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ON TOUCHARD POLYNOMIALS

Ourania CHRYSAPHINOU
Statistical Unit, University of Athens, Athens 621, Greece

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J. Touchard in his work on the cycles of permutations generalized the Bell polynomials in order to study some problems of enumeration of the permutations when the cycles possess certain properties.

In the present paper (considering Touchard's generalization) we introduce and study a class of related polynomials. An exponential generating function, recurrence relations and connections with other well-known polynomials are obtained. In special cases, relations with Stirling number of the first and second kind, as well as with other numbers recently studied are derived. Finally, a combinatorial interpretation is discussed.

1. The Touchard polynomials

Definition. The Touchard polynomials denoted by $T_{n,k} \equiv T_{n,k}(x_1, \ldots; y_1, \ldots)$, $n = 1, 2, \ldots, k = 0, 1, \ldots, n$ are defined by

$$T_{n,k} = \sum \frac{n!}{k_1! \cdots k_n!\, r_1! \cdots r_n!} \left(\frac{x_1}{1!}\right)^{k_1} \cdots \left(\frac{x_n}{n!}\right)^{k_n} \left(\frac{y_1}{1}\right)^{r_1} \cdots \left(\frac{y_n}{n!}\right)^{r_n} \tag{1.1}$$

where the summation is extended over all k_i, $r_i \geq 0$, $i = 1, \ldots, n$ such that $\sum_{i=1}^n k_i = k$, $\sum_{i=1}^n i(k_i + r_i) = n$, and $T_{0,0} = 1$.

It is evident from (1.1) that $T_{n,0} = Y_n(y_1, \ldots, y_n)$, where Y_n are the well-known Bell polynomials, $T_{n,n} = x_1^n = B_{n,n}$, where $B_{n,k}$ the partial Bell polynomials (Comtet [5, p. 133]).

Touchard [9], using a combinatorial interpretation of $T_{n,k}$, obtained the following exponential generating function (e.g.f.):

$$T_k(z) = \sum_{n=k}^{\infty} T_{n,k} \frac{z^n}{n!} = \frac{1}{k!} \{x(z)\}^k e^{y(z)} \tag{1.2}$$

where

$$x(z) = \sum_{j=1}^{\infty} x_j \frac{z^j}{j!}, \qquad y(z) = \sum_{j=1}^{\infty} y_j \frac{z^j}{j!}$$

Using (1.2) we get the generating function

$$T(z, u) = \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} T_{n,k} \frac{z^n}{n!} u^k = e^{ux(z)+y(z)} \tag{1.3}$$

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and the following recurrence relation:

$$T_{n+1,k}=\sum_{l=0}^{n}\binom{n}{l}\{x_{l+1}T_{n-\lambda,k-1}+y_{l+1}T_{n-\lambda,k}\} \tag{1.4}$$

which is useful for tabulation purposes.

2. Connection with other polynomials

Theorem 2.1. *The Touchard polynomials are related with the (exponential) Bell polynomials and the partial Bell polynomials as follows:*

$$\sum_{k=0}^{n} T_{n,k}(x_1,\ldots;y_1,\ldots)=Y_n(x_1+y_1,\ldots,x_n+y_n)$$
$$=\sum_{j=0}^{n}\binom{n}{j}Y_j(x_1,\ldots,x_j)Y_{n-j}(y_1,\ldots,y_{n-j}) \tag{2.1}$$

and

$$T_{n,k}(x_1,\ldots;y_1,\ldots)=\sum_{j=k}^{n}\binom{n}{j}B_{j,k}(x_1,\ldots,x_{j-k+1})Y_{n-j}(y_1,\ldots,y_{n-j}). \tag{2.2}$$

Proof. From the generating function (1.3) we have

$$T(z,u)=\exp\left\{\sum_{k=1}^{\infty}(ux_k+y_k)\frac{z^k}{k!}\right\}$$

or

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}T_{n,k}\frac{z^n}{n!}u^k=\sum_{n=0}^{\infty}Y_n(ux_1+y_1,\ldots,ux_n+y_n)\frac{z^n}{n!}$$

from which, taking $z=u=1$ and using a well-known result (Riordan [8, p. 45]), we get (2.1). (2.2) is derived by expanding the right-hand side of (1.2) and equating the coefficients of $z^n/n!$. □

Remark. For the particular values of $y_1=y_2=\cdots=0$ we obtain from (2.2) that

$$T_{n,k}(x_1,\ldots;0,\ldots)=B_{n,k}(x_1,\ldots) \tag{2.3}$$

and for the case of $y_1=y$, $y_k=0$, $k=2,3,\ldots$ we get

$$T_{n,k}(x_1,\ldots;y,0,\ldots)=\sum_{j=k}^{n}\binom{n}{j}y^{n-j}B_{j,k}. \tag{2.4}$$

Theorem 2.2. *The Touchard polynomials are related with the Rook polynomials as follows:*

$$T_{n,k}(1,2x,0,\ldots;1,0,\ldots)=\binom{n}{k}R_{n-k,k}(x),\quad n=1,2,\ldots,k=1,\ldots,n. \tag{2.5}$$

Proof. The e.g.f. of $T_{n,k}$ is written

$$T_k(z)=\frac{z^k}{k!}(1+xz)^k e^z. \tag{2.6}$$

Since (Riordan [8, p. 174]) the e.g.f. of the Rook polynomials is

$$\sum_{n=0}^{\infty} R_{n,k}(x)\frac{z^n}{n!}=(1+xz)^k e^z,$$

we have from (2.6) that

$$T_k(z)=\sum_{n=0}^{\infty} R_{n,k}\frac{z^{n+k}}{k!\,n!}=\sum_{n=k}^{\infty}\binom{n}{k}R_{n-k,k}\frac{z^n}{n!}$$

from which (2.5) can be derived. □

Remark. Since the Rook polynomials are related with the Laguerre polynomials (Riordan [8, p. 171]) we can get the relation

$$T_{n,k}(1,2x,0,\ldots;1,0,\ldots)=\frac{n!}{k!}x^{n-k}L_{n-k}^{2k-n}(-x^{-1}).$$

3. Some properties of the Touchard polynomials

Theorem 3.1. *The Touchard polynomials satisfy the following relations:*

$$kT_{n,k}=\sum_{r=k-1}^{n}\binom{n}{r}x_{n-r}T_{r,k-1}, \tag{3.1}$$

$$T_{n,k}\left(\frac{x_2}{2},\frac{x_3}{3},\ldots;y_1,\ldots\right)=\frac{n!}{(n+k)!}T_{n+k,k}(0,x_2,x_3,\ldots;y_1,\ldots), \tag{3.2}$$

$$\begin{aligned}T_{n,k}(x_1,\ldots;y_1,\ldots)&=\sum_{r=0}^{k}\binom{n}{r}x_1^rT_{n-r,k-r}(0,x_2,x_3,\ldots;y_1,\ldots)\\&=\sum_{r=0}^{k}\frac{n!}{r!\,(n-k)!}x_1^rT_{n-k,k-r}\left(\frac{x_2}{2},\frac{x_3}{3},\ldots;y_1,\ldots\right),\end{aligned} \tag{3.3}$$

$$T_{n,k}\left(\frac{x_{q+1}}{\binom{q+1}{q}},\frac{x_{q+2}}{\binom{q+2}{q}},\ldots;y_1,\ldots\right)$$

$$=\frac{(q!)^k n!}{(n+kq)!}T_{n+kq,k}(0,\ldots,0,x_{q+1},\ldots;y_1,\ldots), \tag{3.4}$$

$$T_{n,k}(x_1+x_1,\ldots;y_1+y_1,\ldots)$$

$$=\sum_{r=0}^{k}\sum_{s=r}^{n}\binom{n}{s}T_{s,r}(x_1,\ldots;y_1,\ldots)T_{n-s,k-r}(x_1,\ldots;y_1,\ldots). \tag{3.5}$$

Proof. Considering the e.g.f. of $kT_{n,k}$ we have

$$\sum_{n=k}^{\infty} kT_{n,k}\frac{z^n}{n!}=\{x(z)\}\left\{\frac{1}{(k-1)!}\,x(z)^{k-1}e^{y(z)}\right\}$$

or

$$\sum_{n=k}^{\infty} kT_{n,k}\frac{z^n}{n!}=\sum_{j=1}^{\infty} x_j\frac{z^j}{j!}\sum_{n=k-1}^{\infty} T_{n,k-1}\frac{z^n}{n!}$$
$$=\sum_{n=0}^{\infty}\sum_{r=k-1}^{n}\binom{n}{r}x_{n-r}T_{r,k-1}\frac{z^n}{n!}$$

from which (3.1) is derived.

(3.2) is obtained from the e.g.f.

$$T_k(z)=\sum_{n=k}^{\infty} T_{n,k}\left(\frac{x_2}{2},\frac{x_3}{3},\ldots;y_1,\ldots\right)\frac{z^n}{n!}=\frac{1}{k!}\left\{\frac{1}{z}\,(x(z)-x_1z)\right\}^k e^{y(z)}$$

or

$$T_k(z)=\frac{1}{z^k}\sum_{n=k}^{\infty} T_{n,k}(0,x_2,x_3,\ldots;y_1,\ldots)\frac{z^n}{n!}$$
$$=\sum_{n=k}^{\infty}\frac{n!}{(n+k)!}\,T_{n+k,k}(0,x_2,x_3,\ldots;y_1,\ldots)\frac{z^n}{n!}.$$

The e.g.f. (1.2) can be written

$$T_k(z)=\frac{1}{k!}\{x(z)\}^k e^{y(z)}=\frac{1}{k!}\left\{x_1z+\sum_{j=2}^{\infty} x_j\frac{z^j}{j!}\right\}^k e^{y(z)}$$

or

$$T_k(z)=\frac{1}{k!}\sum_{r=0}^{k}\binom{k}{r}x_1^r z^r\left\{\sum_{j=2}^{\infty} x_j\frac{z^j}{j!}\right\}^{k-r} e^{y(z)}$$
$$=\sum_{r=0}^{k}\frac{1}{r!}x_1^r z^r\sum_{n=k-r}^{\infty} T_{n,k-r}(0,x_2,\ldots;y_1,\ldots)\frac{z^n}{n!}$$
$$=\sum_{n=k}^{\infty}\sum_{r=0}^{k}\binom{n}{r}x_1^r T_{n-r,k-r}(0,x_2,\ldots;y_1,\ldots)\frac{z^n}{n!}.$$

Using (3.2) and the last relation we obtain (3.3).

Considering again the e.g.f.

$$T_k(z)=\sum_{n=k}^{\infty} T_{n,k}\left(\frac{x_{q+1}}{\binom{q+1}{q}},\frac{x_{q+2}}{\binom{q+2}{q}},\ldots;y_1,\ldots\right)\frac{z^n}{n!}$$
$$=\frac{(q!)^k}{z^{kq}}\frac{1}{k!}\left\{x(z)-\sum_{j=1}^{q} x_j\frac{z^j}{j!}\right\}^k e^{y(z)}$$

or

$$T_k(z)=\frac{(q!)^k}{z^{kq}}\sum_{n=k}^{\infty} T_{n,k}(0,\ldots,0,x_{q+1},\ldots;y_1,\ldots)\frac{z^n}{n!}$$

$$=(q!)^k\sum_{n=k}^{\infty}\frac{n!}{(n+kq)!}T_{n+kq,k}(0,\ldots,0,x_{q+1},\ldots;y_1,\ldots)\frac{z^n}{n!}$$

we get (3.4).

Finally, we have from (1.2)

$$T_k^*(z)=\sum_{n=k}^{\infty} T_{n,k}(x_1+x_1',\ldots;y_1+y_1',\ldots)\frac{z^n}{n!}$$

$$=\frac{1}{k!}\{x(z)+x'(z)\}^k e^{y(z)+y'(z)}$$

where

$$x'(z)=\sum_{j=1}^{\infty} x_j'\frac{z^j}{j!}\quad\text{and}\quad y'(z)=\sum_{j=1}^{\infty} y_j'\frac{z^j}{j!}.$$

Now, writing the above e.g.f. as follows:

$$T_k^*(z)=\frac{1}{k!}\sum_{r=0}^{k}\binom{k}{r}\{x(z)\}^r\{x'(z)\}^{k-r}e^{y(z)+y'(z)}$$

$$=\sum_{r=0}^{k} T_r(z)T_{k-r}'(z)$$

where

$$T_r(z)=\frac{1}{r!}\{x(z)\}^r e^{y(z)}\quad\text{and}\quad T_{k-r}'(z)=\frac{1}{(k-r)!}\{x'(z)\}^{k-r}e^{y'(z)}$$

we obtain (3.5). □

Theorem 3.2. *The Touchard polynomials are related with the Stirling numbers of the first and second kind for special values of x_j, y_j, $j=1,2,\ldots$ as follows:*

If $x_j=(j-1)!$ and $y_j=0$, $j=1,2,\ldots,$ then

$$T_{n,k}=|s(n,k)|. \tag{3.6}$$

If $x_j=(j-1)!$ and $y_j=(j-1)!$, $j=1,2,\ldots,$ then

$$T_{n,k}=|s(n+1,k+1)|. \tag{3.7}$$

If $x_j=(-1)^{j-1}(j-1)!$ and $y_j=(-1)^{j-1}(j-1)!$, $j=1,2,\ldots,$ then

$$T_{n,k}=s(n,k)+ns(n-1,k). \tag{3.8}$$

If $x_j=1$ and $y_j=0$, $j=1,2,\ldots,$ then

$$T_{n,k}=S(n,k). \tag{3.9}$$

If $x_j=1$, $j=1,2,\ldots$ and $y_1=1$, $y_j=0$, $j=2,3,\ldots,$ then

$$T_{n,k}(1,\ldots;1,0,\ldots)=S(n+1,k+1). \tag{3.10}$$

If $x_j=1$ *and* $y_j=1$, $j=1,2,\ldots$, *then*

$$T_{n,k}=\sum_{j=k}^{n}\binom{n}{j}S(j,k)\sum_{l=0}^{n-j}S(n-j,l). \tag{3.11}$$

Proof. Since $B_{n,k}(0!,1!,2!,\ldots)=|s(n,k)|$ (Comtet [5, p. 135]), (3.6) results immediately from (2.3).

If $x_j=(j-1)!$ and $y_j=(j-1)!$, $j=1,2,\ldots$ the e.g.f. (1.3) is written

$$T(z,u)=(1-z)^{-(u+1)}$$

or

$$\sum_{k=0}^{n}T_{n,k}u^k=(u+1)(u+2)\cdots(u+n)=\sum_{k=0}^{n}|s(n+1,k+1)|\,u^k$$

(Comtet [5, p. 214]) from which (3.7) is implied.

In the case of $x_j=(-1)^{j-1}(j-1)!$, $y_j=(-1)^{j-1}(j-1)!$, $j=1,2,\ldots$ the e.g.f (1.2) is written

$$T_k(z)=\frac{1}{k!}\{\log(1+z)\}^k(1+z)$$

or

$$T_k(z)=\sum_{n=k}^{\infty}s(n,k)\frac{z^n}{n!}+\sum_{n=k}^{\infty}s(n,k)\frac{z^{n+1}}{n!}$$

from which (3.8) results.

Since $B_{n,k}(1,1,\ldots)=S(n,k)$ (Comtet [5, p. 135]), (3.9) follows immediately from (2.3).

From (2.4) we have that

$$T_{n,k}(1,\ldots;1,0,\ldots)=\sum_{j=k}^{n}\binom{n}{j}B_{j,k}(1,\ldots)$$

or

$$T_{n,k}(1,\ldots;1,0,\ldots)=\sum_{j=k}^{n}\binom{n}{j}S(j,k)=S(n+1,k+1)$$

(Riordan [8, p. 43]).

Finally, for $x_j=1$, $y_j=1$, $j=1,2,\ldots$ the e.g.f. (1.2) is written

$$T_k(z)=\frac{1}{k!}\{e^z-1\}^k e^{e^z-1}$$

or

$$T_k(z)=\sum_{n=k}^{\infty}S(n,k)\frac{z^n}{n!}\sum_{n=0}^{\infty}Y_n(1,\ldots,1)\frac{z^n}{n!}$$

where $Y_n(1,\ldots,1)$ are the Bell numbers for which we know that $Y_n(1,\ldots,1)=\sum_{k=0}^{n}S(n,k)$. Thus the last relation (3.11) is derived. □

Theorem 3.3. *If* $x_j=(-1)^{j-1}(j-1)!$ *and* $y_j=(-1)^{j-1}a(j-1)!$, $j=1,2,\ldots$, *then*

$$T_{n,k}(0!\,-1!,\ldots;\alpha,-1!\,\alpha,\ldots)=s_a(n,k) \tag{3.12}$$

where $s_a(n,k)$ *are the non-central Stirling numbers of the first kind* (Koutras [7]).

If $x_j=1$, $j=1,2,\ldots$ *and* $y_1=-a$, $y_j=0$, $j=2,3,\ldots$, *then*

$$T_{n,k}(1,\ldots;-a,0,\ldots)=S_a(n,k) \tag{3.13}$$

where $S_a(n,k)$ *are the non-central Stirling numbers of the second kind* (Koutras [7]).

If $x_j=(r)_j$ *and* $y_j=(-1)^{j-1}s(j-1)!$, $j=1,2,\ldots$, *then*

$$T_{n,k}((r)_1,\ldots;s,-1!\,s,\ldots)=G(n,k;r,s) \tag{3.14}$$

where $G(n,k;r,s)$ *are the Gould–Hopper numbers* [6].

Proof. (3.12), (3.13), (3.14) result by comparing the e.g.f. (1.2) with the e.g.f's

$$\sum_{n=k}^{\infty} s_a(n,k)\frac{z^n}{n!}=\frac{1}{k!}\{\log(1+z)\}^k(1+z)^a,$$

$$\sum_{n=k}^{\infty} S_a(n,k)\frac{z^n}{n!}=\frac{1}{k!}\{e^z-1\}^k e^{-az}$$

and

$$\sum_{n=k}^{\infty} G(n,k;r,s)\frac{z^n}{n!}=\frac{1}{k!}\{(1+z)^r-1\}^k(1+z)^s$$

respectively (Koutras [7], Charalambides and Koutras [4]). ☐

Theorem 3.4. *If* $x_j=1$, $j=1,2,\ldots$ *and* $y_1=\lambda$, $y_j=0$, $j=2,3,\ldots$, *then*

$$T_{n,k}(1,\ldots;\lambda,0,\ldots)=R(n,k,\lambda) \tag{3.15}$$

where $R(n,k,\lambda)=\bar{S}(n,k+1,\lambda)+S(n,k)$ *and* $\bar{S}(n,k,\lambda)$ *are the weighted Stirling numbers of the second kind.*

If $x_j=(j-1)!$ *and* $y_j=\lambda(j-1)!$, $j=1,2,\ldots$, *then*

$$T_{n,k}(0!,1!,\ldots;0!\,\lambda,1!\,\lambda,\ldots)=R_1(n,k,\lambda) \tag{3.16}$$

where $R_1(n,k,\lambda)=\bar{S}_1(n,k+1,\lambda)+|s(n,k)|$ *and* $\bar{S}_1(n,k+1,\lambda)$ *are the weighted Stirling numbers of the first kind.*

Proof. Comparing the e.g.f (1.2) with

$$\sum_{n=k}^{\infty} R(n,k,\lambda)=\frac{1}{k!}(e^z-1)^k e^{\lambda z}$$

(Carlitz [3]) we get (3.15). The (3.16) results from (1.3) and that

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} R_1(n, k, \lambda) \frac{z^n}{n!} u^k = (1-z)^{-\lambda-u}$$

(Carlitz [3]). □

Remark. The definition of $T_{n,k}$ leads to new explicit forms for the numbers we considered in Theorems 3.3 and 3.4 as well as recurrence relations for these numbers. Finally, using Theorems 2.1 and 3.2 we can get some of the relations which have been proved by Carlitz [3] and Koutras [7]. For example,

$$\sum_{k=0}^{n} R(n, k, \lambda) = \sum_{k=0}^{n} T_{n,k}(1, \ldots; \lambda, 0, \ldots) = Y_n(1+\lambda, 1, 1, \ldots)$$

by Theorem 2.1. For $\lambda = 0$ we get the Bell number (cf. [3, 3.17]). For $\lambda = 1$ we have

$$R(n, k, 1) = T_{n,k}(1, \ldots; 1, 0, \ldots) = S(n+1, k+1)$$

by Theorem 3.2 (cf. [3, 3.16]).

The e.g.f of $R_1(n, k, \lambda) = T_{n,k}(0!, 1!, \ldots; 0!\,\lambda, 1!\,\lambda, \ldots)$ is written

$$\sum_{n=k}^{\infty} R_1(n, k, \lambda) \frac{z^n}{n!} = \frac{1}{k!} \{\log(1-z)^{-1}\}^k (1-z)^{-\lambda}.$$

The summation of these numbers is

$$\sum_{k=0}^{\infty} R_1(n, k, \lambda) = Y_n(0!\,(1+\lambda), 1!\,(1+\lambda), \ldots) = (1+\lambda)_n$$

by Theorem 2.1 (cf. [3, 5.7]).

For $\lambda = 1$ we have

$$R_1(n, k, 1) = |s(n+1, k+1)|$$

by Theorem 3.2 (cf. [3, 5.10]).

4. A combinatorial interpretation

We recall the combinatorial interpretation given by Touchard [9]: The number of permutations of n elements, say $u_1, \ldots, u_n$, in which exactly k cycles possess a property A and all the rest a property B is given by $T_{n,k}(x_1, \ldots; y_1, \ldots)$, where x_j, $j = 1, \ldots, n$ is the number of the cyclic permutations of a cycle of length j which possess the property A and y_j, $j = 1, \ldots, n$ is the number of the cyclic permutations of a cycle of length j which possess the property B, provided that any cycle of length j gives the same numbers x_j, y_j. In the sequel we give a special case of the above combinatorial interpretation. Let property A be as follows: the elements

$(u_{i_1}, \ldots, u_{i_j})$ of a cycle of length j, $j=1, \ldots, n$ appear such that $i_1 < i_2 < \cdots < i_j$, and property B: the elements $(u_{i_1}, \ldots, u_{i_j})$ of a cycle of length j, $j=1, 2, \ldots$ appear in any way except that of property A. Then obviously we have $x_j = 1$, $j=1, \ldots, n$ and $y_j = (j-1)! - 1$, $j=1, \ldots, n$ and the number of permutations of n elements in which exactly k cycles possess property A is given by

$$T_{n,k}(1, 1, \ldots; 0, 0, 1, \ldots, (j-1)!-1, \ldots). \tag{4.1}$$

To obtain an explicit formula for (4.1) we use the e.g.f. (1.2) which is written

$$\begin{aligned} T_k(z) &= \frac{1}{k!}\{e^z - 1\}^k \exp\{\log(1-z)^{-1} - (e^z - 1)\} \\ &= (1-z)^{-1} \sum_{n=0}^{\infty} (-1)^n \frac{(e^z-1)^{k+n}}{n!\,k!} \\ &= \sum_{n=0}^{\infty} z^n \sum_{n=0}^{\infty} (-1)^n \binom{n+k}{k} \sum_{l=n+k}^{\infty} S(l, n+k) \frac{z^l}{l!}. \end{aligned}$$

Therefore

$$\begin{aligned} &T_{n,k}(1, 1, \ldots; 0, 0, 1, \ldots, (j-1)!-1, \ldots) \\ &\quad = \sum_{r=0}^{n} (n)_r \sum_{l=0}^{n-r-k} (-1)^l \binom{l+k}{k} S(n-r, l+k) \end{aligned} \tag{4.2}$$

where $S(n, k)$ are the Stirling numbers of the second kind.

In the case of $k=n$ we have $T_{n,n}=1$, as it is expected, and in the case of $k=0$ that

$$T_{n,0} = Y_n(0, 0, 1, \ldots, (j-1)!-1, \ldots).$$

Using an analogous combinatorial interpretation for the Bell polynomials [9] we can conclude that the Bell number $Y_n(1, \ldots, 1)$ enumerates the permutations of n elements with all their cycles having property A.

Finally, comparing (4.2) with the well-known result that the Stirling numbers of the second kind $S(n, k)$ enumerate the permutations of n elements with k cycles when only those cycles with a specified order are permitted [8, p. 76], we could say that the special case we considered above consists a generalization.

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