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Riordan group involutions

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Abstract

We study involutions in the Riordan group, especially those with combinatorial meaning. We give a new determinantal criterion for a matrix to be a Riordan involution and examine several classes of examples. A complete characterization of involutions in the Appell subgroup is developed. In another direction we find several examples that generalize the RNA matrix but are of independent interest.

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1. Introduction

Why are involutions of combinatorial interest? Basic curiosity is certainly supplemented by some intriguing examples. In all these examples we let $D = (d(n,k))_{n,k\geqslant 0}$ and if you change the signs in alternate columns you get a matrix whose square is the identity matrix. The best known

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is Pascal's matrix where $d(n,k)=\binom{n}{k}$. A second example is given by the Lah numbers where $d(n,k)=\binom{n}{k}\frac{(n+1)!}{(k+1)!}$. Yet a third is given by the coefficients of the Laguerre polynomials where $d(n,k)=\binom{n}{k}\frac{n!}{k!}$. These cases have neat closed expressions for $d(n,k)$. Equally interesting are the RNA matrix [10] and Aigner's directed animal matrix [1]. Here are the first few rows of these two matrices:

$$\mathrm{RNA}=\begin{bmatrix}1&0&0&0&0&0\\1&1&0&0&0&0\\1&2&1&0&0&0\\2&3&3&1&0&0\\4&6&6&4&1&0\\8&13&13&10&5&1\\&&\cdots&\cdots&&\end{bmatrix} \tag{1}$$

and

$$\text{Directed animals}=\begin{bmatrix}1&0&0&0&0&0\\1&1&0&0&0&0\\1&2&1&0&0&0\\2&3&3&1&0&0\\4&6&6&4&1&0\\9&13&13&10&5&1\\&&\cdots&\cdots&&\end{bmatrix}. \tag{2}$$

Note that the left hand columns are the number of secondary RNA structures on a chain of length n and the Motzkin numbers.

The concept of the *Riordan group* $(\mathscr{R},*)$ has been introduced by Shapiro et al. [14]. A Riordan matrix $D=[d_{n,k}]_{n,k\geqslant 0}$ is defined by a pair of generating functions $g(z)=g_0+g_1z+g_2z^2+\cdots$ and $f(z)=f_1z+f_2z^2+\cdots$ such that

$$d_{n,k}=[z^n]g(z)(f(z))^k,$$

where $g_0\neq 0$ and $f_1\neq 0$. With little loss of generality we also assume $d_{0,0}=g_0=1$. We denote this matrix as $D=(g(z),f(z))$. One example of a Riordan matrix is the Pascal matrix

$$P=\left(\frac{1}{1-z},\frac{z}{1-z}\right)=\begin{bmatrix}1&0&0&0\\1&1&0&0\\1&2&1&0\\1&3&3&1\\&\cdots&\cdots&\end{bmatrix} \tag{3}$$

for which we have

$$\binom{n}{k}=[z^n]\frac{1}{1-z}\left(\frac{z}{1-z}\right)^k.$$

The Riordan group is the set of all Riordan matrices with the operation being matrix multiplication. In terms of the generating functions this works out as

$$(g(z),f(z))*(G(z),F(z))=(g(z)G(f(z)),F(f(z))).$$

Then it is easy to see that the identity element of the Riordan group is $I=(1,z)$, the usual identity matrix, and the inverse of $(g(z),f(z))$ is $\left(\frac{1}{g(\bar f(z))},\bar f(z)\right)$, where $\bar f(z)$ is the compositional inverse of $f(z)$.

In the present paper, we will focus on involutions in the Riordan group. In combinatorial situations a Riordan matrix will often have all positive entries on and below the main diagonal and cannot itself have order 2. We define an element D in the Riordan group to have pseudo order 2 if DM has order 2 where

$$M = (1, -z) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ & \cdots & \cdots & \end{bmatrix}.$$

Clearly, AMA^{-1} has order 2 for any element A in the Riordan group. An element of (pseudo) order 2 in the Riordan group will be called a (*pseudo*) *Riordan involution* or more briefly a (*pseudo*) *involution*.

In 2001, Shapiro [13] presented several open questions on involutions of the Riordan group:

Q8: Can every element of order 2 in the Riordan group be written as AMA^{-1} for some element A in the Riordan group?

Q8.5: If the element of order 2 or pseudo order 2 has combinatorial significance can we find an A which has a related combinatorial significance?

Q9: If $D = (g(z), f(z))$ has order 2, is there a simple condition for $g(z)$ in terms of $f(z)$?

In [2], Cameron and Nkwanta studied classes of combinatorial matrices having pseudo order 2 in the Riordan group and obtained some partial results on the problem Q8. Recently, Cheon and Kim [3] gave an affirmative answer for Q8 and some positive results concerning the other two questions.

The purpose of this paper is to study the structure of (pseudo) Riordan involutions. In Section 2, we give a useful characterization of a Riordan matrix in terms of the Stieltjes transform [5,7]. In Section 3, we obtain a necessary and sufficient condition to be a Riordan involution. In Section 4, we determine the generating functions for the λ-invariant sequences in each eigenspace of a Riordan involution. Finally, in Section 5, we explore generalized RNA triangles.

2. A characterization of a Riordan matrix

For a Riordan matrix $D = [d_{n,k}]_{n,k\geqslant 0}$, Rogers [11] has found that every element $d_{n+1,k+1}$ can be expressed as a linear combination of the elements in the preceding row starting from the preceding column, and Merlini et al. [9] has found that every element in column 0 can be expressed as a linear combination of all the elements of the preceding row, also see [15]. Because of their importance, these properties are stated as the following theorem.

Theorem 2.1. *Let $D = [d_{n,k}]$ be an infinite triangular matrix. Then D is a Riordan matrix if and only if there exists two sequences $A = \{a_0, a_1, a_2, \ldots\}$ and $Z = \{z_0, z_1, z_2, \ldots\}$ with $a_0 \neq 0$, $z_0 \neq 0$ such that*

(i) $d_{n+1,k+1} = \sum_{j=0}^{\infty} a_j d_{n,k+j} (k, n = 0, 1 \ldots)$,
(ii) $d_{n+1,0} = \sum_{j=0}^{\infty} z_j d_{n,j}$, $(n = 0, 1, \ldots)$.

The coefficients $a_0, a_1, a_2, \ldots$ and $z_0, z_1, z_2, \ldots$ appearing in (i) and (ii) are called by the *A-sequence* and the *Z-sequence* of the Riordan matrix $D = (g(z), f(z))$, respectively. If $A(z)$

and $Z(z)$ are the generating functions of the corresponding sequences then it can be proven that $f(z)$ and $g(z)$ are the solutions of the functional equations, respectively:

$$\begin{aligned} f(z) &= zA(f(z)), \\ g(z) &= g(0)/(1 - zZ(f(z))). \end{aligned} \tag{4}$$

In this section, we give another characterization of a Riordan matrix. Let D be a Riordan matrix and let $\overline{D}$ be the matrix obtained from D by deleting the first row. Since D is a lower triangular matrix with nonzero entries on the main diagonal, there exists a unique matrix S_D such that $DS_D = \overline{D}$. We call the matrix S_D the *Stieltjes transform* of D. See [7] for similar ideas but in the language of continued fractions and [5] where the term production matrix is used instead of Stieltjes matrix. It was pointed out by the referee that the following result also can be found in [6]. But the result has been proved independently.

Theorem 2.2. *Let $D = [d_{n,k}]$ be an infinite lower triangular matrix with $d_{n,n} \neq 0$. Then D is a Riordan matrix if and only if the Stieltjes transform matrix S_D of D has the following form*:

$$S_D = \begin{bmatrix} z_0 & a_0 & & \\ z_1 & a_1 & a_0 & \\ z_2 & a_2 & a_1 & a_0 \\ z_3 & a_3 & a_2 & a_1 \\ & \cdots & \cdots & \end{bmatrix}, \tag{5}$$

where $(a_0, a_1, \ldots)$ is the A-sequence and $(z_0, z_1, \ldots)$ is the Z-sequence of D.

Proof. Let $D = [d_{n,k}]$ be a Riordan matrix. If we write out $DS_D = \overline{D}$ in matrix form

$$\begin{bmatrix} d_{0,0} & 0 & 0 & 0 \\ d_{1,0} & d_{1,1} & 0 & 0 \\ d_{2,0} & d_{2,1} & d_{2,2} & 0 \\ d_{3,0} & d_{3,1} & d_{3,2} & d_{3,3} \\ & \cdots & & \end{bmatrix} \begin{bmatrix} z_0 & a_0 & 0 & 0 \\ z_1 & a_1 & a_0 & 0 \\ z_2 & a_2 & a_1 & a_0 \\ z_3 & a_3 & a_2 & a_1 \\ & & \cdots & \end{bmatrix} = \begin{bmatrix} d_{1,0} & d_{1,1} & 0 & 0 \\ d_{2,0} & d_{2,1} & d_{2,2} & 0 \\ d_{3,0} & d_{3,1} & d_{3,2} & d_{3,3} \\ d_{4,0} & d_{4,1} & d_{4,2} & d_{4,3} \\ & & \cdots & \end{bmatrix}$$

and recall the definitions of the A-sequence $(a_0, a_1, \ldots)$ and the Z-sequence $(z_0, z_1, \ldots)$, we see that they mesh to give the proof. $\square$

Corollary 2.3. *Let D be a Riordan matrix D with A-sequence $(a_0, a_1, \ldots)$ and Z-sequence $(z_0, z_1, \ldots)$, and let S_D be the Stieltjes transform matrix of the form (5). Then the Riordan matrix D is of the form*:

$$D = \sum_{k \geqslant 0} E_{k,0} S_D^k \tag{6}$$

where $E_{k,0}$ is a cell whose $(k, 0)$-entry is 1 and other entries are all zeros, and S_D^0 is the identity matrix.

Proof. Let $D = [d_{n,k}]_{n,k \geqslant 0}$ be a Riordan matrix with A-sequence $(a_0, a_1, \ldots)$ and Z-sequence $(z_0, z_1, \ldots)$. By the definition of the A-sequence and Z-sequence, one can easily show that the top row of S_D is the 1st row of D, and inductively the term $E_{k,0}S_D^k$ in the sum (6) generates the kth row of D. $\square$

3. Structure of a Riordan involution

Let $D = [d_{i,j}]_{i,j\geqslant 0}$ be a Riordan involution with A-sequence $(a_0, a_1, \ldots)$ and Z-sequence $(z_0, z_1, \ldots)$. Then we have $a_0 d_{0,0} = \pm 1$. If $a_0 = 1$ and $d_{0,0} = 1$ (or $d_{0,0} = -1$, respectively) then the only Riordan involution is $I = (1, z)$ (or $-I = (-1, z)$, respectively). Hence we may assume that $d_{0,0} = 1$ and $a_0 = -1$.

Let $D[\alpha]$ be a principal submatrix of order $|\alpha|$ indexed by the rows α and columns α of D and let $\overline{D}[\alpha]$ be the matrix of order $|\alpha| - 1$ obtained from $D[\alpha]$ by deleting the first row and last column. More generally, the matrix obtained from $D[\alpha]$ by deleting the ith row and jth column will be denoted by $D[\alpha](i|j)$.

Theorem 3.1. *Let $D = [d_{i,j}]_{i,j\geqslant 0}$ be a Riordan matrix with A-sequence $(-1, a_1, \ldots)$ and $d_{0,0} = 1$. Then D is a Riordan involution if and only if for any consecutive index set $\alpha = \{i_1, i_2, \ldots, i_k\}$ $(k \geqslant 3)$ of $\{0, 1, 2, \ldots\}$, we have*

$$\det(\overline{D}[\alpha]) = (-1)^{i_2+\cdots+i_{k-1}} d_{i_k,i_1}. \tag{7}$$

Proof. Let $D = [d_{i,j}]_{i,j\geqslant 0}$ be a Riordan involution and let $\alpha = \{i_1, i_2, \ldots, i_k\}$ be any consecutive subset of $\{0, 1, 2, \ldots, \}$. We proceed by induction on $k \geqslant 3$. Since $D[\alpha]$ is a $k \times k$ involutary matrix, we have $\mathbf{r}_i\mathbf{c}_j = \delta_{ij}$, $i, j = 1, 2, \ldots, k$, where $\mathbf{r}_i$ and $\mathbf{c}_j$ are the ith row vector and jth column vector of $D[\alpha]$, respectively and δ_{ij} is the Kronecker symbol. In particular, from $\mathbf{r}_k\mathbf{c}_1 = 0$ we obtain

$$\sum_{j=2}^{k-1} d_{i_k,i_j} d_{i_j,i_1} = ((-1)^{i_1+1} + (-1)^{i_k+1}) d_{i_k,i_1}. \tag{8}$$

First let $k = 3$. From (8) with the consecutive indices i_1, i_2, i_3, we obtain

$$\begin{aligned}\det\overline{D}[\alpha] &= \det\begin{bmatrix} d_{i_2,i_1} & (-1)^{i_2} \\ d_{i_3,i_1} & d_{i_3,i_2}\end{bmatrix} = d_{i_2,i_1}d_{i_3,i_2} + (-1)^{i_2+1}d_{i_3,i_1} \\ &= (-1)^{i_1+1}d_{i_3,i_1} + (-1)^{i_3+1}d_{i_3,i_1} + (-1)^{i_2+1}d_{i_3,i_1} \\ &= (-1)^{i_1+1}d_{i_3,i_1},\end{aligned}$$

which proves (7) for $k = 3$. Now assume that $k \geqslant 4$. Since D is a Riordan matrix with A-sequence $(-1, a_1, \ldots)$ and $d_{0,0} = 1$, we may assume that a matrix $\overline{D}[\alpha]$ of order $k - 1$ has the form

$$\overline{D}[\alpha] = \begin{bmatrix} d_{i_2,i_1} & (-1)^{i_2} & & & \\ d_{i_3,i_1} & d_{i_3,i_2} & (-1)^{i_3} & & \\ \vdots & \vdots & \ddots & \ddots & \\ d_{i_{k-1},i_1} & d_{i_{k-1},i_2} & \cdots & \ddots & (-1)^{i_{k-1}} \\ d_{i_k,i_1} & d_{i_k,i_2} & \cdots & \cdots & d_{i_k,i_{k-1}}\end{bmatrix}, \tag{9}$$

where the unspecified entries are all zeros.

By applying the Laplace expansion on the first column of $\overline{D}[\alpha]$ and then by the induction, we obtain

$$\begin{aligned}\det(\overline{D}[\alpha]) &= \sum_{j=1}^{k-1}(-1)^{j+1}d_{i_{j+1},i_1}\det\overline{D}[\alpha](j|1)\\ &= (-1)^{i_3+\cdots+i_{k-1}}d_{i_k,i_2}d_{i_2,i_1}+\sum_{j=2}^{k-2}(-1)^{j+1}d_{i_{j+1},i_1}\det\overline{D}[\alpha](j|1)\\ &\quad +(-1)^{k+i_2+\cdots+i_{k-1}}d_{i_k,i_1}.\end{aligned}\tag{10}$$

Note that for j with $2 \leqslant j \leqslant k-2$, the matrix $\overline{D}[\alpha](j|1)$ of order $k-2$ has the form

$$\overline{D}[\alpha](j|1)=\begin{bmatrix} D[\alpha'] & O\\ C & \overline{D}[\alpha''] \end{bmatrix},$$

where $\alpha'=\{i_2,\ldots,i_j\}$ and $\alpha''=\{i_{j+1},\ldots,i_k\}$. Since O is $(j-2)\times(k-j)$ zero submatrix of $\overline{D}[\alpha](j|1)$, it follows from the induction that

$$\begin{aligned}\det\overline{D}[\alpha](j|1) &= (\det D[\alpha'])(\det\overline{D}[\alpha''])\\ &= (-1)^{i_2+\cdots+i_j+i_{j+2}+\cdots+i_{k-1}}d_{i_k,i_{j+1}}.\end{aligned}\tag{11}$$

By substituting (11) into (10) and then by applying (8) together with $1+i_j=i_{j+1}$, we obtain

$$\begin{aligned}\det(\overline{D}[\alpha]) &= (-1)^{i_3+\cdots+i_{k-1}}\sum_{j=2}^{k-1}d_{i_k,i_j}d_{i_j,i_1}+(-1)^{k+i_2+\cdots+i_{k-1}}d_{i_k,i_1} \qquad (12)\\ &= \left((-1)^{i_2+\cdots+i_{k-1}}+(-1)^{1+i_3+\cdots+i_k}+(-1)^{2+i_3+\cdots+i_k}\right)d_{i_k,i_1}\\ &= (-1)^{i_2+\cdots+i_{k-1}}d_{i_k,i_1},\end{aligned}$$

which proves (7).

Conversely, suppose that (7) holds for any consecutive index set $\alpha=\{i_1,i_2,\ldots,i_k\}$ $(k\geqslant 3)$ of $\{0,1,2,\ldots,\}$. Note that a Riordan matrix D is involutary if and only if $D[\alpha]$ is involutary for any consecutive set α.

If $k=1,2$ then clearly $D[\alpha]$ is involutary. Hence for a fixed $k\geqslant 3$ it is sufficient to show that $\mathbf{r}_k\mathbf{c}_m=\delta_{km}$ for each $m=1,2,\ldots,k$. If $k=m$ then $\mathbf{r}_k\mathbf{c}_m=(-1)^{i_k}(-1)^{i_m}=(-1)^{2i_k}=1$. Now assume $k\neq m$. Let us consider a consecutive subset $\alpha_m=\{i_m,\ldots,i_k\}$ of $\alpha=\{i_1,i_2,\ldots,i_k\}$. By applying (12) for α_m and then by using $1+i_j=i_{j+1}$ together with $d_{i_k,i_k}=(-1)^{i_k}$, we obtain

$$\det\overline{D}[\alpha_m]=(-1)^{i_{m+2}+\cdots+i_{k-1}}\sum_{j=m+1}^{k}d_{i_k,i_j}d_{i_j,i_m}.\tag{13}$$

Further, by our assumption we have

$$\det\overline{D}[\alpha_m]=(-1)^{i_{m+1}+\cdots+i_{k-1}}d_{i_k,i_m}.\tag{14}$$

Thus it follows from (13) and (14) that

$$(-1)^{i_{m+2}+\cdots+i_{k-1}}\sum_{j=m}^{k}d_{i_k,i_j}d_{i_j,i_m}=0.$$

Since $d_{i_j,i_m} = 0$ for $1 \leqslant j < m$, we obtain

$$\mathbf{r}_k\mathbf{c}_m = \sum_{j=1}^{k} d_{i_k,i_j} d_{i_j,i_m} = \sum_{j=m}^{k} d_{i_k,i_j} d_{i_j,i_m} = 0,$$

which completes the proof. $\square$

Since DM is pseudo involution for a Riordan involution D and $M = (1, -z)$, we have the following corollary.

Corollary 3.2. *Let $D = [d_{i,j}]_{i,j\geqslant 0}$ be a Riordan matrix with $d_{0,0} = 1$. Then D is a pseudo involution if and only if for any consecutive index set $\alpha = \{i_1, i_2, \ldots, i_k\}$ $(k \geqslant 3)$ of $\{0, 1, 2, \ldots\}$, we have*

$$\det(\overline{D}[\alpha]) = d_{i_k,i_1}.$$

For example, the Pascal matrix P given in (3) is a pseudo involution. Indeed,

$$\det\begin{bmatrix} 1 & 1 \\ \mathbf{1} & 2 \end{bmatrix} = 1 \;\; (\alpha = \{0, 1, 2\}), \quad \det\begin{bmatrix} 2 & 1 \\ \mathbf{3} & 3 \end{bmatrix} = 3 \;\; (\alpha = \{1, 2, 3\}),$$

$$\det\begin{bmatrix} 2 & 1 & 0 \\ 3 & 3 & 1 \\ \mathbf{4} & 6 & 4 \end{bmatrix} = 4 \;\; (\alpha = \{1, 2, 3, 4\}), \ldots.$$

One can easily show that each matrix given in (1) and (2) is also a pseudo involution.

As noted in [12], there are some important subgroups of the Riordan group. The *Appell subgroup* is the set $\{(g(z), z)\}$ and the *checkerboard subgroup* is the set $\{(g(z), f(z))\}$ where $g(z)$ is an even function and $f(z)$ is an odd function. In particular, the Appell subgroup is a normal subgroup of the Riordan group.

For the Appell subgroup, we have the following characterization of involutions.

Theorem 3.3. *Let $D = (g(z), -z)$ be a Riordan matrix with a Z-sequence $(z_0, z_1, \ldots)$ where $g(0) = 1$. If D is a Riordan involution then for each $m = 1, 2, \ldots,$*

$$\det\begin{bmatrix} z_0 & 2 & 0 & \cdots & 0 \\ z_1 & z_0 & 2 & \ddots & \vdots \\ z_2 & z_1 & z_0 & \ddots & 0 \\ & \vdots & & \ddots & 2 \\ z_{2m-1} & z_{2m-2} & \cdots & z_1 & z_0 \end{bmatrix} = 0. \tag{15}$$

Conversely, if (15) holds for each $m = 1, 2, \ldots$ then

$$(g(z), z) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ z_0 & 1 & 0 & 0 \\ z_1 & z_0 & 1 & 0 \\ z_2 & z_1 & z_0 & 1 \\ & \cdots & \cdots & \end{bmatrix} \tag{16}$$

is a pseudo involution in the Appel subgroup.

Proof. Let $Z(z)$ be the generating function of a Z -sequence $(z_0, z_1, \ldots)$ in a Riordan involution $D = (g(z), -z)$. Since $(g(z), -z)(g(z), -z) = (1, z)$, we have $g(z)g(-z) = 1$. Hence it follows from (4) that

$$g(z) = \frac{1}{g(-z)} = 1 + zZ(z). \tag{17}$$

By setting $h(z) = \frac{1}{1+g(z)} := \sum_{n=0}^{\infty} h_n z^n$ and using $g(z)g(-z) = 1$, we obtain

$$\begin{aligned} h(z) + h(-z) &= \frac{1}{1+g(z)} + \frac{1}{1+g(-z)} \\ &= \frac{2+g(z)+g(-z)}{1+g(z)+g(-z)+g(z)g(-z)} = 1. \end{aligned} \tag{18}$$

It immediately follows that $h_0 = 1/2$ and $h_{2m} = 0$ for each $m = 1, 2, \ldots$. Further, from (17) we obtain $h(z) = \frac{1}{2+zZ(z)}$. That is, $h(z)$ is the reciprocal inverse of

$$2 + zZ(z) = 2 + z_0 z + z_1 z^2 + \cdots + z_{2m-1} z^{2m} + \cdots$$

Hence, by Wronski's formula (p.17 of [8]) we have

$$h_{2m} = \frac{(-1)^{2m}}{2^{2m+1}} \det \begin{bmatrix} z_0 & 2 & 0 & \cdots & 0 \\ z_1 & z_0 & 2 & \ddots & \vdots \\ z_2 & z_1 & z_0 & \ddots & 0 \\ & \vdots & & \ddots & 2 \\ z_{2m-1} & z_{2m-2} & \cdots & z_1 & z_0 \end{bmatrix}. \tag{19}$$

Since $h_{2m} = 0$ for each $m = 1, 2, \ldots$, the formula (15) follows from (19).

Conversely, suppose that (15) holds for each $m = 1, 2, \ldots$. Let $h(z) = \sum_{n \geqslant 0} h_n z^n = \frac{1}{1+g(z)}$. Since $Z(z)$ is the generating function of a Z-sequence $(z_0, z_1, \ldots)$ in a Riordan matrix $D = (g(z), -z)$, it follows from (4) that

$$h(z) = \frac{1}{1 + \frac{1}{1-zZ(-z)}} = \frac{1 - zZ(-z)}{2 - zZ(-z)} = 1 - \frac{1}{2 - zZ(-z)}.$$

Hence $1 - h(z)$ is the reciprocal inverse of $2 - zZ(-z)$. Thus we obtain $h_0 = \frac{1}{2}$ and then by Wronski's formula

$$-h_{2m} = \frac{(-1)^{2m}}{2^{2m+1}} \det \begin{bmatrix} -z_0 & 2 & 0 & \cdots & 0 \\ z_1 & -z_0 & 2 & \ddots & \vdots \\ -z_2 & z_1 & -z_0 & \ddots & 0 \\ & \vdots & & \ddots & 2 \\ z_{2m-1} & -z_{2m-2} & \cdots & z_1 & -z_0 \end{bmatrix}.$$

From (15), we have $h_{2m} = 0$ for $m \geqslant 1$. Hence, we have

$$1 = h(z) + h(-z) = \frac{1}{1+g(z)} + \frac{1}{1+g(-z)} = \frac{2+g(z)+g(-z)}{1+g(z)+g(-z)+g(z)g(-z)}.$$

It follows $g(z)g(-z) = 1$, which implies that $D = (g(z), -z)$ is a Riordan involution. Hence $(g(z), z)$ is a pseudo involution with the form (16). $\square$

4. Eigenspace of a Riordan involution

In this section, we determine the generating function for the λ-invariant sequence in each eigenspace of a Riordan involution.

Let R^∞ denote the infinite dimensional real vector space of all real sequences $(x_0, x_1, x_2, \ldots)^{\mathrm{T}}$ and let $D = (g(z), f(z)) \neq I$ be a Riordan involution. For an eigenvalue λ of D, $D\mathrm{x} = \lambda\mathrm{x}$ implies that $\mathrm{x} = D^2\mathrm{x} = \lambda(D\mathrm{x}) = \lambda^2\mathrm{x}$. Hence the only eigenvalues of a Riordan involution D are 1 and -1. Let $E_\lambda(D)$ denote the eigenspace of D corresponding to the eigenvalue λ. As in [4] or [16], we call $\mathrm{x} \in R^\infty$ a λ-*invariant sequence* if $\mathrm{x} \in E_\lambda(D)$.

Let us define $E = [\mathrm{x}_0 \vdots \mathrm{x}_1 \vdots \mathrm{x}_2 \vdots \cdots]$ to be the infinite matrix where x_{2m} and $\mathrm{x}_{2m+1}, m = 0, 1, \ldots,$ are eigenvectors corresponding to $\lambda = 1$ and -1, respectively. Then we see that $DE = EM$ or $D = EME^{-1}$. Conversely, if there exists the Riordan matrix A such that $D = AMA^{-1}$ for a Riordan involution D then each column of A is an eigenvector of D corresponding alternately to the eigenvalues 1 and -1.

For example, we consider the RNA triangle D with the minus signs in alternate columns, which is a Riordan involution:

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 2 & -3 & 3 & -1 & 0 \\ 4 & -6 & 6 & -4 & 1 \\ & & \cdots & & \end{bmatrix}.$$

Let A be the Riordan matrix of the form:

$$A = \left(\sqrt{g(z)}, \ln g(z)\right) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 & 0 \\ \frac{3}{8} & 1 & 1 & 0 & 0 \\ \frac{13}{16} & \frac{47}{24} & \frac{3}{2} & 1 & 0 \\ \frac{195}{128} & \frac{47}{12} & \frac{91}{24} & 2 & 1 \\ & & \cdots & & \end{bmatrix}, \tag{20}$$

where $g(z)$ is the same generating function as (21). Then one can see that $D = AMA^{-1}$ or $DA = AM$. Hence each even (or odd, respectively) column of A is an eigenvector of the Riordan involution D corresponding to the eigenvalue 1 (or -1, respectively).

In [3], Cheon and Kim proved the existence for the Riordan matrix A such that $D = AMA^{-1}$ where D is a Riordan involution. We state it as Lemma 4.1. Further, they showed that if $D = (g(z), f(z))$ is a Riordan involution then $g(z)$ may be expressed by $g(z) = \exp[\Phi(z, f(z))]$ for some antisymmetric function $\Phi(x, z)$, i.e., $\Phi(x, z) = -\Phi(z, x)$.

Lemma 4.1 [3]. *Let $D = (g(z), f(z))$ be a Riordan involution. Then there exists the Riordan matrix A such that $D = AMA^{-1}$. In particular, the matrix A is of the form*

$$A = \left(\exp\left[\frac{\Phi(z, f(z))}{2}\right], \Phi(z, f(z))\right)$$

for some antisymmetric function $\Phi(x, z)$.

By Lemma 4.1, we have the following theorem.

Theorem 4.2. *Let $D = (g(z), f(z))$ be a Riordan involution and let $GF(\mathrm{x}_k)$ be the generating function of a λ-invariant sequence x_k for each $k = 0, 1, 2, \dots$. Then*

$$\mathrm{GF}(\mathrm{x}_k) = \exp\left[\frac{\Phi(x, f(x))}{2}\right](\Phi(x, f(x)))^k$$

for some antisymmetric function $\Phi(x, z)$.

Note that if we take $\Phi(x, z) = \ln\left|\frac{z}{x}\right|$ as an antisymmetric function then one can get the matrix A in (20) by Lemma 4.1.

5. Generalized RNA triangles

The RNA triangle given in (1) may be expressed by the Riordan matrix $D = (g(z), zg(z))$ where

$$g(z) = \frac{(1 - z + z^2) - \sqrt{(1 - z + z^2)^2 - 4z^2}}{2z^2}. \tag{21}$$

The simplest combinatorial description for $g(z)$ is that it is the generating function for the number of Motzkin paths with no consecutive UD steps, i.e. with no peaks.

In [2], Cameron and Nkwanta explored $D_n := (g(z)\left(\frac{1-z}{1-zg(z)}\right)^n, zg(z))$ as a generalization of the RNA triangle with $D_0 = D$, and gave combinatorial interpretations for D_1 and D_2. Further they claimed that D_n is a pseudo involution for $n \geqslant 0$, but this does not work for $n \geqslant 1$.

In this section, we generalize the RNA triangle in a similar spirit and some interesting combinatorial items emerge.

Theorem 5.1. *Let $g(z)$ be the same generating function as (21). Then for any generating function $G(z)$ with $G(0) \neq 0$,*

$$E_n := \left(g(z)\left(\frac{G(z)}{G(-zg(z))}\right)^n, zg(z)\right) \tag{22}$$

is a pseudo involution for each $n = 0, 1, 2, \dots$

Proof. First we claim that for any generating function $G(z)$ with $G(0) \neq 0$, $E_n M$ may be factored as

$$E_n M = (G(z), z)^n (g(z), -zg(z))(G(z), z)^{-n},$$

where $M = (1, -z)$. Indeed, we have

$$\begin{aligned}
E_n M &= \left(g(z)\left(\frac{G(z)}{G(-zg(z))}\right)^n, -zg(z)\right) \\
&= \left(g(z)(G(z))^n\left(\frac{1}{G(-zg(z))}\right)^n, -zg(z)\right) \\
&= (g(z)(G(z))^n, -zg(z))\left(\left(\frac{1}{G(z)}\right)^n, z\right) \\
&= ((G(z))^n, z)(g(z), -zg(z))\left(\left(\frac{1}{G(z)}\right)^n, z\right) \\
&= (G(z), z)^n (g(z), -zg(z))(G(z), z)^{-n}.
\end{aligned}$$

Since $(g(z), -zg(z))$ is a Riordan involution, clearly E_nM is Riordan involution. Hence the proof is completed. $\square$

For example, when $G(z) = 1 + z$ we obtain a pseudo involution:

$$E_n = \left(g(z)\left(\frac{1+z}{1-zg(z)}\right)^n, zg(z)\right).$$

We will soon look at E_1 and E_2 in detail but to connect with the bivariate antisymmetric functions we note that

$$\Phi_n(x,z) = \ln\left|\frac{z}{x}\left(\frac{1+x}{1+z}\right)^n\right|$$

is antisymmetric. Thus we have:

$$\begin{aligned}\exp[\Phi_n(z, -zg(z))] &= e^{\ln\left|\frac{-zg(z)}{z}\left(\frac{1+z}{1-zg(z)}\right)^n\right|} = e^{\ln\left|-g(z)\left(\frac{1+z}{1-zg(z)}\right)^n\right|}\\ &= g(z)\left(\frac{1+z}{1-zg(z)}\right)^n.\end{aligned}$$

In particular, for $n = 1, 2$ the matrix E_n has a combinatorial interpretation.

First let us consider the matrix E_1:

$$E_1 = \left(g(z)\left(\frac{1+z}{1-zg(z)}\right), zg(z)\right) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 3 & 1 & 0 & 0 & 0 & 0\\ 6 & 4 & 1 & 0 & 0 & 0\\ 13 & 10 & 5 & 1 & 0 & 0\\ 30 & 24 & 15 & 6 & 1 & 0\\ 71 & 59 & 40 & 21 & 7 & 1\\ & & \cdots & \cdots & & \end{bmatrix}.$$

For the sequence in the first column of E_1, see [A125267] in the Sloane's Encyclopedia of Integer Sequences. Let us consider walks using the steps $U = (1, 1)$, $L = (1, 0)$, and $D = (1, -1)$. We want these paths to start at the origin, not to go below the x-axis, and to have no peaks. We also let the level steps at height 0 be any of three colors. Finally consecutive level steps at height 0 must be distinct colors. Then $(E_1)_{n,k}$ is the number of such paths with n steps ending at height k.

A similar but more intricate combinatorial setting exists for E_2:

$$E_2 = \left(g(z)\left(\frac{1+z}{1-zg(z)}\right)^2, zg(z)\right) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 5 & 1 & 0 & 0 & 0 & 0\\ 15 & 6 & 1 & 0 & 0 & 0\\ 40 & 21 & 7 & 1 & 0 & 0\\ 105 & 62 & 28 & 8 & 1 & 0\\ 275 & 174 & 91 & 36 & 9 & 1\\ & & \cdots & \cdots & & \end{bmatrix}.$$

The same rules as in E_1 apply for heights above zero but at height zero the rules are as follows. The first level step can be any of five colors. The next step can be any of colors A, B, or C. After that an A or B step can be followed by A, B, or C. But a C step can only be followed by a C or a C* step. AC* step can be followed by any of A, B, or C. There also are no down steps from height 1 to height 0. It means that the first column of the matrix enumerates paths made uniquely by horizontal steps. The generating function for the level steps at height 0 is

$$g(z) = \frac{(1+z)^2}{1-3z+z^2}.$$

This sequence is [A054888] in the Sloane's Encyclopedia of Integer Sequences. It also counts layers for the hyperbolic tessellation by regular pentagons of angle $\pi/2$. See the EIS for more information and a link to a beautiful illustration due to Paolo Dominici.

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