ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 32, Number 2, Summer 2002

INVERSION TECHNIQUES AND COMBINATORIAL IDENTITIES: BALANCED HYPERGEOMETRIC SERIES

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Dedicated to my teacher L.C. Hsu on the occasion of his 80th birthday

ABSTRACT. Following the earlier works on *Inversion techniques and combinatorial identities*, the duplicate form of the Gould-Hsu [**18**] inversion theorem is constructed. As applications, several terminating balanced hypergeometric formulas are demonstrated, including those due to Andrews [**3**], which have been the primary stimulation to the present research. Encouraged by the recent work of Standon [**23**], we establish two higher hypergeometric evaluations with three additional parameters, which specialize further to over two hundred hypergeometric identities.

For a complex c and a natural number n, denote the rising shifted-factorial by

$$(c)_0 = 1, \quad (c)_n = c(c+1)\cdots(c+n-1), \quad n = 1, 2, \dots . \tag{0.1a}$$

Following Bailey [**8**], the hypergeometric series, for an indeterminate z and two nonnegative integers m and n, is defined by

$${}_{1+n}F_m \left[\begin{matrix} a_0, & a_1, & \cdots, & a_n \\ & b_1, & \cdots, & b_m \end{matrix} ; z \right] = \sum_{k=0}^{\infty} \frac{(a_0)_k (a_1)_k \cdots (a_n)_k}{k!(b_1)_k \cdots (b_m)_k} z^k, \tag{0.1b}$$

where $\{a_i\}$ and $\{b_j\}$ are complex parameters such that no zero factors appear in the denominators of the summands on the righthand side. When the variable $z = 1$, it will be omitted from the hypergeometric notation. If one of the numerator parameters $\{a_k\}$ is a negative integer, then the series becomes terminating, which reduces to a polynomial in z.

1991 AMS *Mathematics subject classification.* Primary 33C20 and Secondary 05A19.
Keywords and phrases. Hypergeometric series, duplicate inverse series relations.
Received by the editors on July 25, 2000, and in revised form on November 27, 2001.

When $m = n$ and $1+\sum a_k = \sum b_k$, we say that the series is balanced. Recently, Andrews [**3**] discovered the following balanced summation formulae:

(0.2)
$$ {}_5F_4\left[\begin{matrix} -1-2n, & 1+x+n, & x, & z, & \frac{1}{2}+x-z \\ & (x-n)/2, & (1+x-n)/2, & 2z, & 1+2x-2z \end{matrix}\right] \equiv 0, $$

which have an important application to the plane partition enumeration, (see Andrews and Stanton [**7**]). Up to now, there have been three proofs which appeared. The first one is due to Andrews [**3**] himself. To prove this result, he had to prove 20 identities simultaneously through recurrence relation method. The second proof is produced by Zeilberger [**14**] through the "infamous" WZ-method. The third one is due to Stanton [**23**], based on one of Bailey's cubic transformations. By means of inversion techniques, the fourth proof will be included in this paper.

In the development of inversion techniques [**9, 10, 11**], the author has demonstrated that most of the terminating hypergeometric identities are dual formulas of only three hypergeometric summation theorems, named after Chu-Vandermonde-Gauss, Pfaff-Saalschütz and Dougall-Dixon-Kummer. Continuing with our exploration to the power of inversion techniques, we will show that a large class of balanced hypergeometric evaluations, including (0.2) are dual relations of one due to Gessel and Stanton [**17**]. The last identity is in turn dual to the Dougall-Dixon formula, cf. Chu [**11**]. This fact has further strengthened my conviction that in the competition of identity-proving, the inversion techniques would open up "La Terza Via" (the third approach) between the classical series transformations (Pfaff-method) [**2, 3, 4, 5**] and the modern technological WZ-method [**14, 24, 25**].

In order to simplify the notation, the balanced hypergeometric series that appeared in Andrews' work [**3, 4, 5**] will be slightly modified and denoted by

(0.3a)
$$ H^{(\delta)}[a,b;c,d,e] = H^{(\delta)}[a,b;c,d,e \mid n,x,z] $$

(0.3b)
$$ = {}_5F_4\left[\begin{matrix} -\delta-2n, & a+x+n, & b+x, & z, & \frac{1}{2}+x-z \\ & d+2z, & e+2x-2z & \frac{c+x-n}{2}, & \frac{1+c+x-n-2\delta}{2}, \end{matrix}\right], $$

where $\delta = 0$ or 1 corresponds to the finite hypergeometric sums of even and odd terms, respectively. The advantage of this notation is that it expresses the balanced condition for the hypergeometric series as $1+a+b=c+d+e$. In addition, it possesses the following obvious, but useful property:

$$H^{(\delta)}[a,b;c,d,e]_{|z\to\frac{1}{2}+x-z} \Longrightarrow H^{(\delta)}[a,b;c,e-1,1+d]. \tag{0.4}$$

Then the Andrews' identity may be translated into $H^{(1)}[1,0;1,0.1]=0$.

As the fundamentals of the inversion techniques, the first section will introduce the Gould-Hsu inverse series relations and then construct their duplicate analogue. By means of the duplicate inversions, Section 2 will derive several balanced hypergeometric evaluations as dual formulas of the Gessel-Stanton identity [**17**]. The higher hypergeometric evaluations with additional parameters will be established in Section 3, where the other related topics will be discussed briefly.

1. Inverse series relations. For two complex variables x,y and four complex sequences $\{a_k,b_k,c_k,d_k\}_{k\ge 0}$, define two polynomial sequences by

(1.1a)
$$\phi(x;0)\equiv 1,\quad \phi(x;m)=\prod_{i=0}^{m-1}(a_i+xb_i),\quad m=1,2,\dots$$

(1.1b)
$$\psi(y;0)\equiv 1,\quad \psi(y;n)=\prod_{j=0}^{n-1}(c_j+yd_j),\quad n=1,2,\dots .$$

Then there is a celebrated pair of inverse series relations and its generalization.

Lemma 1 [**18**].

$$f(m)=\sum_{k=0}^{m}(-1)^k\binom{m}{k}\phi(k;m)g(k) \tag{1.2a}$$

$$g(m)=\sum_{k=0}^{m}(-1)^k\binom{m}{k}\frac{a_k+kb_k}{\phi(m;k+1)}f(k). \tag{1.2b}$$

Lemma 2 [**9, 11, 19**].

(1.3a)
$$F(m)=\sum_{k=0}^{m}(-1)^k\binom{m}{k}\phi(\lambda+k;m)\phi(-k;m)\frac{\lambda+2k}{(\lambda+m)_{k+1}}G(k)$$

(1.3b)
$$G(m)=\sum_{k=0}^{m}(-1)^k\binom{m}{k}\frac{a_k+(\lambda+k)b_k}{\phi(\lambda+m;1+k)}\frac{a_k-kb_k}{\phi(-m;k+1)}(\lambda+k)_m F(k).$$

Their applications to combinatorial identities and hypergeometric evaluations may be found in [**9, 10, 11**].

In order to adapt the Gould-Hsu inversions to the balanced hypergeometric series (0.3a)–(0.3b), here we present its duplicate form as follows.

Theorem 3 (Duplicate inverse series relations). *With* ϕ *and* ψ*-polynomials defined respectively by* (1.1a) *and* (1.1b)*, the system of equations*

(1.4a)
$$\Omega_n=\sum_{k\geq 0}\binom{n}{2k}\frac{c_k+2kd_k}{\phi(n;k)\psi(n;k+1)}f(k)$$

(1.4b)
$$-\sum_{k\geq 0}\binom{n}{1+2k}\frac{a_k+(1+2k)b_k}{\phi(n;1+k)\psi(n;k+1)}g(k)$$

is equivalent to the system of equations

(1.5a)
$$f(n)=\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\phi(k;n)\psi(k;n)\Omega_k$$

(1.5b)
$$g(n)=\sum_{k=0}^{1+2n}(-1)^k\binom{1+2n}{k}\phi(k;n)\psi(k;n+1)\Omega_k.$$

Proof. For an inverse pair of infinite upper triangular matrices

$$A = (a_{ij})_{0\le i\le j<\infty}$$
$$B = (b_{ij})_{0\le i\le j<\infty}$$

the system of equations

$$F(m) = \sum_{j=0}^{m} a_{jm}G(j), \quad m = 0, 1, 2, \ldots$$

is equivalent to the system

$$G(m) = \sum_{j=0}^{m} b_{jm}F(j), \quad m = 0, 1, 2, \ldots .$$

Therefore, to prove the equivalence between two systems of equations, it suffices to substitute one system into another and then verify the desired result.

Now substituting (1.4a) and (1.4b) into the righthand side of (1.5a), we have

$$S_f(n) - S_g(n) = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \phi(k; n)\psi(k; n)\Omega_k$$

where

$$\begin{aligned} S_f(n) &= \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \phi(k; n)\psi(k; n) \\ &\quad \times \sum_m \binom{k}{2m} \frac{c_m + 2md_m}{\phi(k; m)\psi(k; m+1)} f(m) \\ S_g(n) &= \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \phi(k; n)\psi(k; n) \\ &\quad \times \sum_m \binom{k}{1+2m} \frac{a_m + (1+2m)b_m}{\phi(k; 1+m)\psi(k; m+1)} g(m). \end{aligned}$$

The resulting double sums $S_f(n) - S_g(n)$ should reduce to $f(n)$. This can be accomplished by means of the finite difference method.

For $S_f(n)$, we may rewrite it by interchanging the summation order as

$$\begin{aligned}S_f(n) = \sum_{m=0}^{n} \binom{2n}{2m} \{c_m + 2md_m\} f(m) \\ \times \sum_{k=2m}^{2n} (-1)^k \binom{2n-2m}{k-2m} \frac{\phi(k;n)\psi(k;n)}{\phi(k;m)\psi(k;m+1)}.\end{aligned}$$

When $n > m$, the fraction $\phi(k;n)\psi(k;n)/\phi(k;m)\psi(k;m+1)$ is in fact a polynomial of degree $2n-1-2m$ in k. Therefore its divided difference of order $2(n-m)$ vanishes. This implies that the last sum with respect to k equals zero except for $m = n$, and so $S_f(n) \equiv f(n)$.

Following the same procedure, we assert that

$$\begin{aligned}S_g(n) = \sum_{m=0}^{n-1} \binom{2n}{1+2m} \{a_m + (1+2m)b_m\} g(m) \\ \times \sum_{k=1+2m}^{2n} (-1)^k \binom{2n-1-2m}{k-1-2m} \frac{\phi(k;n)\psi(k;n)}{\phi(k;1+m)\psi(k;m+1)}\end{aligned}$$

is identical with zero, i.e., $S_g(n) \equiv 0$. In conclusion, we have $S_f(n) - S_g(n) \equiv f(n)$.

Similarly, replacing Ω_k in (1.5b) by (1.4a) and (1.4b), we can demonstrate that the resulting double sums reduce to $g(n)$: □

Theorem 4 (the third relation). *For f, g and Ω defined in Theorem 3, let*

$$h(n) = f(n)\frac{(1+2n)\{a_n d_n - b_n c_n\}}{c_n + d_n(1+2n)} + g(n)\frac{a_n + b_n(1+2n)}{c_n + d_n(1+2n)}. \tag{1.6a}$$

Then we have a third relation

$$h(n) = \sum_{k=0}^{1+2n} (-1)^k \binom{1+2n}{k} \phi(k;1+n)\psi(k;n)\Omega_k \tag{1.6b}$$

and the corresponding dual formulae

$$\Omega_n = \sum_{k\geq 0}\binom{n}{2k}\frac{a_k+2kb_k}{\phi(n;1+k)\psi(n;k)}f(k) \tag{1.7a}$$

$$-\sum_{k\geq 0}\binom{n}{1+2k}\frac{c_k+(1+2k)d_k}{\phi(n;1+k)\psi(n;k+1)}h(k). \tag{1.7b}$$

Proof. Substituting (1.5a)–(1.5b) into (1.6a), we can manipulate the result as follows:

$$\begin{aligned}
h(n) &= f(n)\frac{(1+2n)\{a_nd_n-b_nc_n\}}{c_n+d_n(1+2n)}+g(n)\frac{a_n+b_n(1+2n)}{c_n+d_n(1+2n)}\\
&= \frac{(1+2n)\{a_nd_n-b_nc_n\}}{c_n+d_n(1+2n)}\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\phi(k;n)\psi(k;n)\Omega_k\\
&\quad+\frac{a_n+b_n(1+2n)}{c_n+d_n(1+2n)}\sum_{k=0}^{1+2n}(-1)^k\binom{1+2n}{k}\phi(k;n)\psi(k;n+1)\Omega_k\\
&= \sum_{k=0}^{1+2n}(-1)^k\binom{1+2n}{k}\phi(k;n)\psi(k;n)\Omega_k\Big/\{c_n+d_n(1+2n)\}\\
&\quad\times\{\{a_nd_n-b_nc_n\}(1+2n-k)+\{a_n+b_n(1+2n)\}(c_n+kd_n)\}.
\end{aligned}$$

Then the trivial factorization of the last line into $(a_n+kb_n)\{c_n+d_n(1+2n)\}$ leads us to (1.6b). The relation (1.4a)–(1.4b) becomes (1.7a)–(1.7b) under the exchange between $\phi(x;m)$ and $\psi(y;n)$. $\square$

From Theorems 3 and 4, if a known binomial relation fits into one of the two equation system (1.4a)–(1.4b) and (1.7a)–(1.7b), then we can invert it to get three dual relations determined by (1.5a), (1.5b) and (1.6b). That is the philosophy of inversion techniques [**9, 10, 11**]. Next it will be developed for obtaining terminating balanced hypergeometric summation formulas.

2. Gessel-Stanton and dual formulas. Denote the factorial-fractions by

$$\omega_n^{(\delta)}[a,b,c,d,e] = \frac{(\frac{1}{2}+\delta)_n(a+x-2z)_n(b-x+2z)_n}{(c-x)_n(\frac{1}{2}+d+z)_n(e+x-z)_n} \tag{2.1a}$$

$$\Omega_n[a,b;c,d,e] = \frac{(a+x)_n(b+x)_n(z)_n(\frac{1}{2}+x-z)_n}{(\frac{c+x}{2})_n(\frac{1+c+x}{2})_n(d+2z)_n(e+2x-2z)_n}. \tag{2.1b}$$

In Theorems 3 and 4, putting

$$\phi(\tau;m) = (a+x+\tau)_m \tag{2.2a}$$

$$\psi(\tau;m) = (1-c-x-2\tau)_m, \tag{2.2b}$$

then the solutions of the system of equations

$$\Omega_n[a,b;c,d,e] = \sum_{k\geq 0}\binom{n}{2k}\frac{(1-c-x-3k)\times f(k)}{(a+x+n)_k(1-c-x-2n)_{k+1}} \tag{2.3a}$$

$$-\sum_{k\geq 0}\binom{n}{1+2k}\frac{(1+a+x+3k)\times g(k)}{(a+x+n)_{1+k}(1-c-x-2n)_{k+1}} \tag{2.3b}$$

or the system

$$\Omega_n[a,b;c,d,e] = \sum_{k\geq 0}\binom{n}{2k}\frac{(a+x+3k)\times f(k)}{(a+x+n)_{1+k}(1-c-x-2n)_k} \tag{2.4a}$$

$$+\sum_{k\geq 0}\binom{n}{1+2k}\frac{(1+c+x+3k)\times h(k)}{(a+x+n)_{1+k}(1-c-x-2n)_{k+1}}, \tag{2.4b}$$

are given by the following binomial summations

$$f(n)=\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}(a+x+k)_n \times(1-c-x-2k)_n\Omega_k[a,b;c,d,e] \tag{2.5a}$$

$$g(n)=\sum_{k=0}^{1+2n}(-1)^k\binom{1+2n}{k}(a+x+k)_n \times(1-c-x-2k)_{n+1}[a,b;c,d,e] \tag{2.5b}$$

$$h(n)=\sum_{k=0}^{1+2n}(-1)^k\binom{1+2n}{k}(a+x+k)_{1+n} \times(1-c-x-2k)_n\Omega_k[a,b;c,d,e]. \tag{2.5c}$$

In terms of hypergeometric series, we may reformulate them as

$$H^{(0)}[a,b;c,d,e]=H^{(0)}[a,b;c,d,e\mid n,x,z]=\frac{f(n)}{(a+x)_n(1-c-x)_n} \tag{2.6a}$$

$$={}_5F_4\left[\begin{matrix}-2n, & a+x+n, & b+x, & z, & \frac{1}{2}+x-z\\ & \frac{c+x-n}{2}, & \frac{c+1+x-n}{2}, & d+2z & e+2x-2z\end{matrix}\right] \tag{2.6b}$$

$$\begin{aligned}H^{(1)}[a,b;c,d,e]&=H^{(1)}[a,b;c,d,e\mid n,x,z]\\&=\frac{g(n)}{(a+x)_n(1-c-x)_{n+1}}\end{aligned} \tag{2.6c}$$

$$={}_5F_4\left[\begin{matrix}-1-2n, & a+x+n, & b+x, & z, & \frac{1}{2}+x-z\\ & \frac{c+x-n}{2}, & \frac{c-1+x-n}{2}, & d+2z, & e+2x-2z\end{matrix}\right] \tag{2.6d}$$

$$\begin{aligned}H^{(1)}[1+a,b;1+c,d,e]&=H^{(1)}[1+a,b;1+c,d,e\mid n,x,z]\\&=\frac{h(n)}{(a+x)_{1+n}(1-c-x)_n}\end{aligned} \tag{2.6e}$$

$$(2.6\text{f}) \quad = {}_5F_4\left[\begin{matrix}-1-2n, & 1+a+x+n, & b+x, & z, & \frac{1}{2}+x-z \\ & \frac{c+x-n}{2}, & \frac{1+c+x-n}{2}, & d+2z, & e+2x-2z\end{matrix}\right]$$

where, with reference to (1.6a), the following three term relation holds
(2.7)

$$h(n) = f(n)\frac{(1+2n)(1+2a-c+x+3n)}{1+c+x+3n} - g(n)\frac{1+a+x+3n}{1+c+x+3n}.$$

If a binomial relation matching with (2.3a), (2.3b) or (2.4a), (2.4b) exists, we can get three dual identities through (2.6a), (2.6c) and (2.6e). Fortunately, one hypergeometric formula due to Gessel and Stanton [**17**] fits into our scheme. We may reproduce it, (see also [**11**]), as
(2.8a)

$${}_7F_6\left[\begin{matrix}1+\frac{2a}{3}, & 2a, 2b, 1-2b, & a-d, & \frac{1}{2}+a+d+m, -m \\ \frac{2a}{3}, & 1+a-b, \frac{1}{2}+a+b, & 1+2d, & 1+2a+2m, -2d-2m\end{matrix}\right]$$

$$(2.8\text{b}) \qquad = \frac{(1+2a)_{2m}}{(1+2d)_{2m}}\frac{(1+d-b)_m(\frac{1}{2}+d+b)_m}{(1+a-b)_m(\frac{1}{2}+a+b)_m},$$

whose reformulation, under parameter replacements $a \to x/2$, $b \to z - x/2$ and $d \to y - 2m - x/2$, reads as
(2.9a)

$${}_7F_6\left[\begin{matrix}1+\frac{2}{3}, 1+x-2z, 2z-x, x-y+2m, \frac{1}{2}+y-m, & x, -m \\ \frac{x}{3}, 1+x-z, \frac{1}{2}+z, x-2y+2m, 1+x+2m, & 1-x+2y-4m\end{matrix}\right]$$

$$(2.9\text{b}) \quad = \frac{(x-2y)_{2m}(1+x)_{2m}(z-y)_{2m}(\frac{1}{2}+x-y-z)_{2m}}{(x-2y)_{4m}(\frac{1}{2}+z)_m(z-y)_m(1+x-z)_m(\frac{1}{2}+x-y-z)_m}.$$

With $y \to -\delta$ and then $\delta + 2m \to n$, we may further specify it to
(2.10a)

$${}_6F_5\left[\begin{matrix}x, & 1+\frac{x}{3}, & 1+x-2z, 2z-x, & -\frac{n}{2}, & \frac{1-n}{2} & \\ & \frac{x}{3}, & 1+x-z, & \frac{1}{2}+z, & 1+x+n, & 1-x-2n\end{matrix}\right]$$

$$= \frac{(1+x)_n(x)_n(z)_n(\frac{1}{2}+x-z)_n}{(\frac{x}{2})_n(\frac{1+x}{2})_n(2z)_n(1+2x-2z)_n} \tag{2.10b}$$

which can be expressed in terms of (2.3a)–(2.3b) as a binomial identity

$$\sum_k \binom{n}{2k} \frac{-x(x+3k)}{(x+n)_{k+1}(-x-2n)_{1+k}} \frac{\frac{1}{2}{}_k(x)_k(1+x-2z)_k(2z-x)_k}{(\frac{1}{2}+z)_k(1+x-z)_k} \tag{2.11a}$$

$$= \frac{(x)_n(x)_n(z)_n(\frac{1}{2}+x-z)_n}{(\frac{1+x}{2})_n(\frac{2+x}{2})_n(2z)_n(1+2x-2z)_n}. \tag{2.11b}$$

This will be our starting point for attacking the balanced hypergeometric series.

2.1 *Case* [10101]. It is obvious that (2.11a)–(2.11b) reads also as

$$\Omega_n[1,0;1,0,1] = \sum_k \binom{n}{2k} \frac{-(x+3k)\times(x)_k(-x)_k}{(x+n)_k(-x-2n)_{k+1}} \omega_k^{(0)}[1,0,0,0,1],$$

which is the case $a=c=e=1$ and $b=d=0$ of (2.3a)–(2.3b) with

$$\begin{aligned} f(n) &= \omega_n^{(0)}[1,0,0,0,1]\times(x)_n(-x)_n \\ g(n) &= 0 \\ h(n) &= \omega_n^{(1)}[1,0,0,0,1]\times(x)_n(-x)_n, \end{aligned}$$

where the last evaluation is derived from the first two via (2.7). In view of (2.6a), (2.6c) and (2.6e), we may write down directly the following dual formulas:

$$H^{(0)}[1,0;1,0,1] = \frac{x}{x+n}\omega_n^{(0)}[1,0,0,0,1] \tag{2.12a}$$

$$H^{(1)}[1,0;1,0,1] = 0, \quad \text{[Andrews } \mathbf{3}\text{, (4.2)]} \tag{2.12b}$$

$$H^{(1)}[2,0;2,0,1] = \frac{x}{x+n}\frac{\omega_n^{(1)}[1,0,0,0,1]}{1+x+n}. \tag{2.12c}$$

2.2 *Case* [00101]. Splitting the factor

$$x + 3k = (x + n + k) - (n - 2k)$$

we may restate (2.11a)–(2.11b) as

$$\begin{aligned}\Omega_n[0,0;1,0,1] = \sum_k \binom{n}{2k} \frac{-(x+3k) \times f(k)}{(x+n)_k(-x-2n)_{k+1}} \\ - \sum_k \binom{n}{1+2k} \frac{(1+x+3k) \times g(k)}{(x+n)_{1+k}(-x-2n)_{k+1}},\end{aligned}$$

which corresponds to the case $a = b = d = 0$ and $c = e = 1$ of (2.3a)–(2.3b) with

$$f(n) = x\frac{(x)_n(-x)_n}{x+3n}\omega_n^{(0)}[1,0,0,0,1] \tag{2.13a}$$

$$g(n) = -x\frac{(x)_n(-x)_n}{1+x+3n}\omega_n^{(1)}[1,0,0,0,1] \tag{2.13b}$$

$$h(n) = 2x\frac{(x)_n(-x)_n}{2+x+3n}\omega_n^{(1)}[1,0,0,0,1], \tag{2.13c}$$

where the last evaluation is obtained from the first two via (2.7). In view of (2.6a), (2.6c) and (2.6e), we may have the following two-balanced formulas:

(2.14a)
$$H^{(0)}[0,0;1,0,1] = \frac{x}{x+3n}\omega_n^{(0)}[1,0,0,0,1]$$

(2.14b)
$$H^{(1)}[0,0;1,0,1] = \frac{1}{1+x+3n}\omega_n^{(1)}[1,0,1,0,1]$$

(2.14c)
$$H^{(1)}[1,0;2,0,1] = \frac{2x}{2+x+3n}\frac{\omega_n^{(1)}[1,0,0,0,1]}{(x+n)}.$$

2.3 *Case* [01101]. According to the factor-splitting

$$x + 3k = \frac{x+2k}{x+n}\{x+n+k\} + \frac{k}{x+n}\{n-2k\},$$

the relation (2.11a)–(2.11b) may be expressed as

$$\Omega_n[0,1;1,0,1] = \sum_k \binom{n}{2k} \frac{-(x+3k)\times f(k)}{(x+n)_k(-x-2n)_{k+1}} \\ - \sum_k \binom{n}{1+2k} \frac{(1+x+3k)\times g(k)}{(x+n)_{1+k}(-x-2n)_{k+1}},$$

which corresponds to the case $a = d = 0$ and $b = c = e = 1$ of (2.3a)–(2.3b) with

(2.15a)
$$f(n) = \frac{x+2n}{x+3n}\omega_n^{(0)}[1,0,0,0,1](x)_n(-x)_n$$
(2.15b)
$$g(n) = \frac{n}{1+x+3n}\omega_n^{(1)}[1,0,0,0,1](x)_n(-x)_n$$
(2.15c)
$$h(n) = \frac{x+n}{2+x+3n}\omega_n^{(1)}[1,0,0,0,1](x)_n(-x)_n,$$

where the last evaluation is obtained from the first two via (2.7). In view of (2.6a), (2.6c) and (2.6e), we may have the following balanced formulas:

(2.16a)
$$H^{(0)}[0,1;1,0,1] = \frac{x+2n}{x+3n}\omega_n^{(0)}[1,0,0,0,1]$$
(2.16b)
$$H^{(1)}[0,1;1,0,1] = \frac{-n/x}{1+x+3n}\omega_n^{(1)}[1,0,1,0,1]$$
(2.16c)
$$H^{(1)}[1,1;2,0,1] = \frac{1}{2+x+3n}\omega_n^{(1)}[1,0,0,0,1].$$

2.4 *Case* [00001]. By means of the factor-splitting

$$x+3k = -\frac{x-1+3k}{1-x-2n+k}\{x+n+k\} - \frac{x+1+3}{1-x-2n+k}\{n-2k\},$$

we may rewrite (2.11a)–(2.11b) as follows:

$$\Omega_n[0,0;0,0,1] = \sum_k \binom{n}{2k} \frac{(1-x-3k)\times f(k)}{(x+n)_k(1-x-2n)_{k+1}} - \sum_k \binom{n}{1+2k} \frac{(1+x+3k)\times g(k)}{(x+n)_{1+k}(1-x-2n)_{k+1}},$$

which corresponds to the case $a=b=c=d=0$ and $e=1$ of (2.3a)–(2.3b) with

$$f(n) = \omega_n^{(0)}[1,0,0,0,1] \times (x)_n(-x)_n$$
$$g(n) = \omega_n^{(1)}[1,0,0,0,1] \times (x)_n(-x)_n.$$

In view of (2.6a) and (2.6c), we may have the following balanced formulas:

(2.17a)
$$H^{(0)}[0,0;0,0,1] = \omega_n^{(0)}[1,0,1,0,1], \quad [\mathbf{3}]$$
(2.17b)
$$H^{(1)}[0,0;0,0,1] = \omega_n^{(1)}[1,0,2,0,1]/(1-x),$$

where the evaluation derived from the third relation (2.6e) is identical with (2.12b) and so has been omitted.

2.5 *Stanton* [**23**]. Now we consider a more general case. For a complex number α, the factor-splitting

$$x+3k = \frac{\alpha+2k}{\alpha+n}\{x+n+k\} - \frac{\alpha-x-k}{\alpha+n}\{n-2k\}$$

leads us to reformulating the relation (2.11a)–(2.11b) as

(2.18a)
$$\frac{\alpha+n}{\alpha}\Omega_n[0,0;1,0,1] = \sum_k \binom{n}{2k} \frac{-(x+3k)\times f(k)}{(x+n)_k(-x-2n)_{k+1}}$$
(2.18b)
$$- \sum_k \binom{n}{1+2k} \frac{(1+x+3k)\times g(k)}{(x+n)_{1+k}(-x-2n)_{k+1}},$$

with

$$\text{(2.19a)} \quad f(n) = \frac{x(\alpha+2n)}{\alpha(x+3n)}\omega_n^{(0)}[1,0,0,0,1](x)_n(-x)_n$$

$$\text{(2.19b)} \quad g(n) = \frac{x(x-\alpha+n)}{\alpha(1+x+3n)}\omega_n^{(1)}[1,0,0,0,1](x)_n(-x)_n.$$

Comparing (2.18a)–(2.18b) with (1.4a)–(1.4b), we may derive from (1.5a)–(1.5b) the dual relations:

(2.20a)

$$f(n) = \sum_{k=0}^{2n}(-1)^k \binom{2n}{k} (x+k)_n(-x-2k)_n \frac{\alpha+k}{\alpha}\Omega_k[0,0;1,0,1]$$

(2.20b)

$$g(n) = \sum_{k=0}^{1+2n}(-1)^k \binom{1+2n}{k} (x+k)_n(-x-2k)_{n+1} \frac{\alpha+k}{\alpha}\Omega_k[0,0;1,0,1].$$

According to Corollary 4, we also get the third relation as follows:

(2.21a)

$$h(n) = \frac{x(2\alpha-x+n)}{\alpha(2+x+3n)}(x)_n(-x)_n\omega_n^{(1)}[1,0,0,0,1]$$

(2.21b)

$$= \sum_{k=0}^{1+2n}(-1)^k \binom{1+2n}{k} (x+k)_{1+n}(-x-2k)_n \frac{\alpha+k}{\alpha}\Omega_k[0,0;1,0,1].$$

Then (2.19a) = (2.20a), (2.19b) = (2.20b) and (2.21a) = (2.21b) read respectively as the following hypergeometric identities due to Stanton [**23**, Theorem 1]:

$$\text{(2.22a)} \qquad {}_6F_5\left[\begin{matrix} -2n, & 1+\alpha, & x+n, & x, & z, & \frac{1}{2}+x-z \\ & \alpha, & \frac{1+x-n}{2}, & \frac{2+x-n}{2}, & 2z, & 1+2x-2z \end{matrix}\right]$$

$$\text{(2.22b)} \qquad = \omega_n^{(0)}[1,0,0,0,1] \times \frac{x(\alpha+2n)}{\alpha(x+3n)} \quad [\mathbf{23}]$$

$$(2.22c)\qquad {}_6F_5\left[\begin{matrix}-1-2n, & 1+\alpha, & x+n, & x, & z, & \frac{1}{2}+x-z\\ & \alpha, & \frac{x-n}{2}, & \frac{1+x-n}{2}, & 2z, & 1+2x-2z\end{matrix}\right]$$

$$(2.22d)\qquad = \omega_n^{(1)}[1,0,1,0,1] \times \frac{\alpha-x-n}{\alpha(1+x+3n)} \quad [\mathbf{23}, \text{A2}]$$

$$(2.22e)\qquad {}_6F_5\left[\begin{matrix}-1-2n, & 1+\alpha, & 1+x+n, & x, & z, & \frac{1}{2}+x-z\\ & \alpha, & \frac{1+x-n}{2}, & \frac{2+x-n}{2}, & 2z, & 1+2x-2z\end{matrix}\right]$$

$$(2.22f)\qquad = \frac{\omega_n^{(1)}[1,0,0,0,1]}{x+n} \times \frac{x(2\alpha-x+n)}{\alpha(2+x+3n)} \quad [\mathbf{23}, \text{A3}],$$

which may be considered as linear combinations of any two cases among [10101], [01101], [00101] and [00001] demonstrated previously.

As indicated by Stanton, they include nine hypergeometric identities as their limiting cases. Among them, the first four cases have been demonstrated in detail.

$$\begin{aligned} \alpha = x+n &\Longrightarrow [10101]: \ \text{(see Case 2.1)}\\ \alpha \to \infty &\Longrightarrow [00101]: \ \text{(see Case 2.2)}\\ \alpha = x &\Longrightarrow [01101]: \ \text{(see Case 2.3)}\\ \alpha = \frac{x-n}{2} &\Longrightarrow [00001]: \ \text{(see Case 2.4).}\end{aligned}$$

The remaining five cases may be displayed as follows:

2.6 *Case* [00100]. $\alpha = 2x-2z$:

(2.23a)

$$H^{(0)}[0,0;1,0,0] = \frac{x}{x+3n}\omega_n^{(0)}[1,0,0,0,0]$$

(2.23b)

$$H^{(1)}[0,0;1,0,0] = \frac{x-2z}{1+x+3n}\,\frac{\omega_n^{(1)}[1,1,1,0,1]}{2(x-z)}$$

(2.23c)

$$H^{(1)}[1,0;2,0,0] = \frac{x(3x-4z+n)}{2+x+3n}\,\frac{\omega_n^{(1)}[1,0,0,0,1]}{2(x+n)(x-z)}.$$

The identity (2.23b) in the middle was discovered by Andrews [**3**].

2.7 *Case* [001-11]. $\alpha = 2z - 1$:

$$H^{(0)}[0,0;1,-1,1] = \frac{x}{x+3n}\omega_n^{(0)}[1,0,0,-1,1] \tag{2.24a}$$

$$H^{(1)}[0,0;1,-1,1] = \frac{1+x-2z}{1+x+3n}\frac{\omega_n^{(1)}[2,0,1,0,1]}{1-2z} \tag{2.24b}$$

$$H^{(1)}[1,0;2,-1,1] = \frac{x(2+x-4z-n)}{2+x+3n}\frac{\omega_n^{(1)}[1,0,0,0,1]}{(1-2z)(x+n)}. \tag{2.24c}$$

The identity (2.24b) in the middle is due to Andrews [**3**].

2.8 *Case* [-1-100-1]. $\alpha = 1/2 + x - z \mid x \to x - 1$:

(2.25a)

$$H^{(0)}[-1,-1;0,0,-1] = \omega_n^{(0)}[0,1,1,0,0] \times \frac{(x-1)(1-2x+2z-4n)}{(x-1+3n)(1-2x+2z)}$$

(2.25b)

$$H^{(1)}[-1,-1;0,0,-1] = \omega_n^{(1)}[0,1,2,0,0] \times \frac{(2z-1+2n)}{(2z+1-2x)(x+3n)}$$

(2.25c)

$$H^{(1)}[0,-1;1,0,-1] = \frac{\omega_n^{(1)}[1,1,1,0,0]}{x-1+n} \times \frac{2(1-x)(x-2z)}{(1+x+3n)(1-2x+2z)},$$

where the last identity (2.25c) is due to Andrews [**3**].

2.9 *Case* [-1-10-21]. $\alpha = z \mid x \to x-1,\ z \to z-1$:

(2.26a)

$$H^{(0)}[-1,-1;0,-2,1] = \omega_n^{(0)}[2,-1,1,-1,1] \times \frac{(1-x)(z-1+2n)}{(1-z)(x-1+3n)}$$

(2.26b)

$$H^{(1)}[-1,-1;0,-2,1] = \omega_n^{(1)}[2,-1,2,-1,0] \times \frac{x-z}{(1-z)(x+3n)}$$

(2.26c)

$$H^{(1)}[0,-1;1,-2,1] = \frac{\omega_n^{(1)}[2,0,1,-1,1]}{1-x-n} \times \frac{(1-x)(1+x-2z)}{(1-z)(1+x+3n)}.$$

2.10 *Case* [-1-10-10] and [0-11-10]. $\alpha \to 0 \mid x \to x-2,\ z \to z-1$:

(2.27a)
$$H^{(0)}[-1,-1;0,-1,0] = \omega_n^{(0)}[1,0,1,-1,0] \times \tfrac{x-1}{x-1+3n}$$
(2.27b)
$$H^{(0)}[0,-1;1,-1,0] = \tfrac{\omega_n^{(0)}[1,0,0,-1,0]}{x+3n} \times \tfrac{x(1-x)(2-x+n)}{(1-x-n)(2-x-n)}$$
(2.27c)
$$H^{(1)}[0,-1;1,-1,0] = \tfrac{\omega_n^{(1)}[2,1,1,0,1]}{1+x+3n} \times \tfrac{(1-x)(x-2z)(1+x-2z)}{(1-x-n)(x-z)(1-2z)}$$
$$H^{(1)}[-1,-1;0,-1,0] = \omega_n^{(1)}[1,0,2,0,1]$$
(2.27d)
$$\times \frac{\left\{\begin{array}{c}(1+x-2z+n)(x-2z-n)\\ -(x-z+n)(1-2z-2n)\end{array}\right\}}{(1-2z)(x-z)(x+3n)}$$
(2.27e)
$$H^{(1)}[1,-1;2,-1,0] = \omega_n^{(1)}[1,0,0,0,1] \times \tfrac{x(1-x)}{2+x+3n}$$
(2.27f)
$$\times \frac{\left\{\begin{array}{c}(x-z+n)(1-2z-2n)(3x+n-6)\\ (1-x-n)(2-x-n)(x+3n)\end{array}\right\}}{(1-2z)(x-z)(-x-n)_3}$$

Among the identities just displayed, (2.27a) and (2.27c) are in the list of 20 identities by Andrews [**3**].

Besides these nine cases, there is the tenth case, which specifies $\alpha = -\delta - 2n$. However, the resulting identities are included in Section 2.1 and 2.4.

3. Further development. Let α, β and γ be three complex parameters. Similar to (0.3a), (0.3b), further hypergeometric series with additional parameters are denoted as follows:

(3.1a)
$$H_\alpha^{(\delta)}[a,b;c,d,e] = H_\alpha^{(\delta)}[a,b;c,d,e \mid n,x,z]$$
(3.1b)
$$= {}_6F_5\left[\begin{array}{cccccc} -\delta-2n, & 1+\alpha, & a+x+n, & b+x, & z, & \frac{1}{2}+x-z \\ \alpha, & \frac{c+x-n}{2}, & \frac{1+c+x-n-2\delta}{2} & d+2z, & e+2x-2z & , \end{array}\right]$$

(3.2a)

$$H^{(\delta)}_{\alpha\gamma}[a,b;c,d,e] = H^{(\delta)}_{\alpha\gamma}[a,b;c,d,e \mid n,x,z]$$

(3.2b)

$$= {}_7F_6\left[\begin{matrix} -\delta-2n, & 1+\alpha, & 1+\gamma, & a+x+n, & b+x, & z, & \frac{1}{2}+x-z \\ & \alpha, & \gamma, & \frac{c+x-n}{2}, & \frac{1+c+x-n-2\delta}{2} & d+2z, & e+2x-2z, \end{matrix}\right]$$

(3.3a)

$$H^{(\delta)}_{\alpha\beta\gamma}[a,b;c,d,e] = H^{(\delta)}_{\alpha\beta\gamma}[a,b;c,d,e \mid n,x,z]$$

(3.3b)

$$= {}_8F_7\left[\begin{matrix} -\delta-2n, & 1+\alpha, & 1+\beta, & 1+\gamma, & a+x+n, & b+x, & z, & \frac{1}{2}+x-z \\ & \alpha, & \beta, & \gamma, & \frac{c+x-n}{2}, & \frac{1+c+x-n-2\delta}{2}, & d+2z, & e+2x-2z \end{matrix}\right]$$

3.1 *Hypergeometric evaluations with three additional parameters.* The hypergeometric series $H^{(\delta)}_{\alpha\beta\gamma}[a,b;c,d,e]$ may be considered as a function in α,β and γ. Then it is easy to see that $(\alpha\beta\gamma)H^{(\delta)}_{\alpha\beta\gamma}[a,b;c,d,e]$ is symmetric in α,β,γ and linear with respect to each of them. Therefore, there is a linear function

$$\mathcal{L}(\alpha,\beta,\gamma) = H^{(\delta)}_{\alpha\beta\gamma}[a,b;c,d,e] = A + \frac{B}{\alpha\beta\gamma} + C\frac{\alpha+\beta+\gamma}{\alpha\beta\gamma} + D\frac{\alpha\beta+\alpha\gamma+\beta\gamma}{\alpha\beta\gamma}$$

with A,B,C,D to be determined.

By means of contiguous recurrence relations satisfied by hypergeometric series, Stanton [**23**, Theorem 2] derived an expression for $H^{(1)}_{\alpha\beta\gamma}[1,0;2,2,1]$ in terms of lower hypergeometrics. Here we will establish two explicit ${}_8F_7$-formulas:

Theorem 5 (Two ${}_8F_7$-summation formulas).

$$[\mathbf{A}] \quad H^{(0)}_{\alpha\beta\gamma}[0,0;1,1,2] = \frac{2nxz(1+2x-2z)\omega^{(0)}_n[0,-1,0,0,1]}{\alpha\beta\gamma(x+3n)(x-2z)^2(1+x-2z)^2}\mathcal{U}(\alpha,\beta,\gamma)$$

$$[\mathbf{B}] \quad H^{(1)}_{\alpha\beta\gamma}[1,1;2,2,2] = \frac{z(1+x+n)(1+2x-2z)\omega^{(1)}_n[0,0,0,1,2]}{2\alpha\beta\gamma(1+z)(1+x-z)(2+x+3n)(x-2z)^2}\mathcal{V}(\alpha,\beta,\gamma)$$

where $\mathcal{U}$ and $\mathcal{V}$ are symmetric polynomials in α, β and γ given, respectively, by

$$\begin{aligned}
&\mathcal{U}(\alpha,\beta,\gamma)\\
&= \tfrac{\alpha\beta\gamma}{n}\Big\{\tfrac{(1+x)(x+3n)(x-2z+n)(1+x-2z-n)}{2z(1+2x-2z)} - (x-2z)(1+x-2z)\Big\}\\
&\quad - (\alpha\beta+\alpha\gamma+\beta\gamma)\{(x+n)(3+2x-3n) - 4z(1+2x-2z)\}\\
&\quad + (\alpha+\beta+\gamma)\{x(x+n)(2+x-3n) - 2z(x-n)(1+2x-2z)\}\\
&\quad + \{x(x+n)(n-x+4nx) - 2nz(1+2x-2z)(3x+n)\}
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{V}(\alpha,\beta,\gamma)\\
&= \frac{2\alpha\beta\gamma}{1+x+n}\Big\{\tfrac{(2x-3z)(x-2z+n)(1+x+n)(2+x+3n)}{2z(1+2x-2z)} - (1+z)(x-2z)\Big\}\\
&\quad - (\alpha\beta+\alpha\gamma+\beta\gamma)\{(x-2z)(x-2z+3n) + n(2+x+3n)\}\\
&\quad + (\alpha+\beta+\gamma)\{(1+2z)(x-2z+n)(2+x+3n)\\
&\quad - 2(1+z)(x-2z)(1+x+n)\} + \{(1+2n)(x-2z)\\
&\qquad\qquad \times (2z-x+n+4nz) - n(1+2z)(1+2x-2z)(2+x+3n)\}.
\end{aligned}$$

Proof. For $H^{(0)}_{\alpha\beta\gamma}[0,0,1,1,2]$, we have the following specific examples

$$\begin{aligned}
\mathcal{L}(x{+}n, 2z, 1{+}2x{-}2z) &= H^{(0)}[1,0;1,0,1 \mid n,x,z]\\
\mathcal{L}(-2n, 2z, 1{+}2x{-}2z) &= H^{(1)}[1,0;1,0,1 \mid n{-}1,x,z]\\
\lim_{\alpha\to 0}\alpha\mathcal{L}(\alpha,x,-2n) &= H^{(0)}[0,0;0,0,1 \mid n{-}1,2{+}x,1{+}z]\\
&\quad\times\frac{(1{-}2n)z(1{+}x)(x{+}n)(1{+}2x{-}2z)}{(1{+}x{-}n)(2{+}x{-}n)(1{+}2z)(1{+}x{-}z)}\\
\lim_{\alpha\to 0}\alpha\mathcal{L}(-\alpha,x,\infty) &= H^{(1)}[0,0;1,0,1 \mid n{-}1,2{+}x,1{+}z]\\
&\quad\times\frac{2nz(1{+}x)(x{+}n)(1{+}2x{-}2z)}{(1{+}x{-}n)(2{+}x{-}n)(1{+}2z)(1{+}x{-}z)}.
\end{aligned}$$

Replacing the right members by their explicit expressions and then resolving the system of equations, we get the first hypergeometric identity stated in the theorem.

Similarly, we may specify $H^{(1)}_{\alpha\beta\gamma}[1,1;2,2,2]$ to the previously established hypergeometric evaluations as follows:

$$\begin{aligned}
\mathcal{L}(2z, 1+2z, 1+2x-2z) &= H^{(1)}[1,1;2,0,1 \mid n, x, z]\\
\mathcal{L}(2z, 1+2z, x-z+1/2) &= H^{(1)}[0,0;1,0,0 \mid n, 1+x, z]\\
\mathcal{L}(-1-2n, z, 1+2x-2z) &= H^{(0)}[0,0;0,0,1 \mid n, 1+x, 1+z]\\
\mathcal{L}(1+x+n, z, 1+2x-2z) &= H^{(1)}[1,0;1,0,1 \mid n, 1+x, 1+z],
\end{aligned}$$

whose solution leads us to the second formulae stated in the theorem. □

From Theorem 5 we can derive $20 = 2\binom{10}{1}$ hypergeometric evaluations for ${}_7F_6$-series with two additional parameters; $110 = 2\binom{11}{2}$ for ${}_6F_5$-series with one additional parameter; and $440 = 2\binom{12}{3}$ for ${}_5F_4$-series. For the limit of space, one selection about two hundreds of such identities will appear elsewhere.

3.2 *Null q-analogue.* With ϕ and ψ-polynomials defined respectively by (1.1a) and (1.1b), we can establish, by means of finite q-differences, the q-analogues of Theorem 3.

Theorem 6 (q-duplicate inverse series relations). *The system of equations*

$$\omega(n) = \sum_{k\geq 0} \begin{bmatrix} n \\ 2k \end{bmatrix} \frac{c_k + q^{2k} d_k}{\phi(q^n;k)\psi(q^n;k+1)} f(k) \tag{3.4}$$

$$- \sum_{k\geq 0} \begin{bmatrix} n \\ 1+2k \end{bmatrix} \frac{a_k + q^{1+2k} b_k}{\phi(q^n;1+k)\psi(q^n;k+1)} g^{(k)} \tag{3.5}$$

is equivalent to the system

(3.6)
$$f(n) = \sum_{k=0}^{2n} (-1)^k \begin{bmatrix} 2n \\ k \end{bmatrix} q^{\binom{2n-k}{2}} \phi(q^k; n)\psi(q^k; n)\omega(k)$$
(3.7)
$$g(n) = \sum_{k=0}^{1+2n} (-1)^k \begin{bmatrix} 1+2n \\ k \end{bmatrix} q^{\binom{1+2n-k}{2}} \phi(q^k; n)\psi(q^k; n+1)\omega(k).$$

Replacing q by its inverse, we may reformulate the theorem as:

Theorem 7 (q-duplicate inverse series relations). *The system of equations*

(3.8)
$$\Omega(n) = \sum_{k\geq 0} q^{\binom{n-2k}{2}} \begin{bmatrix} n \\ 2k \end{bmatrix} \tfrac{c_k + q^{-2k} d_k}{\phi(q^{-n}; k)\psi(q^{-n}; k+1)} F(k)$$
(3.9)
$$-\sum_{k\geq 0} q^{\binom{n-1-2k}{2}} \begin{bmatrix} n \\ 1+2k \end{bmatrix} \tfrac{a_k + q^{-1-2k} b_k}{\phi(q^{-n}; 1+k)\psi(q^{-n}; k+1)} G(k)$$

is equivalent to the system

(3.10)
$$F(n) = \sum_{k=0}^{2n} (-1)^k \begin{bmatrix} 2n \\ k \end{bmatrix} \phi(q^{-k}; n)\psi(q^{-k}; n)\Omega(k)$$
(3.11)
$$G(n) = \sum_{k=0}^{1+2n} (-1)^k \begin{bmatrix} 1+2n \\ k \end{bmatrix} \phi(q^{-k}; n)\psi(q^{-k}; n+1)\Omega(k).$$

Unfortunately, we have not found the q-counterpart of Gessel-Stanton (2.8a)–(2.8b) which fits in our q-inverse series relations.

3.3 *Multiplicate inversions.* Based on the double sections of natural numbers, we derived the duplicate inversion Theroem 3. It may be extended to the multi-sections of natural numbers. For this reason, let

$$\{A_{ij} \mid B_{ij}\}, \quad i = 0, 1, 2, \dots , \quad l; j = 0, 1, 2, \dots ,$$

be complex sequences, and define the corresponding polynomials

$$\phi_i(x;0) \equiv 1, \quad \phi_i(x;n) = \prod_{k=0}^{n-1} (A_{ik} + xB_{ik}), \quad n = 1, 2, \dots , \tag{3.12}$$

with more compact notation

$$\Phi(x;n) = \prod_{i=0}^{l} \phi_i(x;[(i+n)/(1+l)]) \tag{3.13}$$

$$\varepsilon(k) = A_{l-k \pmod{1+l},[k/(1+l)]} \tag{3.14}$$

$$+ B_{l-k \pmod{1+l},[k/(1+l)]}, \tag{3.15}$$

where $[x]$ denotes the integer part for a real number x.

Then, the same approach to Theorem 3 may be used to demonstrate the following multiplicate inversion formulas:

Theorem 8 (Multiplicate inverse series relations).

$$\Xi(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \Phi(k;n)\Theta(k) \tag{3.16}$$

$$\Theta(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\varepsilon(k)}{\Phi(n;k+1)} \Xi(k). \tag{3.17}$$

Up to now, we have not found any applications to hypergeometric identities.

3.4 *Hypergeometric reversals.* Reversing the summation order of hypergeometric series (0.3a)–(0.3b) and then performing parameter replacements

$$x \longrightarrow 1-c-\delta-u-3n$$

and

$$z \longrightarrow \frac{1}{2}-\delta-u+v-2n,$$

we may simplify the result as the following:

Theorem 9 (The reversal of $H^{(\delta)}[a,b;c,d,e \mid n,x,z]$).

(3.18)
$$ {}_5F_4\left[\begin{matrix}-\delta-2n, & \frac{u}{2},\frac{1+u}{2}, & 2c-\delta-e+2v,\delta-d+2u-2v+2n\\ & c-a+u, & c-b+u+n,c-\delta+v-n,\frac{1}{2}+u-v\end{matrix}\right]$$

(3.19)
$$=(-1)^{\delta}H^{(\delta)}[a,b;c,d,e \mid n,1-c-\delta-u-3n,\frac{1}{2}-\delta-u+v-2n]$$

(3.20)
$$\times\frac{(\frac{u}{2})_{\delta+2n}(\frac{1+u}{2})_{\delta+2n}(2c-\delta-e+2v)_{\delta+2n}(\delta-d+2u-2v+2n)_{\delta+2n}}{(c-a+u)_{\delta+2n}(c-b+u+n)_{\delta+2n}(c-\delta+v-n)_{\delta+2n}(\frac{1}{2}+u-v)_{\delta+2n}}.$$

Two examples from (2.12a) and (2.12b) may be displayed respectively as

$$ {}_5F_4\left[\begin{matrix}-2n, & \frac{u}{2},\frac{1+u}{2}, & 1+2v, & 2u-2v+2n\\ & u,1+u+n, & 1+v-n, & \frac{1}{2}+u-v\end{matrix}\right] \tag{3.21}$$

$$=\frac{\frac{1}{2}{}_n(u-2v)_n(u-2v+n)_n}{(\frac{1}{2}+u-v)_n(1+u+n)_n(-v)_n} \tag{3.22}$$

(3.23)
$$ {}_5F_4\left[\begin{matrix}-1-2n, & \frac{u}{2}, & \frac{1+u}{2}, & 2v, & 1+2u-2v+2n\\ & u,1+u+n, & v-n, & \frac{1}{2}+u-v & \end{matrix}\right]=0,$$

In the same way all the hypergeometric formulas established in this paper can be transformed to another large class of identities.

3.5 *Gessel-Stanton* [**17**]. To make the paper self-contained, here we reproduce a proof of Gessel-Stanton formulae (2.8a)–(2.8b) by means of inversion machinery, cf., Chu [**11**].

For a nonnegative integer m and complex numbers b, c, d, e with $1 + a + 2m = b + c + d + e$, the Dougall theorem [**8**, Section 5.1] states that

(3.24)
$$ {}_7F_6\left[\begin{matrix} a, 1+(a/2), & b, & c, & d, & e, & -m \\ (a/2), & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a+m \end{matrix}\right] $$

(3.25)
$$ = \frac{(1+a)_m(1+a-b-c)_m(1+a-b-d)_m(1+a-c-d)_m}{(1+a-b)_m(1+a-c)_m(1+a-d)_m(1+a-b-c-d)_m}, $$

which may be specified as

$$ {}_7F_6\left[\begin{matrix} a+d+\frac{1}{2}, \frac{3+a+d}{4}, b+d+\frac{1}{2}, 1+d-b, a+\frac{m}{2}, a+\frac{1+m}{2}, -m \\ \frac{1+a+d}{4}, 1+a-b, a+b+\frac{1}{2}, d-\frac{3-m}{2}, d-\frac{2-m}{2}, a+d+m+\frac{3}{2} \end{matrix}\right] $$
$$ = \frac{(2b)_m(1-2b)_m(a-d)_m(a+d+3/2)_m}{(2+2d)_m(-1-2d)_m(1+a-b)_m(a+b+1/2)_m}. $$

It may be restated in terms of (1.3a) as

$$ \frac{(2a)_m(2b)_m(1-2b)_m(a-d)_m(a+d+1/2)_m}{(1+a-b)_m(a+b+1/2)_m(2+2d)_m} $$
$$ = \sum_{k=0}^{m}(-1)^k\binom{m}{k}(2a+2k)_m(-1-2d-2k)_m\frac{2k+a+d+1/2}{(m+a+d+1/2)_{k+1}} $$
$$ \times\frac{(a)_k(a+1/2)_k(a+d+1/2)_k(b+d+1/2)_k(1+d-b)_k}{(1+d)_k(d+3/2)_k(a+b+1/2)_k(1+a-b)_k}, $$

with

$$ \lambda = a+d+1/2, \quad \phi(x;n) = (2x-1-2d)_n $$
$$ F(k) = \frac{(2a)_k(2b)_k(1-2b)_k(a-d)_k(a+d+1/2)_k}{(1+a-b)_k(a+b+1/2)_k(2+2d)_k} $$
$$ G(m) = \frac{(a)_m(a+1/2)_m(a+d+1/2)_m(b+d+1/2)_m(1+d-b)_m}{(1+d)_m(d+3/2)_m(a+b+1/2)_m(1+a-b)_m}. $$

Then the corresponding dual relation (1.3b) reads as

(3.26)
$$\frac{(a)_m(a+1/2)_m(a+d+1/2)_m(b+d+1/2)_m(1+d-b)_m}{(1+d)_m(d+3/2)_m(a+b+1/2)_m(1+a-b)_m}$$

(3.27)
$$= \sum_{k=0}^{m}(-1)^k \binom{m}{k} \tfrac{2a+3k}{(2a+2m)_{1+k}} \tfrac{-1-2d-k}{(-1-2d-2m)_{k+1}} (k+a+d+1/2)_m$$

(3.28)
$$\times \frac{(2a)_k(2b)_k(1-2b)_k(a-d)_k(a+d+1/2)_k}{(1+a-b)_k(a+b+1/2)_k(2+2d)_k},$$

whose reformulation in terms of hypergeometric series leads us to (2.8a)–(2.8b), due to Gessel-Stanton [**17**].

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