choices for B.

Theorem 3.1. *Let D be a Riordan involution and B a Riordan matrix such that $D = BMB^{-1}$. Then for any element C of the checkerboard subgroup, D may be expressed in terms of BC, i.e.*

$$D = (BC)M(BC)^{-1}. \tag{17}$$

Proof. By Theorem 2.5, there exists the Riordan matrix B such that $D = BMB^{-1}$. Now let $C = (g(x), f(x))$ be a checkerboard matrix, i.e. $g(x)$ and $f(x)$ are both even functions. Thus

$$\begin{aligned} g(x) = g(-x) \quad \text{and} \quad f(x) = f(-x) &\Leftrightarrow (g(x), -f(x)) = (g(-x), -f(-x)) \\ &\Leftrightarrow (g(x), f(x))(1, -1) = (1, -1)((g(x), f(x)) \\ &\Leftrightarrow CM = MC. \end{aligned}$$

It follows that

$$D = BMB^{-1} = B(CMC^{-1})B^{-1} = (BC)M(BC)^{-1},$$

which completes the proof. □

Theorem 3.1 tells us that if D is a Riordan involution then there are infinitely many Riordan matrices X of the form $X = BC$ such that $D = XMX^{-1}$ where B is the Riordan matrix given by (7) and C is any checkerboard matrix. It also allows a possibility of finding some X of combinatorial significance for a suitable choice of C, where $D = XMX^{-1}$.

For a combinatorial consideration of a Riordan involution, it is natural to focus on pseudo-involutions with nonnegative integer entries. By virtue of Corollary 2.4, there is no pseudo-involution with integer entries in the Appell subgroup. So we concentrate our focus to pseudo-involutions in the Bell subgroup. A *unit* Riordan matrix has all ones on the main diagonal.

Theorem 3.2. *Let $D = (g(x), f(x))$ be a pseudo-involution with nonnegative integer entries. If there exists a unit Riordan matrix B with integer entries such that $DM = BMB^{-1}$ then all the* *entries below the main diagonal of D are even numbers. Equivalently*, $[x^n]g(x)$ *and* $[x^n]f(x)$ *are even for* $n \geqslant 1$ *and* $g(0) = f(0) = 1$.

Proof. Let $D = [d_{n,k}]_{n,k\geqslant 0} = (g(x), f(x))$ be a pseudo-involution with nonnegative integer entries. Since the main diagonal entries of D are all equal to 1, it is obvious that $g(0) = f(0) = 1$. Suppose that there exists a unit Riordan matrix B with integer entries such that $DM = BMB^{-1}$. Then we have

$$(-1)^k d_{n,k} = \sum_{j=0}^{n}(-1)^j b_{n,j}\hat{b}_{j,k} = \sum_{j=0}^{n} b_{n,2j}\hat{b}_{2j,k} - \sum_{j=0}^{n} b_{n,2j+1}\hat{b}_{2j+1,k}, \tag{18}$$

where $B = [b_{n,k}]_{n,k\geqslant 0}$ and $B^{-1} = [\hat{b}_{n,k}]_{n,k\geqslant 0}$. Since $\sum_{j=0}^{n} b_{n,j}\hat{b}_{j,k} = \delta_{nk}$, where δ_{nk} is the Kronecker delta function, for nonnegative integers n, k with $n \neq k$ we have $\sum_{j=0}^{n} b_{n,2j}\hat{b}_{2j,k} = -\sum_{j=0}^{n} b_{n,2j+1}\hat{b}_{2j+1,k}$. It follows from (18) that

$$(-1)^k d_{n,k} = 2\sum_{j=0}^{n} b_{n,2j}\hat{b}_{2j,k}, \quad n \neq k. \tag{19}$$

Since B is a unit Riordan matrix with integer entries, B^{-1} is so. Hence each term of the right-hand side of (19) is integer and so $d_{n,k} \in 2Z$, which implies that all the entries below the main diagonal of D are even.

Moreover, since $g(x)(xf(x))^k$ is the kth column generating function of D, we have $g(x) = 1 + \sum_{n\geqslant 1} d_{n,0}x^n$ when $k = 0$, which implies that $g(x)$ has even coefficients except $g(0) = 1$. If $k = 1$ then $g(x)(xf(x)) = x + \sum_{n\geqslant 2} d_{n,1}x^n$. By the Vandermonde convolution, $f(x)$ must have even coefficients except $f(0) = 1$, which completes the proof. □

We note that by Theorem 3.2, there is no unit Riordan matrix B with integer entries such that $(BMB^{-1})M$ is the Pascal matrix, or Nkwanta's RNA matrix given by (15), or Aigner's directed animal matrix given by (16). However, we will show that the question Q8.1 has an affirmative answer for these pseudo-involutions, see Corollary 3.5.

Lemma 3.3. *Let* $D = (g(x), g(x))$ *be a pseudo-involution of the Bell subgroup. Then the Riordan matrix of the form:*

$$C = \left(e^{-\frac{x}{2}}\phi(x), \frac{1}{x}\bar{g}(e^x)\phi(x)\right) \tag{20}$$

is the checkerboard matrix where $\phi(x) = e^{\frac{x}{2}}\left(\cosh\frac{x}{2}\right)^n$ *for any* $n \in Z$ *and* $\bar{g}(x)$ *is the compositional inverse of* $g(x)$.

Proof. Since $\cosh\frac{x}{2}$ is an even function, clearly $e^{-\frac{x}{2}}\phi(x) = \left(\cosh\frac{x}{2}\right)^n$ is even for any $n \in Z$. We only need to show that $\bar{g}(e^x)\phi(x)$ is odd function for any $n \in Z$, equivalently, $e^{\frac{x}{2}}\bar{g}(e^x) = -e^{-\frac{x}{2}}\bar{g}(e^{-x})$ or $t\bar{g}(t) = -\bar{g}\left(\frac{1}{t}\right)$ where $t = e^x$. Since $DM = (g(x), -g(x))$ is a Riordan involution, we have $g(x)g(-xg(x)) = 1$ which implies $xg(x) = -\bar{g}\left(\frac{1}{g(x)}\right)$. By setting $g(x) = t$, we obtain $t\bar{g}(t) = -\bar{g}\left(\frac{1}{t}\right)$ which implies that $e^{\frac{x}{2}}\bar{g}(e^x)$ is odd function. Hence $\bar{g}(e^x)\phi(x) = e^{\frac{x}{2}}\bar{g}(e^x)\left(\cosh\frac{x}{2}\right)^n$ is odd function for any $n \in Z$, which completes the proof. □

Theorem 3.4. *Let $D = (g(x), g(x))$ be a pseudo-involution with nonnegative integer entries in the Bell subgroup. Then there exists a unit Riordan matrix $X = (\psi(x), \psi(x))$ such that $DM = XMX^{-1}$, where*

$$\psi(x) = \left(\frac{1+g(x)}{2}\right)^{2m+1}\left(\frac{1}{g(x)}\right)^{m}, \quad m \in Z. \tag{21}$$

In particular, if $g(x)$ has even coefficients except $g(0) = 1$ then the Riordan matrix X given by (21) is of integer entries for any integer m.

Proof. By Corollary 2.6, there exists the Riordan matrix B given by (14) such that $DM = BMB^{-1}$. By Theorem 3.1 and Lemma 3.3, we have $DM = (BC)M(BC)^{-1}$ where C is the checkerboard matrix given by (20). By a matrix multiplication given by (1), we obtain

$$\begin{aligned} X := BC &= \left(\sqrt{g(x)}, \frac{\ln g(x)}{x}\right) * \left(\mathrm{e}^{-\frac{x}{2}}\phi(x), \frac{\bar{g}(\mathrm{e}^x)}{x}\phi(x)\right) \\ &= (\phi(\ln g(x)), \phi(\ln g(x))). \end{aligned}$$

By choosing $n = 2m+1$ for $m \in Z$, it follows from $\phi(x) = \mathrm{e}^{\frac{x}{2}}\left(\cosh\frac{x}{2}\right)^{2m+1}$ that

$$\begin{aligned} \psi(x) := \phi(\ln g(x)) &= \sqrt{g(x)}\left(\cosh(\ln\sqrt{g(x)})\right)^{2m+1} \\ &= \sqrt{g(x)}\left(\frac{1}{2}\left(\sqrt{g(x)} + \frac{1}{\sqrt{g(x)}}\right)\right)^{2m+1} \\ &= \left(\frac{1+g(x)}{2}\right)^{2m+1}\left(\frac{1}{g(x)}\right)^{m}. \end{aligned} \tag{22}$$

Since $g(0) = 1$, we have $\psi(0) = 1$ for any $m \in Z$. Hence $X = (\psi(x), \psi(x))$ is a unit Riordan matrix for any $m \in Z$.

In particular, let $g(x) = \sum_{n \geqslant 0} g_n x^n$, where $g_0 = 1$ and $g_n \in 2Z$ for $n \geqslant 1$. It follows that $\frac{g(x)+1}{2}$ has integer coefficients. Moreover, $\frac{1}{g(0)} = 1$ and each coefficient $\hat{g}_n$ for $n \geqslant 1$ of the reciprocal $\frac{1}{g(x)}$ of $g(x)$ has the determinantal expression (see p. 157, [2]) given by $\hat{g}_n = (-1)^n \det H$ where $H = [h_{i,j}]$ is the $n \times n$ lower Hessenberg matrix defined by $h_{i,j} = g_{i-j+1}$ for $1 \leqslant i, j \leqslant n$ with $g_k = 0$ for $k < 0$. Since g_k for $k \geqslant 0$ is integer, it is obvious that $\hat{g}_n$ is integer for $n \geqslant 1$. It follows from (22) that $\psi(x)$ has integer coefficients for any $m \in Z$. Hence $X = (\psi(x), \psi(x))$ is a matrix of integer entries for any $m \in Z$, which completes the proof. □

Now, we give an affirmative answer for the question Q8.1 if D is a pseudo-involution in the Bell subgroup.

Corollary 3.5 (Q8.1). *Let $D = (g(x), g(x))$ be a pseudo-involution with nonnegative integer entries in the Bell subgroup. If $D = [d_{n,k}]_{n,k\geqslant 0}$ has some combinatorial property P then there exists the Riordan matrix $X = [x_{n,k}]_{n,k\geqslant 0}$ with the related property P such that $DM = XMX^{-1}$, where*

$$X = \left(\frac{1+g(x)}{2}, \frac{1+g(x)}{2}\right). \tag{23}$$

In particular, if $d_{n,k}$ counts some configuration with the property P then $x_{n,n} = 1$ for $n \geqslant 0$ and $x_{n,k}$ for $n \neq k$ is given by

$$x_{n,k} = \frac{1}{2^{k+1}} \sum_{\ell=0}^{k} \binom{k+1}{\ell+1} d_{n-k+\ell,\ell}. \tag{24}$$

Proof. By Theorem 3.4, if we choose $m = 0$ we obtain the Riordan matrix X given by (23) such that $DM = XMX^{-1}$. Now we show that if D has some combinatorial property P then X has also the related property P.

First note that X given by (23) may be expressed by

$$X = (1 + g(x), 1 + g(x)) * \left(\frac{1}{2}, \frac{1}{2}\right). \tag{25}$$

Let $G = (1 + g, 1 + g) = [g_{n,k}]_{n,k\geqslant 0}$, where $g = g(x)$. We observe that

$$g_{n,k} = [x^n](1 + g)(x(1 + g))^k = [x^{n-k}](1 + g)^{k+1} = \sum_{\ell=0}^{k+1} \binom{k+1}{\ell} [x^{n-k}]g^{\ell}. \tag{26}$$

If $n = k$ then we have $g_{n,n} = 2^{n+1}$ for $n \geqslant 0$. Now let $n \neq k$. It follows from (26) that

$$g_{n,k} = \sum_{\ell=1}^{k+1} \binom{k+1}{\ell} [x^{n-k+\ell-1}]x^{\ell-1}g^{\ell}. \tag{27}$$

Since $d_{n,k} = [x^n]g(xg)^k = [x^n]x^k g^{k+1}$, from (27) we have

$$g_{n,k} = \sum_{\ell=0}^{k} \binom{k+1}{\ell+1} d_{n-k+\ell,\ell}. \tag{28}$$

It implies that if $d_{n,k}$ counts some configuration with the property P then $g_{n,k}$ counts the related configuration with the property P. Moreover, the entry $x_{n,k}$ of X given by (24) can be obtained from (25) and (28). Hence D has some combinatorial property P then the Riordan matrix X such that $DM = XMX^{-1}$ should be have the related property P, which completes the proof. □

Example 1. Let us consider the pseudo-involution (see A081696, [8]) given by $D = ((1 + xC(x))C(x), (1 + xC(x))C(x))$, where $C(x) = \frac{1-\sqrt{1-4x}}{2x}$ is the generating function for the Catalan numbers:

$$D = \left(\frac{1 - x - \sqrt{1 - 4x}}{x}, \frac{1 - x - \sqrt{1 - 4x}}{x}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 4 & 4 & 1 & 0 & 0 \\ 10 & 12 & 6 & 1 & 0 \\ 28 & 36 & 24 & 8 & 1 \\ & & \cdots & & \end{bmatrix}.$$

Interestingly, from (23) we obtain the Catalan triangle [6]:

$$X = (C(x), C(x)) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 \\ 5 & 5 & 3 & 1 & 0 \\ 14 & 14 & 9 & 4 & 1 \\ & & \cdots & & \end{bmatrix}.$$

Now, one can easily see that $DM = XMX^{-1}$. Obviously, this example shows an affirmative answer for the question Q8.1.

Example 2. Let us consider the RNA matrix $(g(x), g(x))$ given by (15):

$$\mathrm{RNA} = [r_{n,k}]_{n,k\geqslant 0} = (g(x), g(x)) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 2 & 3 & 3 & 1 & 0 & 0 \\ 4 & 6 & 6 & 4 & 1 & 0 \\ 8 & 13 & 13 & 10 & 5 & 1 \\ & & \cdots & & & \end{bmatrix}.$$

It is known [1] that RNA matrix has a lattice path interpretation. That is, $r_{n,k}$ counts the number of NES-paths of length n and terminal height k where NES-path is a counting path which starts at the origin (0, 0) and take unit steps, $N(0, 1)$, $E(1, 0)$, and $S(0, -1)$ with the following restrictions: no paths pass below the x-axis and no S step immediately follows an N step.

Now, consider the Riordan matrix X obtained from (25):

$$X = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 & 0 & 0 \\ 1 & 4 & 8 & 0 & 0 & 0 \\ 2 & 5 & 12 & 16 & 0 & 0 \\ 4 & 10 & 18 & 32 & 32 & 0 \\ 8 & 21 & 37 & 56 & 80 & 64 \\ & & \cdots & & & \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2^3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2^4} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2^5} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2^6} \\ & & \cdots & & & \end{bmatrix}.$$

By (28), we obtain

$$g_{n,k} = \sum_{\ell=0}^{k} \binom{k+1}{\ell+1} r_{n-k+\ell,\ell}.$$

For example

$$g_{3,2} = \binom{3}{1} r_{1,0} + \binom{3}{2} r_{2,1} + \binom{3}{3} r_{3,2} = 3 \cdot 1 + 3 \cdot 2 + 1 \cdot 3 = 12.$$

Hence the Riordan matrix X has also lattice path interpretation by NES-paths as well as the RNA matrix, which shows an affirmative answer for the question Q8.1.

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