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Summation formulae on reciprocal sequences

Wenchang Chu[1], Pierluigi Magli

Dipartimento di Matematica, Università degli Studi di Lecce, Lecce-Arnesano, P.O. Box 193, 73100 Lecce, Italy

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Abstract

By means of series rearrangement, we prove an algebraic identity on the symmetric difference of bivariate Ω-polynomials associated with an arbitrary complex sequence. When the sequence concerned is ε-*reciprocal*, we find some unusual recurrence relations with binomial polynomials as coefficients. As applications, several interesting summation formulae are established for Bernoulli, Fibonacci, Lucas and Genocchi numbers.

In view of the classical binomial transform, a sequence $\{a_n\}_{n=0}^{\infty}$ is said to be ε-*reciprocal* iff the following relation holds:

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}a_k=\varepsilon a_n \qquad \text{for } n\in\mathbb{N}_0.$$

When $\varepsilon=\pm 1$, we shall simply call the corresponding sequences *reciprocal* and *minus-reciprocal* respectively. There are several classical combinatorial sequences satisfying ε-*reciprocity*, for example, the sequence $\{\frac{1}{2^n}\}_{n=0}^{\infty}$, the Bernoulli numbers, the Fibonacci numbers, the Lucas numbers and the Genocchi numbers.

Motivated by the recent works due to Sun [7,8], we shall investigate symmetric differences of bivariate Ω-polynomials. Several summation formulae will be established that are generally valid for ε-*reciprocal* sequences. They lead us, subsequently, to numerous interesting old and new combinatorial identities on the classical numbers just mentioned.

The paper is organized as follows. In the first section, we shall define, for an arbitrary complex sequence, the associated polynomials and prove a symmetric difference theorem

E-mail addresses: chu.wenchang@unile.it (W. Chu), pierluigi.magli@unile.it (P. Magli).
[1] Tel.: +39 0832 297409; fax: +39 0832 297594.

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on bivariate Ω-polynomials. Then in the second section, the *ε-reciprocal* sequences will be characterized by their associated polynomials and exponential generating functions. In particular, several summation formulae for the *ε-reciprocal* sequences will be derived from the symmetric difference theorem. As applications, numerous interesting old and new identities on the sequence $\{\frac{1}{2^n}\}_{n=0}^{\infty}$, the Bernoulli numbers, the Fibonacci numbers, the Lucas numbers and the Genocchi numbers will be displayed in the third section.

1. Algebraic identity on the symmetric difference

With $\ell \in \mathbb{Z}$ and $c \in \mathbb{C}$, let $(c)_\ell$ stand for the shifted factorial defined explicitly by

$$(c)_\ell = \begin{cases} c(c+1)\cdots(c+\ell-1), & \ell = 1, 2, \ldots; \\ 1, & \ell = 0; \\ \dfrac{1}{(c-1)(c-2)\cdots(c+\ell)}, & \ell = -1, -2, \ldots. \end{cases}$$

For any given complex sequence $\{a_n\}_{n\in\mathbb{N}_0}$ with $\mathbb{N}_0$ being the set of non-negative integers, the associated polynomials are defined by

$$A_n(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} a_k x^{n-k} \qquad \text{for } n \in \mathbb{N}_0. \tag{1.1}$$

Define further the Ω-functions with two variables x, y and three integer parameters m, n, ℓ with $m, n \in \mathbb{N}_0$ and $\ell \in \mathbb{Z}$ through

$$\Omega_{m,n,\ell}(x, y) := \sum_{k=0}^{m} \binom{m}{k} \frac{A_{n+k+\ell}(x)}{(n+k+1)_\ell} (y-x)^{m-k}. \tag{1.2}$$

Then we establish the following fundamental theorem.

Theorem 1 (*Symmetric Difference*). *Let the Ω-polynomials be defined by (1.2). Then the symmetric difference $\Omega_{m,n,\ell}(x, y) - \Omega_{n,m,\ell}(y, x)$ vanishes for $\ell \le 0$ and is equal, for $\ell > 0$, to the following expression:*

$$(-1)^m \frac{m!n!}{(m+n+\ell)!} \sum_{k=1}^{\ell} \binom{m+n+\ell}{\ell-k} \binom{m+k-1}{m} A_{\ell-k}(y)(x-y)^{m+n+k}.$$

When $\ell = 0, 1$, this theorem recovers the main results due to Sun [8, Eqs. 1.4–1.5].

Proof. Recalling the definitions (1.1) and (1.2), we have

$$\begin{aligned} \Omega_{m,n,\ell}(x, y) &= \sum_{k=0}^{m} \binom{m}{k} \frac{(y-x)^{m-k}}{(n+k+1)_\ell} A_{n+k+\ell}(x) \\ &= \sum_{k=0}^{m} \binom{m}{k} \frac{(y-x)^{m-k}}{(n+k+1)_\ell} \sum_{i=0}^{n+k+\ell} (-1)^i a_i \binom{n+k+\ell}{i} x^{n+k+\ell-i}. \end{aligned}$$

Rewriting the last monomial according to the binomial theorem

$$\{(x-y)+y\}^{n+k+\ell-i} = \sum_{j=i}^{n+k+\ell} \binom{n+k+\ell-i}{j-i} (x-y)^{n+k+\ell-j} y^{j-i}$$

we get the following triple sum expression:

$$\Omega_{m,n,\ell}(x,y) = \sum_{k=0}^{m} \binom{m}{k} \frac{(y-x)^{m-k}}{(n+k+1)_\ell} \sum_{i=0}^{n+k+\ell} (-1)^i \binom{n+k+\ell}{i} a_i \times \sum_{j=i}^{n+k+\ell} \binom{n+k+\ell-i}{j-i} (x-y)^{n+k+\ell-j} y^{j-i}.$$

Interchanging the summation order and then applying the binomial coefficient relation

$$\binom{n+k+\ell}{i}\binom{n+k+\ell-i}{j-i} = \binom{n+k+\ell}{j}\binom{j}{i}$$

we can further reformulate the triple sum as

$$\Omega_{m,n,\ell}(x,y) = \sum_{k=0}^{m} \binom{m}{k} \frac{(y-x)^{m-k}}{(n+k+1)_\ell} \times \sum_{j=0}^{n+k+\ell} \binom{n+k+\ell}{j} (x-y)^{n+k+\ell-j} \sum_{i=0}^{j} (-1)^i a_i \binom{j}{i} y^{j-i}$$

which reduces, in view of (1.1), to the following double sum:

$$\Omega_{m,n,\ell}(x,y) = \sum_{j=0}^{m+n+\ell} A_j(y)(x-y)^{m+n+\ell-j} \tag{1.3a}$$

$$\times \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \frac{\binom{n+k+\ell}{j}}{(n+k+1)_\ell}. \tag{1.3b}$$

By means of the well-known Chu–Vandermonde convolution formula on binomial coefficients, it is not difficult to evaluate (1.3b) as follows.

- When $\ell \le 0$, the result reads as

$$\text{Eq. (1.3b)} = \begin{cases} 0, & 0 \le j < m+\ell; \\ \dfrac{(j-\ell)!}{j!} \dbinom{n}{j-m-\ell}, & m+\ell \le j \le m+n+\ell. \end{cases}$$

- When $\ell \ge 0$, the result reads as

$$\text{Eq. (1.3b)} = \begin{cases} \dfrac{(-1)^m (\ell-j)_m}{j!(1+n)_{m+\ell-j}}, & 0 \le j < \ell; \\ 0, & \ell \le j < m+\ell; \\ \dfrac{(j-\ell)!}{j!} \dbinom{n}{j-m-\ell}, & m+\ell \le j \le m+n+\ell. \end{cases}$$

Therefore for $\ell \le 0$, the Ω-function displayed in (1.3a) and (1.3b) can be simplified as

$$\begin{aligned} \Omega_{m,n,\ell}(x,y) &= \sum_{j=m+\ell}^{m+n+\ell} \frac{(j-\ell)!}{j!} \binom{n}{j-m-\ell} A_j(y)(x-y)^{m+n+\ell-j} \\ &= \sum_{k=0}^{n} \binom{n}{k} \frac{A_{m+k+\ell}(y)}{(m+k+1)_\ell} (x-y)^{n-k} = \Omega_{n,m,\ell}(y,x) \end{aligned}$$

which corresponds to the case $\ell \le 0$ of Theorem 1.

When $\ell > 0$, the Ω-function displayed in (1.3a) and (1.3b) can be written as two sums:

$$\Omega_{m,n,\ell}(x,y) = \sum_{j=m+\ell}^{m+n+\ell} \frac{(j-\ell)!}{j!} \binom{n}{j-m-\ell} A_j(y)(x-y)^{m+n+\ell-j} + \sum_{j=0}^{\ell-1} \frac{(-1)^m(\ell-j)_m}{j!(1+n)_{m+\ell-j}} A_j(y)(x-y)^{m+n+\ell-j}.$$

Shifting the summation index for the first by $j \to m+\ell+k$ and inverting for the second by $j \to \ell-k$, we reduce the last expression to

$$\Omega_{m,n,\ell}(x,y) = \sum_{k=0}^{n} \binom{n}{k} \frac{A_{m+k+\ell}(y)}{(m+k+1)_\ell}(x-y)^{n-k} \qquad [\Longrightarrow \Omega_{n,m,\ell}(y,x)]$$
$$+ \frac{(-1)^m m!\,n!}{(m+n+\ell)!} \sum_{k=1}^{\ell} \binom{m+n+\ell}{\ell-k} \binom{m+k-1}{m} A_{\ell-k}(y)\,(x-y)^{m+n+k}$$

which is exactly the case $\ell > 0$ of Theorem 1. $\square$

For the applications in the next section, it is convenient for us to state the identity in Theorem 1 separately in accordance with $\ell \le 0$ and $\ell > 0$.

For $\ell \le 0$, the identity stated in Theorem 1 reads as $\Omega_{m,n,\ell}(x,y) = \Omega_{n,m,\ell}(y,x)$. Replacing ℓ by $-\ell$ and applying the binomial relation

$$\frac{1}{(m+1)_{-\ell}} = \ell!\binom{m}{\ell} \qquad \text{where } \ell \in \mathbb{N}_0$$

we can explicitly reformulate the identity in Theorem 1 as the following corollary.

Corollary 2 (*Symmetric Formula*). *For $m, n, \ell \in \mathbb{N}_0$, the following identity holds:*

$$\sum_{i=0}^{m} \binom{m}{i}\binom{n+i}{\ell} A_{n+i-\ell}(x)(y-x)^{m-i} \tag{1.4a}$$

$$= \sum_{j=0}^{n} \binom{n}{j}\binom{m+j}{\ell} A_{m+j-\ell}(y)(x-y)^{n-j}. \tag{1.4b}$$

As pointed out by an anonymous referee, the symmetric formula stated in Corollary 2 can also be proved directly by means of umbral calculus and the generating function method, which will not be reproduced.

When $\ell \ge 1$, exchanging parameters $m \rightleftharpoons n$ and $x \rightleftharpoons y$ in Theorem 1, we derive another interesting relation.

Corollary 3 (*Symmetric Formula*). *For $m, n, \ell \in \mathbb{N}_0$, the following identities hold:*

$$\frac{(m+n+\ell)!}{m!n!}\{\Omega_{m,n,\ell}(x,y) - \Omega_{n,m,\ell}(y,x)\} \tag{1.5a}$$

$$= \sum_{k=1}^{\ell} (-1)^m \binom{m+n+\ell}{\ell-k}\binom{m+k-1}{m} A_{\ell-k}(y)(x-y)^{m+n+k} \tag{1.5b}$$

$$= \sum_{k=1}^{\ell}(-1)^{1+n}\binom{m+n+\ell}{\ell-k}\binom{n+k-1}{n}A_{\ell-k}(x)(y-x)^{m+n+k}. \tag{1.5c}$$

2. ε-Reciprocal sequences and summation formulae

For ε-reciprocal sequences, this section will prove two functional equations satisfied by their associated polynomials and exponential generating functions. Then we shall specialize Theorem 1 to derive four combinatorial identities on ε-reciprocal sequences.

Lemma 4. *The sequence* $\{a_n\}_{n=0}^{\infty}$ *is* ε-reciprocal *iff the associated polynomials satisfy the following functional equation:*

$$A_n(1+x) = (-1)^n\varepsilon A_n(-x) \qquad \text{for } n \in \mathbb{N}_0. \tag{2.1}$$

Proof. This can easily be verified by means of the binomial theorem as follows:

$$\begin{aligned} A_n(1+x) &= \sum_{k=0}^{n}(-1)^k\binom{n}{k}a_k(1+x)^{n-k} \\ &= \sum_{k=0}^{n}(-1)^k\binom{n}{k}a_k\sum_{j=k}^{n}\binom{n-k}{n-j}x^{n-j} \\ &= \sum_{j=0}^{n}\binom{n}{j}x^{n-j}\sum_{k=0}^{j}(-1)^k\binom{j}{k}a_k \\ &= \varepsilon\sum_{j=0}^{n}\binom{n}{j}a_jx^{n-j} = (-1)^n\varepsilon A_n(-x). \end{aligned}$$

Conversely, the case $x = 0$ of the associated polynomial relation stated in Lemma 4 coincides with the ε-*reciprocity* of $\{a_n\}_{n=0}^{\infty}$. □

Now we introduce the exponential generating function for the sequence $\{a_n\}_{n=0}^{\infty}$ via

$$\widehat{A}(z) = \sum_{n\geq 0}a_n\frac{z^n}{n!}. \tag{2.2}$$

Then the following theorem characterizes the ε-*reciprocity* through the exponential generating function.

Theorem 5. *The sequence* $\{a_n\}_{n=0}^{\infty}$ *is* ε-reciprocal *iff the following exponential generating function equation holds:*

$$\widehat{A}(z) = \varepsilon e^z\,\widehat{A}(-z). \tag{2.3}$$

Proof. According to the definition of the ε-*reciprocal* sequence, we have

$$\begin{aligned} \varepsilon\widehat{A}(z) &= \sum_{n=0}^{\infty}\varepsilon a_n\frac{z^n}{n!} = \sum_{n=0}^{\infty}\frac{z^n}{n!}\sum_{k=0}^{n}(-1)^k\binom{n}{k}a_k \\ &= \sum_{k=0}^{\infty}a_k\frac{(-z)^k}{k!}\sum_{n=k}^{\infty}\frac{z^{n-k}}{(n-k)!} = e^z\widehat{A}(-z) \end{aligned}$$

which gives the relation displayed in Theorem 5 under replacement $z \to -z$. □

There are numerous ε-*reciprocal* functions. Here we display four classes of such functions for subsequent application.

Proposition 6 (ε-*Reciprocal Sequences*). *For two complex numbers α and γ subject to $\alpha+\gamma=1$, define four functions by*

$$\widehat{A}(z) = u(z)\{e^{\alpha z}+e^{\gamma z}\}, \tag{2.4a}$$

$$\widehat{B}(z) = v(z)\{e^{\alpha z}-e^{\gamma z}\}; \tag{2.4b}$$

$$\widehat{C}(z) = \frac{u(z)e^{z}}{e^{\alpha z}+e^{\gamma z}}, \tag{2.4c}$$

$$\widehat{D}(z) = \frac{v(z)e^{z}}{e^{\alpha z}-e^{\gamma z}}. \tag{2.4d}$$

Then $\widehat{A}(z)$ and $\widehat{C}(z)$ are reciprocal (or minus-reciprocal) *when $u(z)$ is an even* (or odd) *function. Similarly, $\widehat{B}(z)$ and $\widehat{D}(z)$ are* reciprocal (or minus-reciprocal) *when $v(z)$ is an odd* (or even) *function.*

Under the condition of ε-reciprocity, Theorem 1 can be specified by the following finite summation formulae.

Putting $y=1+x$ in Corollary 2, we find the following identity.

Proposition 7. *Let $\{a_n\}_{n=0}^{\infty}$ be ε*-reciprocal. *Then for $\ell\in\mathbb{N}_0$, the following holds:*

$$(-1)^{n+\ell}\sum_{i=0}^{m}\binom{m}{i}\binom{n+i}{\ell}A_{n+i-\ell}(x)$$
$$=(-1)^{m}\varepsilon\sum_{j=0}^{n}\binom{n}{j}\binom{m+j}{\ell}A_{m+j-\ell}(-x).$$

Instead, putting $y=1+x$ in Corollary 3, we will find another identity.

Proposition 8. *Let $\{a_n\}_{n=0}^{\infty}$ be ε*-reciprocal. *Then for $\ell\in\mathbb{N}$, the following holds:*

$$(-1)^{n+\ell}\sum_{i=0}^{m}\binom{m}{i}\frac{A_{n+i+\ell}(x)}{(n+i+1)_\ell}-(-1)^{m}\varepsilon\sum_{j=0}^{n}\binom{n}{j}\frac{A_{m+j+\ell}(-x)}{(m+j+1)_\ell}$$
$$=\frac{m!\,n!\,\varepsilon}{(m+n+\ell)!}\sum_{k=1}^{\ell}\binom{m+n+\ell}{\ell-k}\binom{m+k-1}{m}A_{\ell-k}(-x).$$

In particular for $\varepsilon=\pm1$, we can further establish two identities on *reciprocal* and *minus-reciprocal* sequences.

Corollary 9. *Let $n\in\mathbb{N}_0$ and λ be an indeterminate. For a* reciprocal *sequence $\{a_n\}_{n=0}^{\infty}$, the following summation formula holds:*

$$\sum_{k=0}^{2n-1}\binom{2\lambda}{k}\binom{n-\lambda-1}{2n-k-1}a_k=0,$$

Proof. Note that $A_n(0)=(-1)^n a_n$. When $\{a_n\}_{n=0}^{\infty}$ is *reciprocal*, specifying Proposition 8 with $x\to0$, $n\to m$ and $\ell\to2n$ and then inverting the summation order, we get the following identity:

$$\sum_{k=0}^{2n-1}\binom{2m+2n}{k}\binom{-m-1}{2n-k-1}a_k=0.$$

Observing that this is a polynomial identity in m, we obtain the identity stated in Corollary 9 from it by making the replacement $m\to\lambda-n$. □

Similarly, when $\{a_n\}_{n=0}^{\infty}$ is *minus-reciprocal*, specifying Proposition 8 with $x\to 0$, $n\to m$ and $\ell\to 2n+1$ and then inverting the summation order, we find the following identity:

$$\sum_{k=0}^{2n}\binom{1+2m+2n}{k}\binom{-m-1}{2n-k}a_k=0.$$

This is again a polynomial identity in m and leads us, under the replacement $m\to\lambda-n$, to the following identity.

Corollary 10. *Let $n\in\mathbb{N}_0$ and λ be an indeterminate. For a* minus-reciprocal *sequence $\{a_n\}_{n=0}^{\infty}$, the following summation formula holds:*

$$\sum_{k=0}^{2n}\binom{1+2\lambda}{k}\binom{n-\lambda-1}{2n-k}a_k=0.$$

We remark that both identities displayed in the last two corollaries can also be verified by means of a binomial transformation due to Gessel [3, Eq. 7.16]. The interested reader can work out the details as an exercise.

3. Identities on classical combinatorial numbers

As applications of the identities displayed for *reciprocal* and *minus-reciprocal* sequences, we are now going to derive several interesting summation formulae for the sequence $\{\frac{1}{2^n}\}_{n=0}^{\infty}$, the Bernoulli numbers, the Fibonacci numbers, the Lucas numbers and the Genocchi numbers.

3.1. Binomial identities

By means of the exponential function expansion

$$e^{z/2}=\sum_{n\geq 0}\frac{z^n}{2^n\cdot n!}$$

the sequence $\{1/2^n\}_{n=0}^{\infty}$ is *reciprocal* in view of (2.4a). Then Corollary 9 produces the following binomial identity:

$$\sum_{k=0}^{2n-1}\frac{1}{2^k}\binom{2\lambda}{k}\binom{n-\lambda-1}{2n-k-1}=0. \tag{3.1}$$

This is a special cases of the hypergeometric series identity due to Kummer [6] (cf. also Bailey [1, Section 2.4]):

$$\sum_{n=0}^{\infty}\frac{(a)_n(c)_n}{2^n\cdot n!\left(\frac{1+a+c}{2}\right)_n}=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1+a+c}{2}\right)}{\Gamma\left(\frac{1+a}{2}\right)\Gamma\left(\frac{1+c}{2}\right)}.$$

The case $x = 0$ of Proposition 7 gives additionally the following symmetric formula:

$$(-1)^\ell \sum_{i=0}^{m} \frac{(-1)^i}{2^{n+i}} \binom{m}{i} \binom{n+i}{\ell} = \sum_{j=0}^{n} \frac{(-1)^j}{2^{m+j}} \binom{n}{j} \binom{m+j}{\ell}.$$

3.2. Bernoulli numbers and polynomials

Recall that the Bernoulli numbers and polynomials are defined (cf. [4, Section 7.6]) respectively by

$$\frac{z}{e^z - 1} = \sum_{n \geq 0} B_n \frac{z^n}{n!} \qquad \text{and} \qquad \frac{z e^{zx}}{e^z - 1} = \sum_{n \geq 0} B_n(x) \frac{z^n}{n!}.$$

In view of (2.4d), the sequence $\{(-1)^n B_n\}_{n=0}^{\infty}$ is *reciprocal* with the exponential generating function being given by $z e^z/(e^z - 1)$. According to Corollary 9, we have the following identity:

$$\sum_{k=0}^{2n-1} (-1)^k \binom{2\lambda}{k} \binom{n-\lambda-1}{2n-k-1} B_k = 0.$$

Note that $B_{1+2k} = 0$ except for $B_1 = -1/2$. Replacing n by $n+1$ and then singling out the term corresponding to $k = 1$, we can reformulate the last identity as the following very interesting summation formula on Bernoulli numbers:

$$\sum_{k=0}^{n} \binom{2\lambda}{2k} \binom{n-\lambda}{2n-2k+1} B_{2k} = \lambda \binom{n-\lambda}{2n} \tag{3.2}$$

which does not seem to be among the known identities of Bernoulli numbers.

Noting that the associated polynomials of the reciprocal sequence $\{(-1)^n B_n\}_{n=0}^{\infty}$ coincide with the Bernoulli polynomials

$$B_n(x) = \sum_{k=0}^{n} B_k \binom{n}{k} x^{n-k}$$

we find through Proposition 7 the following symmetrical identity:

$$(-1)^{m+\ell} \sum_{i=0}^{m} \binom{m}{i} \binom{n+i}{\ell} B_{n+i-\ell}(-x) \tag{3.3a}$$

$$= (-1)^n \sum_{j=0}^{n} \binom{n}{j} \binom{m+j}{\ell} B_{m+j-\ell}(x). \tag{3.3b}$$

When $\ell = 0$, this identity reduces to the symmetric formula (cf. Gessel [3, Lemma 7.2])

$$\sum_{i=0}^{m} \binom{m}{i} B_{n+i} = (-1)^{m+n} \sum_{j=0}^{n} \binom{n}{j} B_{m+j}$$

which leads easily to an identity due to Kaneko [5] (cf. also Zagier [11, Eq. 14]):

$$\sum_{k=0}^{n+1} \binom{n+1}{k} (n+k) B_{n+k} = 0. \tag{3.4}$$

When $\ell = 1$, the identity (3.3) contains a recent result that appeared in [8, Eq. 1.3] and [10, Eq. 8] as a very special case. In fact, performing the replacements $m \to 1+m$, $n \to 1+n$ and $\ell \to 1$ and then separating the last term from each sum, we get immediately the identity just mentioned:

$$(-1)^n(m+n+1)(m+n+2)x^{m+n} = (-1)^m \sum_{i=0}^{m} \binom{m+1}{i} (n+i+1) B_{n+i}(-x) + (-1)^n \sum_{j=0}^{n} \binom{n+1}{j} (m+j+1) B_{m+j}(x).$$

3.3. *Fibonacci numbers*

The Fibonacci numbers (cf. [9, Section 2.3]) are defined by means of the recurrence relation

$$\begin{cases} F_0 = 0 \qquad \text{and} \qquad F_1 = 1 \\ F_{n+2} = F_{n+1} + F_n \qquad \text{for } n \in \mathbb{N}_0 \end{cases}$$

with the following explicit formula:

$$F_n = \frac{\alpha^n - \gamma^n}{\sqrt{5}} \qquad \text{for } \alpha, \gamma = \frac{1 \pm \sqrt{5}}{2} \quad \text{and} \quad n \in \mathbb{N}_0.$$

It is trivial to compute the exponential generating function

$$\sum_{n=0}^{\infty} F_n \frac{z^n}{n!} = \frac{e^{\alpha z} - e^{\gamma z}}{\sqrt{5}}.$$

According to (2.4b), the sequence $\{F_n\}_{n=0}^{\infty}$ is *minus-reciprocal*. Therefore Corollary 10 yields the following identity:

$$\sum_{k=0}^{2n} \binom{1+2\lambda}{k} \binom{n-\lambda-1}{2n-k} F_k = 0. \tag{3.5}$$

Letting $x = 0$ in Proposition 7, we also get the symmetric formula

$$(-1)^{\ell} \sum_{i=0}^{m} (-1)^i \binom{m}{i} \binom{n+i}{\ell} F_{n+i-\ell} + \sum_{j=0}^{n} (-1)^j \binom{n}{j} \binom{m+j}{\ell} F_{m+j-\ell} = 0.$$

3.4. *Lucas numbers*

The Lucas numbers (cf. [4, p. 312]) are defined by means of the recurrence relation

$$\begin{cases} L_0 = 2 \qquad \text{and} \qquad L_1 = 1 \\ L_{n+2} = L_{n+1} + L_n \qquad \text{for } n \in \mathbb{N}_0 \end{cases}$$

with the following explicit formula:

$$L_n = \alpha^n + \gamma^n \quad \text{for } \alpha, \gamma = \frac{1 \pm \sqrt{5}}{2} \quad \text{and} \quad n \in \mathbb{N}_0.$$

It is easy to compute the exponential generating function

$$\sum_{n=0}^{\infty} L_n \frac{z^n}{n!} = e^{\alpha z} + e^{\gamma z}.$$

According to (2.4a), the sequence $\{L_n\}_{n=0}^{\infty}$ is *reciprocal*. Therefore Corollary 9 yields the following identity:

$$\sum_{k=0}^{2n-1} \binom{2\lambda}{k} \binom{n-\lambda-1}{2n-k-1} L_k = 0. \tag{3.6}$$

In addition, letting $x = 0$ in Proposition 7, we find also the symmetric formula

$$(-1)^{\ell} \sum_{i=0}^{m} (-1)^i \binom{m}{i} \binom{n+i}{\ell} L_{n+i-\ell} = \sum_{j=0}^{n} (-1)^j \binom{n}{j} \binom{m+j}{\ell} L_{m+j-\ell}.$$

3.5. Genocchi numbers

The Genocchi numbers (cf. [2, Section I-14]) are defined by the following exponential generating function:

$$\frac{2z}{1+\mathrm{e}^z} = \sum_{n\geq 0} G_n \frac{z^n}{n!}$$

and are related to the Bernoulli numbers by $G_{2n} = 2(1-2^{2n})B_{2n}$. Then the sequence $\{(-1)^n G_n\}_{n=0}^{\infty}$ generated by $2z\mathrm{e}^z/(1+\mathrm{e}^z)$ is *minus-reciprocal* in view of (2.4c). By means of Corollary 10, we get the following identity:

$$\sum_{k=1}^{n} \binom{1+2\lambda}{2k} \binom{n-\lambda-1}{2n-2k} G_{2k} = (1+2\lambda) \binom{n-\lambda-1}{2n-1} \tag{3.7}$$

where we have appealed to the property $G_{1+2k} = 0$ except for $G_1 = 1$.

Moreover, letting $x = 0$, we can derive from Proposition 7 the symmetric formula

$$\sum_{i=0}^{m} (-1)^i \binom{m}{i} \binom{n+i}{\ell} G_{n+i-\ell} + (-1)^{\ell} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \binom{m+j}{\ell} G_{m+j-\ell} = 0.$$

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