NUMER. FUNCT. ANAL. AND OPTIMIZ., 5(1), 1-20 (1982)

CHOLESKY FACTORIZATION OF POSITIVE DEFINITE BI-INFINITE MATRICES

C. K. Chui, J. D. Ward
Department of Mathematics
Texas A&M University
College Station, TX 77843

P. W. Smith
Department of Mathematics
Old Dominion University
Norfolk, VA 23508

ABSTRACT

A Cholesky factorization for positive definite bi-infinite matrices is presented. Applications to block Toeplitz matrices, signal processing, and spline interpolation are then derived.

INTRODUCTION

There has been much recent interest in bi-infinite matrices stimulated initially by de Boor [1,2] and Micchelli [16], who were interested in determining bounds for spline interpolation operators. In this and more general settings the question of factoring such matrices is of much interest [3, 5, 6, 8, 9]. We confront in this paper the problem of factoring a positive definite bi-infinite matrix A as LL^T where L is lower triangular and $L_{ii} > 0$ for all i. It is well known that such factorizations are unique if A is finite, however, this is not the case if A is bi-infinite.

We will show that every positive definite bi-infinite matrix A has a Cholesky factorization which is unique if we additionally require L to have a lower triangular inverse. These results are presented in sections two and three. In section four, we relate these results to the classical work of Wiener, Hopf, and Krein on convolution equations, and in section five we present an application of these results to digital signal processing and spline interpolation. In particular, we obtain a new result concerning the null eigensplines for a periodic mesh thus consolidating and extending many results observed in special cases.

The questions addressed in this paper could be categorized as infinite dimensional numerical analysis. Many of the stability problems in spline analysis can be addressed by using these techniques.

2. PRELIMINARIES

Throughout this paper, all matrices are assumed to have real entries unless specified otherwise, and A will always denote a bounded positive definite bi-infinite matrix, i.e., $A = BB^T$ for some invertible matrix B. We say that A has a Cholesky factorization if we can write

$$A = LL^T$$

where L is a bounded lower triangular bi-infinite matrix with respect to some orthonormal basis $\{e_i\}$, $i = \ldots, -1, 0, 1, \ldots$,; that is, $\langle Le_i, e_j \rangle = 0$ for all i and j with with $j < i$. Here and throughout, the superscript T denotes transposition of a matrix.

It was already mentioned in the above section that the existence of a Cholesky factorization for a bi-infinite matrix A is unclear. It will be shown that unlike the infinite (that is, one-sided infinite) case, a bi-infinite matrix A may have several Cholesky factorizations. We first give an alternate proof of the Cholesky Factorization Theorem for finite matrices

which will motivate our results on bi-infinite matrices. This approach is undoubtedly well-known to the experts, see for example Lawson & Hanson [14], page 125.

Lemma 2.1. *Let R be a positive definite $n \times n$ matrix with real entries. Then R has a Cholesky factorization $R = LL^T$, where L is a lower triangular $n \times n$ matrix with real entries. Furthermore, $L^T = UR^{1/2}$ where U is a unitary matrix.*

Proof. Since R is invertible, so is its (positive definite) square root $R^{1/2}$. In particular, the columns of $R^{1/2}$ form a basis of $\mathbb{R}^n$. We define a unitary matrix U in the following fashion. The first row of U is the normalization of the first column of $R^{1/2}$. Assume that the first k rows of U have been defined. Then the (k+1)st row of U is a linear combination of the first (k+1)st columns $a_1, \ldots, a_{k+1}$, respectively, of $R^{1/2}$, such that it is orthogonal to $a_1, \ldots, a_k$ and is normalized to have unit length. It is easy to see that U is unitary and that $L^T = UR^{1/2}$ is upper triangular. Furthermore

$$LL^T = (R^{1/2})^T U^T UR^{1/2} = (R^{1/2})^T U^{-1} UR^{1/2} = R.$$

Since L has real entries, L^T is the same as the conjugate transpose L* of L. Hence.

$$||R|| = ||LL^*|| = ||L||^2,$$

so that $||L|| = ||R^{1/2}||$. Also, since finite matrices have unique Cholesky factorizations, we have the following.

Corollary 2.1. *Let R be a positive definite $n \times n$ matrix with Cholesky factorization $R = LL^T$. Then $||L|| = ||R^{1/2}||$.*

We now consider banded bi-infinite matrices.

Proposition 2.1. *Let A be a positive definite invertible 2m - banded bi-infinite matrix with real entries.*

Then A has an m - banded Cholesky factorization.

Proof. Let A^N denote the compression of A to the orthonormal set $\{e_i: i = -N,\dots,N\}$; that is, $A^N = PAP$ where P is the orthogonal projection onto span $\{e_i: i = -N,\dots,N\}$. Let L^N be the lower triangular Cholesky factor of A^N. Then L^N is m-banded and

$$\sum_{j=i-m}^{i} (\ell_{ij}^{(N)})^2 = \sum_{j=1}^{i} (\ell_{ij}^{(N)})^2 = a_{ii}^{(N)} \le ||A||,$$

where $a_{ii}^{(N)}$ is the ith diagonal entry of A^N and $L^N = [\ell_{ij}^{(N)}]$. Thus, the ℓ_2-norm of each row of L^N is uniformly bounded. It follows by a standard diagonalization argument that there is a subsequence $N_k \to \infty$, so that for each i and j

$$\ell_{ij}^N \to \ell_{ij}$$

as $N = N_k \to \infty$. It is clear that $LL^T = A$ where $L = [\ell_{ij}]$.

A Cholesky factorization for bi-infinite matrices is, however, not necessarily unique, as seen in the following.

Proposition 2.2. Let $A = [a_{ij}]$ be a positive definite tridiagonal Toeplitz bi-infinite matrix with $a_{ii} = r$ and $a_{i,i-1} = a_{i,i+1} = s$ for all i, such that $r^2 > 4s^2$. Then A has two distinct Toeplitz Cholesky factorizations.

Proof. Let $A = LL^T$ where $L = [\ell_{ij}]$ is a one-banded Toeplitz lower triangular matrix with $\ell_{ii} = d$ and $\ell_{i,i-1} = \ell$ for all i. Then

$$\ell^2 + d^2 = r \text{ and } \ell d = s$$

where $d > 0$. Hence

$$d^4 - rd^2 + s^2 = 0,$$

or

$$d = \sqrt{\frac{r \pm \sqrt{r^2-4s^2}}{2}} .$$

That is, there are two distinct solutions for d.

Although there is no uniqueness in Cholesky factorization for bi-infinite matrices in general, it turns out that our (unique) Cholesky factors of compressions of A converge to a "preferred" Cholesky factor, in the case when A is a Toeplitz matrix. That is, we have the following.

Proposition 2.3. Let A satisfy the conditions as given in the above proposition and A^N be a compression of A as defined in the proof of Proposition 2.1. Then the Nth row of the Cholesky factor L^N of A converges to a row vector that defines a Cholesky factor $L = [\ell_{ij}]$ of A with

$$\ell_{ii} = d = \sqrt{\frac{r + \sqrt{r^2-4s^2}}{2}} .$$

That is, L^N converges to the most diagonally dominant Toeplitz Cholesky factor L of A.

Proof. The equations for Cholesky factorization of A^N are

$$\ell_{-N,-N}^{(N)} = \sqrt{r}, \quad \ell_{-N+1,-N}^{(N)} = s/\sqrt{r} ,$$

and

$$\ell_{-N+1,-N+1}^{(N)} = \sqrt{r - (s/\ell_{-N,-N}^{(N)})^2} ,$$

and, in general, we obtain the iteration formula for the (i+1)st diagonal entry:

$$d_i = \sqrt{r - (s/d_{i-1})^2}$$

where $d_i = \ell_{-N+i,\,-N+i}^{(N)}$. Let $f(x) = \sqrt{r - (s/x)^2}$. The function f(x) has two fixed points:

$$\bar{d}_1 = \sqrt{\frac{r - \sqrt{r^2 - 4s^2}}{2}} \quad \text{and} \quad \bar{d}_2 = \sqrt{\frac{r + \sqrt{r^2 - 4s^2}}{2}} ,$$

where $s/\sqrt{r} < \bar{d}_1 < \bar{d}_2 < \sqrt{r}$. Since $f(x) > x$ for all x between $\bar{d}_1$ and $\bar{d}_2$ and $\ell^{(N)}_{-N,-N} = \sqrt{r}$, the bounded sequence $\{\ell^{(N)}_{-N+i,-N+i}\}$ has a subsequence that converges to $\bar{d}_2$. Since $\bar{d}_2$ is the unique limit point, the whole sequence converges to $\bar{d}_2$, and this holds for each i. That is, L^N converges to the most diagonally dominant Toeplitz Cholesky factor L of A.

3. STABLE CHOLESKY FACTORIZATION

In this section, the existence and uniqueness of a particular Cholesky factorization for an arbitrary positive definite bi-infinite matrix A will be given. We will call the factorization appearing in Theorem 3.1 *stable*, since this Cholesky factor is the most diagonally dominant one when A is a tridiagonal Toeplitz matrix. Throughout this section, the strong and weak operator topologies will be used extensively. Recall that a sequence of operators T_n is said to converge to an operator T strongly (notation: $T_n \overset{S}{\to} T$) if $||T_n x - Tx|| \to 0$ for all x, and the convergence is called weak convergence (notation: $T_n \overset{W}{\to} T$) if $\langle T_n x, y\rangle \to \langle Tx, y\rangle$ for all x and y. Of course if $T_n \overset{S}{\to} T$, then $T_n \overset{W}{\to} T$. If T_n and T are positive definite operators on a Hilbert space and $T_n \overset{S}{\to} T$, it is known that $T_n^{\frac{1}{2}} \overset{S}{\to} T^{\frac{1}{2}}$. To see this note that using [Problem 93, 9] it is easily established that $P(T_n) \overset{S}{\to} P(T)$ for any polynomial of fixed degree. Let $M = \sup_n ||T_n||$, then

$$||T_n^{\frac{1}{2}} f - T^{\frac{1}{2}} f|| \le ||(T_n^{\frac{1}{2}} - P(T_n))f|| + ||(P(T_n) - P(T))f|| + ||(P(T) - T^{\frac{1}{2}})f|| \le 3\varepsilon$$

where f is any fixed unit vector, n is sufficiently large, and P is a polynomial which approximates the square root function within ε uniformly on [0,M]. We need the following well known results.

Proposition 3.1. *Let U be a bi-infinite matrix whose rows u_i form a complete orthonormal set for the Hilbert space ℓ^2. Then U is a unitary matrix.*

Proof. Let $Ux = y$. Then

$$||y|| = ||(\ldots, \langle u_0, x\rangle, \langle u_1, x\rangle, \ldots)|| = ||x||.$$

Hence, U is an isometry with closed range in ℓ^2. Also, $Uu_i = (\ldots,0,1,0,\ldots) = e_i$ for each i, so that U has dense range. That is, U is an onto isometry, and is therefore, unitary.

Proposition 3.2. *Let A be a positive definite matrix and p_i the ith column vector of A. For each index set K, set $V_K = \operatorname{span}\{p_j : j \in K\}$. Then there exists a positive ε, such that*

$$\inf\{||p_i - v|| : v \in V_K\} =: d(p_i, V_K) \ge \varepsilon$$

for all i and K with $i \notin K$.

Remark: The Proposition follows from properties of Riesz bases and is true assuming only that A is invertible. For the reader's convenience, we provide a simple proof.

Proof. Assume the contrary. Then there exist a sequence of integers i_n and index sets K_n such that $i_n \notin K_n$, and $d(p_{i_n}, V_{K_n}) < \frac{1}{n}$ for all n. Select $v_n \in V_{K_n}$ such that $d(p_{i_n}, v_n) < \frac{1}{n}$, and since $i_n \notin K_n$ there is an u_n such that $A u_n = v_n$ and $\langle e_{i_n}, u_n\rangle = 0$ where $Ae_{i_n} = p_{i_n}$.

In addition, since $||e_{i_n}|| = 1$, we have $||e_{i_n} - u_n|| \ge 1$. However, $||A(e_{i_n} - u_n)|| = ||p_{i_n} - v_n|| < 1/n$ which implies that A is not invertible, a contradiction.

We are now ready to generalize Lemma 2.1 to infinite matrices.

Lemma 3.1. *Let R be a positive definite infinite matrix with real entries. Then R has a Cholesky factorization $R = LL^T$ where L is a lower triangular matrix. Furthermore, $L^T = UR^{\frac{1}{2}}$ where U is unitary.*

Proof. From Proposition 3.2, the columns of R form a basis of ℓ^2. If we follow the proof of Lemma 2.1 to define a matrix U, then the rows of U form an orthonormal basis; and by Proposition 3.1, U is unitary, and $L^T = L^* = UR^{1/2}$.

We remark that the above Cholesky factorization is unique and that L is also invertible.

Next, we will go to bi-infinite matrices A which are bounded and positive definite. As we have seen in the previous section, such a matrix A may have more than one, and in fact infinitely many, Cholesky factorizations in general. Our final goal is to obtain a particular Cholesky factorization $A = LL^T$ for A that is unique if the diagonal elements of L are positive. Let $\{e_i\}$, $i = \ldots,-1,0,1,\ldots$, be an orthonormal basis of a Hilbert space H and $A^N = P_{-N}AP_{-N}$ where P_{-N} is the orthogonal projection onto $H_N := \text{span}\{e_i : i = -N,-N+1,\ldots,\}$. We have the following.

Lemma 3.2. *If A is positive definite, then A^N is invertible relative to H_N.*

Proof. Let $x \in H_N$, $||x|| = 1$. Then $\langle A^N x,x\rangle = \langle P_{-N}AP_{-N}x,x\rangle = \langle Ax,x\rangle \geq 1/||A^{-1}|| > 0$. Since A^N is positive, A^N is bounded below by $1/||A^{-1}||$ and is hence invertible.

Note that A^N may be viewed as an infinite matrix (relative to H_N) or a bi-infinite matrix (relative to $\text{span}\{e_i : i = \ldots,-1,0,1,\ldots\}$). In the latter case, A^N converges strongly to A. By Lemma 3.1, $A^N = L_N L_N^T$ where $L_N^T = U_N(A^N)^{1/2}$ and U_N is unitary. In the sequel, we will denote the "mth" row of U_N by $u_{N,m}$. We have the following result.

Lemma 3.3. *For each m, $u_{N,m} \to u_m$ in H.*

Proof. The proof proceeds in the same manner as that of Lemma 2.1. It suffices to prove the lemma for $m = 0$. In this case, set

$$M^L_{i,N} = \text{span}\{a^L_{i-1},\ldots,a^L_{i-N}\}$$

where a^L_j is the jth column of $(A^L)^{1/2}$ and let $P_{i,N,L}(x)$ be the orthogonal projection of x onto $M^L_{i,N}$. From Lemma 3.1, we have

$$u_{N,0} = \frac{a_0^N - [P_{0,N,N}(a_0^N)]}{||a_0^N - [P_{0,N,N}(a_0^N)]||} .$$

Here, by Proposition 3.2, $||a_0^N - [P_{0,N,N}(a_0^N)]|| \geq \eta > 0$ for all N. Hence, if a_i^∞ denotes the ith column of $A^{1/2}$, it is clear that as $N \to \infty$,

$$||[P_{i,N,\infty}(a_i^\infty)] - [P_{i,\infty,\infty}(a_i^\infty)]|| = ||[P_{i-N+1,\infty,\infty}(a_i^\infty)]|| \to 0.$$

Let $\varepsilon > 0$ be given. Then we have

$$||[P_{0,N,N}(a_0^N)] - [P_{0,N,\infty}(a_0^N)]|| < \varepsilon \quad \text{and}$$

$$||[P_{0,N,\infty}(a_0^N)] - [P_{0,\infty,\infty}(a_0^N)]|| < \varepsilon$$

for all sufficiently large N. Since $a_0^N \to a_0^\infty$, this shows that $\{u_{N,0}\}$ is a Cauchy sequence in H, and hence, converges to some u_0 in H. However, since the sequence of scaling factors associated with $\{u_{N,0}\}$ (chosen so that $\ell_{ii} = a_{ii}^{1/2}$) also converges, we can define a bi-infinite matrix U whose rows form an orthonormal basis of H. By Proposition 3.1, U is unitary. This completes the proof of the lemma.

Let U be the unitary operator having rows u_m as defined in the above lemma. We have the following.

Lemma 3.4. $U_N \overset{W}{\to} U$.

Proof. We must show that $\langle U_N x,y\rangle \to \langle Ux,y\rangle$ for all x and y in H. If suffices to prove this for any x with finite support, but this is obvious by using Lemma 3.3.

Lemma 3.5. $L_N \overset{W}{\to} A^{1/2}U^T$.

Proof. Since $\langle U_N x,y\rangle \to \langle Ux,y\rangle$ and $||(A^N)^{1/2}y - A^{1/2}y|| \to 0$ for all x and y, we have

$$\langle (A^N)^{1/2}U_N^T x,y\rangle = \langle U_N^T x, (A^N)^{1/2}y\rangle$$

$$= \langle U_N^T x, A^{1/2} y\rangle + \langle U_N^T x, ((A^N)^{1/2} - A^{1/2}) y\rangle$$
$$\to \langle U^T x, A^{1/2} y\rangle = \langle A^{1/2} U^T x, y\rangle,$$

or

$$L_N = (A^N)^{1/2} U_N^T \to A^{1/2} U^T.$$

Let $L := A^{1/2}U^T$. The above lemma says that $L_N \overset{W}{\to} L$. We will show that L is a Cholesky factor of A.

Lemma 3.6. $A = LL^T$ where L is lower triangular.

Proof. Since $L_N = (A_N)^{1/2} U_N^T \overset{W}{\to} A^{1/2}U^T = L$, and $U^T = U^*$, we have $LL^T = A^{1/2}U^T U A^{1/2} = A^{1/2}U^* U A^{1/2} = A$.

At this point, we have shown that L^T is upper triangular and is invertible. To establish the uniqueness result, we also need $(L^T)^{-1}$ be upper triangular. We first prove this result for $(L_N^T)^{-1}$.

Lemma 3.7. The columns of U_N converge to the columns of U in H. $(L_N^T)^{-1} \overset{W}{\to} (L^T)^{-1}$ and $(L^T)^{-1}$ is upper triangular.

Proof. To prove the convergence of the columns of U_N, it is sufficient to consider the "zeroth" column. The "zeroth" column of U is

$$\hat{u}_0 = Ue_0 = [\ldots, \langle u_i, e_0\rangle, \ldots,]^T$$

where u_i are the rows of U. For each $\varepsilon > 0$, pick N_0 so large that

$$\sum_{|i| \le N_0} |\langle u_i, e_0\rangle|^2 > 1 - \varepsilon.$$

Also, let M_0 be chosen such that

$$||u_{N,i} - u_i|| \le (\varepsilon/N_0)^2$$

for $|i| \le M_0$. Hence, $||\hat{u}_{N,0} - \hat{u}_0|| < \varepsilon$ for all sufficiently large N. To show that $(L_N^T)^{-1} \overset{W}{\to} (L^T)^{-1}$, we first note that $L_N^T = U_N (A^N)^{1/2}$ where $(A^N)^{1/2} \overset{S}{\to} A^{1/2}$. Since the rows of U_N^T converge (in norm) to the rows of U^T, we also have $U_N^T \overset{S}{\to} U^T$. In addition, since A is positive definite, A^{-1} is a power series in $I - A$ (in case

$||A|| < 1$ which we may assume without loss of generality) so that $(A^N)^{-1/2} \overset{S}{\to} A^{-1/2}$. Hence, following the proof of Lemma 3.5, we have $(L_N^T)^{-1} = (A^N)^{-1/2} U_N^T \overset{W}{\to} A^{-1/2}U^T = (L^T)^{-1}$. Finally, since weak operator limits of upper triangular matrices are upper triangular, it follows that $(L^T)^{-1}$ is also upper triangular.

We can now state the main result of the section, namely:

Theorem 3.1. Let $A = LL^T$ where $L = [\ell_{ij}]$ and L^{-1} are lower triangular as obtained above. If $\ell_{ii} > 0$ for all i, then this factorization is unique.

Proof. Suppose $A = LL^T = \tilde{L}\tilde{L}^T$ where $\tilde{L} = [\ell_{ij}]$ is lower triangular and $\ell_{ii} > 0$ for all i. Then $\tilde{L}^{-1}L = \tilde{L}^T(L^T)^{-1}$ where $\tilde{L}^{-1}L$ is lower triangular and $\tilde{L}^T(L^T)^{-1}$ is upper triangular by Lemma 3.7. Hence, they are both equal to a diagonal operator D. Hence,

$$\tilde{L}^{-1}L = \tilde{L}^T(L^T)^{-1} = D = D^T = L^T(\tilde{L}^{-1})^T,$$

where D is a diagonal operator with nonnegative entries, so that

$$D^2 = (L^T(\tilde{L}^{-1})^T)\tilde{L}^T(L^T)^{-1} = I.$$

Since D is a positive operator, we have $D = I$. That is, $L = \tilde{L}$.

4. APPLICATION TO BLOCK TOEPLITZ MATRICES

Let $T = [t_{ij}]$, $t_{ij} = c_{i-j}$ be a bi-infinite Toeplitz matrix of real numbers. We define the symbol of T by $T(z) = \Sigma c_j z^j$. It is well known [17, p. 184] that T is a bounded operator on ℓ_2 if and only if the sequence $\{c_k\}$ is a sequence of Fourier coefficients of some L^∞ function f on the unit circle. Note that if $T(z) = \sum_{j=0}^{\infty} c_j z^j$ then T is lower triangular. The function $T(\cdot)$ is called the symbol of the Toeplitz matrix T. Almost everything that is known on bi-infinite Toeplitz matrices can be obtained by studying their associated symbols. In particular, we include the next proposition and proof for the reader's convenience.

Proposition 4.1. Let A be a $2n$-banded positive definite real Toeplitz bi-infinite matrix. Then $A = LL^T$ where L is an n-banded lower triangular Toeplitz matrix.

Proof. Write $A = [a_{i-j}]$ and let

$$A(z) = a_0 + a_1[z+z^{-1}] + \dots + a_n[z^n + z^{-n}]$$

be its symbol. Since the a_i are real we see that the non-real roots of A appear in conjugate pairs. In addition, since A is positive definite, $A(z) > 0$ for $|z| = 1$. Finally $A(z) = A(z^{-1})$ so that the roots appear in reciprocal pairs. Let $|z_1| \le \dots \le |z_{2n}|$ be the roots of A. Set $q(z) = c_0 \prod_{i=1}^{n} (1 - \frac{z}{z_{n+i}})$ with $c_0 > 0$ chosen so that $q(z)q(z^{-1}) = A(z)$. Note that q has real coefficients. Then $L = c_0 G_1 \dots G_n$ where G_i are lower triangular 1-banded Toeplitz matrices with 1 on the diagonal and $-\frac{1}{z_{n+i}}$ on the sub-diagonal. Since $|z_{n+i}| > 1$ for $i = 1,\dots,n$ we see that the G_i are invertible as lower triangular matrices and we have $A = LL^T$ where the L agrees with the Cholesky factor in Theorem 3.1.

Let us compare our results with those in the literature. Let R denote the class of all functions on the unit circle having absolutely convergent Fourier series. A continuous function on the unit circle is said to have a factorization if it can be written as a product of two functions, one holomophic in $|z| < 1$ and continuous on $|z| \le 1$ and the other holomorphic in $|z| > 1$. The factorization is said to be proper, if one of these two factors is zero-free in the appropriate region $|z| \le 1$ or $|z| \ge 1$ and is said to be canonical, if both are zero-free in the corresponding regions $|z| \le 1$ and $|z| \ge 1$ (cf. [13; p. 197]). Two factorizations $g(z) = g_+(z)g_-(z)$ and $g(z) = \tilde{g}_+(z)\tilde{g}_-(z)$ are

said to be essentially similar, if $g_+(z) = \alpha\tilde{g}_+(z)$ and $g_-(z) = \alpha^{-1}\tilde{g}_-(z)$ for some complex constant α. If $g(z) \ne 0$ for $|z| = 1$, ind g will denote the index of g along the unit circle. Then if $g \in R$ admits a canonical factorization, g must satisfy:
(i) $g(z) \ne 0$ for $|z| = 1$ and (ii) ind $g = 0$. The converse of this also holds. Indeed, if $g \in R$ is zero-free on $|z| = 1$, then by a theorem of Wiener and Levy, [see 12] $\ln g$ is also in R. Hence, we may define

$$g_+(z) = \exp\{\beta_0/2 + \sum_{j=1}^{\infty} \beta_j z^j\}$$

for $|z| \le 1$ and

$$g_-(z) = \exp\{\beta_0/2 + \sum_{j=-\infty}^{-1} \beta_j z^j\}$$

for $|z| \ge 1$, where $\ln g(z) = \Sigma_{-\infty}^{\infty} \beta_j z^j$. Then g_+ and g_- are in R, zero-free in $|z| \le 1$ and $|z| \ge 1$ respectively, and $g(z) = g_+(z)g_-(z)$ for $|z| = 1$. More precisely, there is the following result of Krein [12].

Theorem (Krein). For a function $g \in R$ to admit a canonical factorization, it is necessary and sufficient that $g(z) \ne 0$ for $|z| = 1$ and ind $g = 0$. Furthermore, if g admits a canonical factorization, then all canonical factorizations of g are essentially similar.

Hence, in view of Theorem 3.1, we have the following.

Corollary 4.1. Let $g \in R$ be the symbol of a positive operator A which admits a canonical factorization. Let $g(z) = g_+(z)g_-(z)$ for $|z| = 1$ as discussed above. Then $g_+(z)$ and $g_-(z)$ are the symbols of the Cholesky factors L and L^T of A obtained in Theorem 3.1.

We next discuss factorization of block Toeplitz matrices. A matrix $T = [t_{i,j}]$ is said to be block Toeplitz, if $t_{i,j} = t_{i+r,j+r}$ for some positive integer r and all i and j. The smallest integer r is called the size of the block Toeplitz matrix. We have the following result.

Theorem 4.1. Let A be a positive definite bi-infinite block Toeplitz matrix of size r. Then the Cholesky factors L and L^T of A obtained in Theorem 3.1 are also block Toeplitz of size r.

Proof. As in Section 3 let A^N denote the compression of A onto the coordinates [-N,-N+1,...,). Then we have seen that

$$A^N = L^N(L^N)^T$$

where L^N is the Cholesky factorization of A^N and in particular we know that $L^N_{i,j} \to L_{i,j}$ where $LL^T = A$. Thus we need to verify that $L_{i,j} = L_{i+r,j+r}$, but this follows from the argument below. Recalling that the Cholesky factorization of A^N can be generated from the top down we see that for fixed i and j

$$L^N_{i,j} = L^{N+r}_{i-r,j-r}$$

since A^N is block Toeplitz of order r. Now sending N to ∞ yields the result.

As a corollary we obtain a result which we can use in the next section in the context of spline interpolation.

Corollary 4.2. If T is a block Toeplitz positive definite matrix of size r, that is, $T = [A_{i-j}]$ where the A_i are r×r matrices with $A_R = A^T_{-R}$, then the symbol T(z) of T is defined by $T(z) = \sum_{j=-\infty}^{\infty} A_j z^j$ and the symbol can be factored as $T(z) = L(z)L^T(z) = L(z)L(z^{-1})$ where L is any bounded lower triangular block Toeplitz matrix so that $LL^T = A$.

We remark that this factorization for any block Toeplitz matrix whose symbol is positive definite but does not have absolutely convergent Fourier series appears to be new.

5. FURTHER APPLICATIONS

In [11], Holmes gave an operator-theoretic overview of discrete-time single-channel signal processing. In particular, he discussed the notion of scalar bi-sequences and stationary bi-sequences.

Let H be an arbitrary Hilbert space. Fix an $x_0 \in H$ and a unitary operator U on H. Define

$$x_n = U^n(x_0) \quad n = 0, \pm 1, \pm 2, \dots \quad .$$

It is easily seen that

$$\langle x_{m+k}, x_{n+k} \rangle = \langle x_m, x_n \rangle$$

for all integers m, n and k. Such a bi-infinite sequence (bi-sequence in Holmes' terminology) is called stationary. On the other hand, suppose a scalar bi-infinite sequence $\{r_n\}$ has the properties

i) $|r_n| \le r_0$, ii) $r_{-n} = \overline{r}_n$ for any n, iii) $[\hat{r}_{i,j}] \ge 0$ where $\hat{r}_{i,j} = r_{i-j}$. Such a sequence is said to be a positive bi-sequence. It is easily seen that via the rule $r_n = \langle x_n, x_0 \rangle$ any stationary sequence x_n generates a positive bi-sequence r_n. Holmes [11] gave a heuristic argument why a positive bi-sequence should evolve from a stationary bi-sequence. In what follows, we prove that a positive bi-sequence $\{r_n\}$ for which $[\hat{r}_{i,j}]$ is positive definite is generated by some stationary sequence $\{x_n\}$.

Proposition 5.1. Suppose the positive bi-sequence $\{r_n\}$ satisfies the property that the bi-infinite matrix $[\hat{r}_{i,j}]$ is positive definite. Then $r_n = \langle x_n, x_0 \rangle$ for some stationary sequence $\{x_n\}$.

Proof. Write $R := [\hat{r}_{i,j}] = LL^*$ where L is lower triangular and Toeplitz. Now if $\{w_n\}$ is an orthonormal bi-sequence, set

$$x_n = \sum_{j=-\infty}^{\infty} \ell_{n,j} w_j \quad \text{where} \quad L = [\ell_{i,j}].$$

It is straightforward to verify that $\langle x_{n+k}, x_n \rangle = r_k$ for all k and n.

We now turn our attention to a problem in spline interpolation. Let $\underline{t}$ be a bi-infinite partition of the real line which is periodic of order r; that is, $t_{i+r} = t_i + 1$ and $t_{i+1} > t_i$ for all i. Let $S^m_{\underline{t}}$ be the linear space of C^{m-2} piecewise polynomials of order m with breakpoints at $\underline{t}$. One of the fundamental questions in spline interpolation concerns the existence and unicity of a bounded spline interpolant to given bounded data. In particular, many results exist for the bounded interpolation problem: Find $s \in S^{2k}_{\underline{t}} \cap L_\infty$ so that

(*) $\quad s(t_i) = x_i, \quad -\infty < i < \infty,$

with $||x||_\infty < \infty$.

In this section we will show that the subspace of splines vanishing at all the knots has a basis of exponentially growing or decaying splines and hence (*) has a unique solution. More precisely we will show that the growth rates occur in reciprocal pairs. That (*) has a unique solution was shown in [1] and [7].

First recall that each $s \in S^{2k}_{\underline{t}}$ has a unique representation as a linear combination of normalized B-splines

$$s(\tau) = \sum_{j=-\infty}^{\infty} \alpha_j N_{j,2k,\underline{t}}(\tau)$$

where $N_{j,2k,\underline{t}}(\tau) := (t_{j+2k} - t_j)[t_j, \ldots, t_{j+2k}](\cdot - \tau)_+^{2k-1}$.
This representation allows us to rewrite the interpolation problem as

$$A\,\underline{\alpha} = \underline{x}$$

where $A_{i,j} := (N_{j,2k,\underline{t}}(t_i))$. Note that A is a block banded Toeplitz matrix with block size r. Furthermore, it is clear that s is bounded if and only if $\underline{\alpha}$ is bounded. Thus the bounded interpolation problem is equivalent to determining if A is boundedly invertible on ℓ_∞. We know from the Toeplitz theory that A is boundedly invertible on ℓ_∞ if and only if the determinant of its symbol, $A(z)$, does not vanish on the unit circle $|z| = 1$. It will be shown that assuming a periodic knot sequence, $\det A(z)$ never vanishes on the unit circle. It has been shown in [6] that if $\det A(z_0) = 0$, then z_0 is real and $(-1)^r z_0 \geq 0$. Furthermore it was established in [7, 15] that the singular values z_0 have multiplicity one. It will be established here that $\det A(z)$ vanishes for reciprocal pairs; that is, $\det A(z) = 0$ implies $\det A(z^{-1}) = 0$. This fact together with the above implies $\det A(z)$ is never zero for $|z| = 1$ and hence there exists a unique solution for (*).

Let z_0 be a complex number such that $\det A(z_0) = 0$. Then there is a nonzero r-vector $\underline{w}$ so that

$$A(z_0)\underline{w} = 0$$

and thus $\underline{\alpha}^* := (\ldots, z_0^{-1}\underline{w}, \underline{w}, z_0\underline{w}, z_0^2\underline{w}, \ldots)^T$ satisfies $A\underline{\alpha}^* = 0$ and hence

$$s(\tau) := \sum_j \alpha_j^* N_{j,2k,\underline{t}}(\tau)$$

is a <u>null spline</u> with exponential decay/growth determined by z_0. Such a spline is an example of an eigenspline for $\underline{t}$ with growth rate λ, that is $s(x+1) = \lambda s(x)$ for all x. In fact most people attack the bounded interpolation problem from the study of nullsplines. We will prove

<u>Theorem 5.2</u>. <u>The space of nullsplines</u> $\eta := \{s \in S^{2k}_{\underline{t}} : s(t_i) = 0\}$ <u>is spanned by the eigensplines with distinct growth rates</u> $\{\lambda_i\}_{i=1}^{2k-2}$ <u>which satisfy</u> $\lambda_i \lambda_{2k-2-i+1} = 1$, $i = 1, \ldots, k-1$. <u>Consequently problem</u> (*) <u>is well posed</u>.

<u>Remark</u>: It was shown by Friedland and Micchelli [7] that the product of the λ_i is 1 and further it was known that if the nullsplines were preserved under the change of variables $t \to -t$ then the growth rates came

in reciprocal pairs. Our contribution is to demonstrate that this observation is generally true whether or not the nullsplines are preserved under the change of variable $t \to -t$.

Proof. The proof of this result follows by relating the interpolation problem to a least squares problem in which the nullsplines for each problem have a clear relation to each other as will be shown below. Since we can factor the symbol for the least squares problem (Cor. 4.2) $S(z) = L(z)L(z^{-1})$ we see that $\det S(z_0) = 0$ implies $\det S(z_0^{-1}) = 0$ and thus the null vectors can be paired with reciprocal growth rates. Note that $s \in \eta$ if and only if

$$\int_R s^{(k)}(\tau) N_{i,k,\underline{t}}(\tau)\, d\tau = 0$$

by the Peano Kernel Theorem. Now $s^{(k)} \in S_{\underline{t}}^{k}$ and hence $s^{(k)} = \Sigma \beta_i N_{i,k,\underline{t}}$ where the vector $\underline{\beta}$ is a null vector for the bi-infinite least squares matrix

$$s_{ij} := \int_R N_{i,k,\underline{t}}(\tau)\; N_{j,k,\underline{t}}(\tau)\, d\tau \quad .$$

S is Toeplitz with block size r and is 2k-2 strictly banded and hence has a nullspace of dimension 2k-2 [2]. Thus we see that there is a one-to-one correspondence between the null vectors for A and the null vectors for S. However, S is positive definite and symmetric which guarantees that the growth rates of the null vectors come in reciprocal pairs. This feature will be inherited by the null vectors for A (when we integrate back) and this completes the proof.

We remark that these results could be obtained by noting that the symmetric positive definite matrix S is totally positive. Thus the results of [4] imply that $S = LL^T$ where this is a stable factorization. Since this factorization may be achieved by factoring compressions of S we conclude as in Theorem 4.1 that L is block Toeplitz and the result follows.

ACKNOWLEDGEMENT

The work of these authors was supported by the U.S. Army Research Office under Contract No. DAAG 29-81-K-0013.

REFERENCES

[1] de Boor, C., Odd-degree spline interpolation at a bi-infinite knot sequence; in Approximation Theory, Bonn, 1976, R. Schaback and K. Scherer, eds., Lecture Notes Math. 556, Springer, Heidelberg, 1976, 30-53.

[2] de Boor, C., What is the main diagonal of a bi-infinite band matrix?, in Quantitative Approximation, R. DeVore and K. Scherer, eds., Academic Press, 1980, 11-23.

[3] de Boor, C. and Pinkus, A., The approximation of a totally positive band matrix by a strictly banded totally positive one, Linear Algebra and Applications, to appear.

[4] de Boor, C., Jia, R., and Pinkus, A., Structure of invertible (bi)infinite totally positive matrices, MRC TSR 2311, 1981; Linear Algebra and Applications, to appear.

[5] Cavaretta, A., Dahmen, W., Micchelli, C. A. and Smith, P. W., A factorization theorem for band matrices, Linear Algebra and Applications, to appear.

[6] __________, On the solvability of certain systems of linear difference equations, SIAM J. of Math. Analysis, to appear.

[7] Friedland S. and Micchelli, A., Bounds on the solutions of difference equations and spline interpolation at knots, Linear Algebra and Its Applications (1978) 20, 219-251.

[8] Gohberg, I. C. and Krein M. G., Introduction to the Theory of Linear Nonself-adjoint Operators, Translations of mathematical monographs, Vol. 18.

[9] __________, Theory and Application of Volterra Operators in Hilbert Space, Translations of mathematical monographs, Vol. 24.

[10] Halmos, P. R., A Hilbert Space Problem Book, Van nostrand, New York, 1967.

[11] Holmes, R. B., Mathematical foundations of signal processing, SIAM Review, Vol. 21 (1979), Number 3, 361-389.

[12] Katznelson, Y., An Introduction to Harmonic Analysis, Dover Publications, Inc., New York, 1968.

[13] Krein, M. G., Integral equations on the half-line with kernel depending upon the difference of the arguments, AMS Translations (2) 22, 1962, 163-288.

[14] Lawson, C. L. and Hanson, R. J., Solving Least Squares Problems, Prentice-Hall Inc., Englewood Cliffs, N. J., 1974.

[15] Lee, S. L., Micchelli, C. A., Sharma, A., and Smith, P. W., Some properties of periodic B-spline collocation matrices, IBM Research Report, RC 8816 (#38643).

[16] Micchelli, C. A., Infinite spline interpolation, in Approximation in Theorie und Praxis. Ein Symposimsbericht, G. Meinardus ed., Bibliographisches Institut, Mannheim, 1979, 209-238.

[17] Widom, H., Toeplitz matrices, Studies in Real and Complex Analysis, Vol. 3, MAA studies in mathematics, U.S.A.

Received: December 1981

Final Revision: July 1982

NUMER. FUNCT. ANAL. AND OPTIMIZ., 5(1), 21-46 (1982)

A LOCAL VERSION OF HAAR'S THEOREM IN APPROXIMATION THEORY

Günther Nürnberger

Institut für Angewandte Mathematik
Universität Erlangen-Nürnberg
8520 Erlangen, West-Germany

ABSTRACT

We give a complete characterization of those functions in $C_o(T)$, where T is a locally compact subset of the real line, which have a strongly unique best approximation from an n-dimensional weak Chebyshev subspace of $C_o(T)$ and apply this result to spline functions of degree n with k fixed knots. Furthermore we prove similar results for unique best approximations.

0. INTRODUCTION. We consider the following approximation problem. If $G=\text{span}\{g_1,\dots,g_n\}$ is an n-dimensional subspace of $C_o(T)$, where T is a locally compact subset of the real line, and $f\in C_o(T)$, then a function $g_o\in G$ is called a best approximation of f from G, if $\|f-g\| \geq \|f-g_o\|$ for all $g\in G$. A function $g_o\in G$ is called strongly unique best approximation of f from G, if there exists a constant K>0 such that

$\|f-g\| \geq \|f-g_o\| + K\|g-g_o\|$ for all $g\in G$.

A fundamental result in approximation theory says, that the following three conditions are equivalent:

(1) Every $f\in C_o(T)$ has a unique best approximation from G. (2) Every $f\in C_o(T)$ has a strongly unique best approximation from G. (3) For all distinct