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Some remarks on Rota's umbral calculus

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SUMMARY

It is shown that Rota's theory of Sheffer polynomials can be generalized to the quotient field of the ring of formal power series in $\frac{1}{x}$. As a special case we give some applications to the classical theories of factorial series and of Laguerre polynomials.

1. INTRODUCTION

The purpose of this note is to show that Rota's theory of polynomials of binomial type and more generally of Sheffer polynomials (cf. [2], [4], [1]) can be generalized to include "polynomials of negative degree". This generalization makes it possible to include the theory of factorial series into Rota's theory and thus solve a problem posed in [4], p. 753.

In order to avoid repetitions of well-known facts we follow the notation and terminology of [4] if not stated otherwise. The starting point of this investigation was the observation that there are a lot of functions which may be called Sheffer polynomials of negative degree. Simple examples are the powers x^n or $(1+x)^n$ and the (lower) factorials

$$(x)_n = \frac{\Gamma(x+1)}{\Gamma(x-n+1)}$$

which are defined for all $n \in Z$. The problem of connection constants for these examples leads to binomial and factorial series. Since convergence questions would obscure the simple formalism we want to develop, we have chosen to work in the quotient field F of formal power series in $\frac{1}{x}$.

2. OPERATORS ON F

Let F be the quotient field of the ring of formal power series in $\frac{1}{x}$ (over a field of characteristic zero). An element $f \in F$ has a unique representation in the form

$$f(x) = \sum_{k \in Z} a_k x^k$$

where $a_k = 0$ for all $k > n$. The largest integer n such that $a_n \neq 0$ is called the degree deg f of f.

If $f(x) = \sum a_k x^k$ and $g(x) = \sum b_l x^l$ are elements of F, then multiplication is defined by $(fg)(x) = \sum c_n x^n$ with $c_n = \sum_{k+l=n} a_k b_l$. Since all coefficients with sufficiently high index vanish, this sum is always finite.

A finite or infinite sequence $(f_n)_1^\infty$ of elements $f_n \in F$ is called summable if deg $f_n \leqslant M < \infty$ for all n and only finitely many elements of the sequence have the same degree.

For each summable sequence (f_n), $f_n = \sum a_{nk} x^k$, we define the sum $\sum f_n$ by

$$(\sum f_n)(x) = \sum_k (\sum_n a_{nk}) x^k.$$

Let us denote by P the ring of all polynomials in x. Then P is a subring of F.

By multiplying both sides with $(x+a)^n$ for $n > 0$, it is clear that formulas such as

$$\frac{1}{(x+a)^n} = \sum_{k=0}^{\infty} \binom{-n}{k} a^k \frac{1}{x^{n+k}}$$

hold in F.

DEFINITION: A linear mapping $S: F \to F$ is called operator if for each summable sequence (f_n) the sequence (Sf_n) is also summable and $S(\sum f_n) = \sum (Sf_n)$. The set of all operators on F is denoted by $L(F)$.

EXAMPLES:

1. Let $g \in F$. Then $f \to gf$ defines an operator on F, the multiplication operator g.

(From the context it is always clear whether we mean the element $g \in F$ or the operator g).

2. Define D by

$$D(\sum a_k x^k) = \sum a_k k x^{k-1}.$$

Then clearly $D \in L(F)$. We call it the differentiation operator. In order to have a suggestive terminology we call an element $p_n \in F$ of the form

$$p_n(x) = \sum_{k=0}^{\infty} a_k\, D^k\, x^n, \quad n \in Z, \quad a_0 \neq 0,$$

a polynomial of degree n. For $n \geqslant 0$ this gives of course an ordinary polynomial.

3. Let now $\sum_{k=0}^{\infty} \frac{a_k}{k!} D^k$ be a formal power series in D. This defines an operator $a(D) \in L(F)$ by setting $a(D)f = \sum_{k=0}^{\infty} \frac{a_k}{k!}(D^k f)$. The element $a(D)f$ is well defined since $(D^k f)$ is a summable family. It is also clear that $\deg a(D)f \leqslant \deg f$. Therefore $a(D) \in L(F)$.

REMARK: In P we have

$$(E^a p)(x) = p(x+a) = \left(\sum_{k=0}^{\infty} \frac{a^k}{k!} D^k\right) p(x).$$

Since for $n > 0$ we also have

$$E^a \frac{1}{x^n} = \frac{1}{(x+a)^n} = \sum_{k \geqslant 0} \binom{-n}{k} a^k \frac{1}{x^{n+k}} = \left(\sum_{k \geqslant 0} \frac{a^k}{k!} D^k\right) \frac{1}{x^n},$$

we may define

$$f(x+a) = (E^a f)(x) = (e^{aD} f)(x)$$

for each $f \in F$ and get thus an extension of the translation operator E^a from P to F.

4. We call a set $(p_n)_{n \in Z}$ of polynomials $p_n \in F$ admissible if each p_n has degree n and $p_0 \equiv 1$. It is clear that each $f \in F$ has a unique representation of the form

$$f(x) = \sum_k a_k p_k(x).$$

Given an admissible set $p = (p_n)$, we define an operator

$$T = T(p) \in L(F) \text{ by } T(\sum a_k p_k) = \sum a_k p_{k+1}.$$

We call T the admissible operator corresponding to p. We may then write $p_n(x) = T^n 1$, $n \in Z$.

5. For any admissible set p with corresponding operator T denote by $F(p)$ the quotient field of formal power series in T^{-1}. Defining $\varepsilon_p : F \to F(p)$ by

$$\varepsilon_p (\sum a_k p_k) = \sum a_k T^k$$

we can interpret F as a vector space over the field $F(p)$ by setting

$$\lambda\, \varepsilon_p^{-1}(\mu) = \varepsilon_p^{-1}(\lambda\mu) \quad \text{for } \lambda, \mu \in F(p).$$

It is then obvious that each element $f(T) \in F(p)$ defines an operator $f(T) \in L(F)$ by

$$f(T)\, \varepsilon_p^{-1}(g(T)) = \varepsilon_p^{-1}(f(T)g(T))$$

for each $g(T) \in F(p)$.

6. Given any admissible set p with corresponding operator T there exists a uniquely determined operator R such that $R1 = 0$ and $RT - TR = I$ where I denotes the identity operator.

PROOF: Suppose that R exists satisfying $RT - TR = I$. Then we have for all $n \in Z$

$$RT^n = T^n R + nT^{n-1}.$$

This follows by induction for $n > 0$.

By multiplying both sides of $RT - TR$ with T^{-1} we get $T^{-1}R - RT^{-1} = T^{-2}$ or $RT^{-1} = T^{-1}R - T^{-2}$. Again by induction we get the desired equation for all $n \in Z$.

Now we use $R1 = 0$. This gives us

$$Rp_n = RT^n 1 = (T^n R + nT^{n-1})1 = nT^{n-1}1 = np_{n-1}.$$

Therefore R must satisfy $R \sum a_k p_k = \sum a_k k p_{k-1}$. But it is clear that this R satisfies indeed $R1 = 0$ and $RT - TR = I$.

In the special case $p = (x^n)$ we have $T = x$ and $R = D$. Therefore we call $R = R(p)$ the p-differentiation operator.

7. Let p be an admissible set and R p-differentiation. Then for each formal power series $\sum_{k=0}^{\infty} \frac{a_k}{k!} R^k$ we get an operator $a(R)$ on F by

$$a(R)f = \sum_{k=0}^{\infty} \frac{a_k}{k!}(R^k f).$$

3. THE PINCHERLE DERIVATIVE

The Pincherle derivative is the mapping $\frac{\partial}{\partial D} : L(F) \to L(F)$ defined by

$$\frac{\partial S}{\partial D} = Sx - xS.$$

It is in a certain sense dual to differentiation with respect to x, defined by

$$\frac{\partial S}{\partial x} = DS - SD.$$

The equation $Dx - xD = I$ means $\frac{\partial D}{\partial D} = I$ and $\frac{\partial x}{\partial x} = I$. Furthermore we have $x\frac{\partial}{\partial D} = \frac{\partial}{\partial D}x$ and $D\frac{\partial}{\partial x} = \frac{\partial}{\partial x}D$.

Now we want to generalize this situation:

Let p be an admissible set of polynomials and T the corresponding operator. Let R be p-differentiation.

We can now define two linear operators on $L(F)$, the Pincherle derivatives $\frac{\partial}{\partial R}$ and $\frac{\partial}{\partial T}$.

DEFINITION: Let T be an admissible operator and R the uniquely defined operator satisfying $R1 = 0$ and $RT - TR = I$. Then we define $\frac{\partial S}{\partial R} = ST - TS$ and $\frac{\partial S}{\partial T} = RS - SR$ for each $S \in F(S)$. (Note the asymmetry in this definition).

It is obvious that $\frac{\partial R}{\partial R} = I$ and $\frac{\partial T}{\partial T} = I$ and that $T\frac{\partial}{\partial R} = \frac{\partial}{\partial R}T$ and $R\frac{\partial}{\partial T} = \frac{\partial}{\partial T}R$ holds.

Furthermore we have shown in 2. Example 6. that

$$\frac{\partial T^n}{\partial T} = nT^{n-1} \text{ for } n \in Z.$$

In the same way we can show that $\frac{\partial R^n}{\partial R} = nR^{n-1}$ for $n = 1, 2, 3, \ldots$. (Note that R is not invertible).

LEMMA: For the Pincherle derivative the Leibniz formula

$$\left(\frac{\partial}{\partial T}\right)^n (f(T)g(T)) = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{\partial}{\partial T}\right)^k f(T) \left(\frac{\partial}{\partial T}\right)^{n-k} g(T)$$

holds for $n = 1, 2, 3, \ldots$.

PROOF: It suffices to show this formula for $n = 1$. Then it follows easily by induction. But for $n = 1$ it is trivial:

$$\begin{aligned} \frac{\partial}{\partial T}(f(T)g(T)) &= Rf(T)g(T) - f(T)g(T)R = \\ &= (Rf(T) - f(T)R)g(T) + f(T)(Rg(T) - g(T)R) = \\ &= \frac{\partial}{\partial T}f(T) \cdot g(T) + f(T)\frac{\partial}{\partial T}g(T). \end{aligned}$$

We can now prove a very useful formula.

THEOREM 1: Let p be an admissible set of polynomials, T the corresponding admissible operator and R p-differentiation. Then for each formal power series $a(R)$ and each $f \in F(p)$ we have

$$a(R)f(T) = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{\partial}{\partial T}\right)^k f(T)\left(\frac{\partial}{\partial R}\right)^k a(R).$$

PROOF: First observe that

$$\varepsilon_p^{-1}\left(\frac{\partial f(T)}{\partial T}\right) = \frac{\partial f(T)}{\partial T}\,1 = (Rf(T) - f(T)R)1 = Rf(T)1 = R\,\varepsilon_p^{-1}(f(T)).$$

Now let $a(R) = \sum \frac{a_k}{k!} R^k$ and $g(T) \in F(p)$. Then we have

$$\begin{aligned}
a(R)f(T)g(T)1 &= \sum_{0}^{\infty} \frac{a_n}{n!} R^n f(T)g(T)1 = \\
&= \sum_n \frac{a_n}{n!}\left[\left(\frac{\partial}{\partial T}\right)^n (f(T)g(T))\right]1 \\
&= \sum_n \frac{a_n}{n!}\left[\sum_{k=0}^{n}\binom{n}{k}\left(\frac{\partial}{\partial T}\right)^k f(T)\left(\frac{\partial}{\partial T}\right)^{n-k} g(T)\right]1 \\
&= \sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{\partial}{\partial T}\right)^k f(T)\left(\sum_{n=k}^{\infty}\frac{a_n}{(n-k)!}\left(\frac{\partial}{\partial T}\right)^{n-k} g(T)\right)1 \\
&= \sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{\partial}{\partial T}\right)^k f(T)\left(\frac{\partial}{\partial R}\right)^k a(R)\,g(T)1
\end{aligned}$$

Since this holds for each $g(T) \in F(p)$ and thus for each $g \in F$, the theorem is proved.

COROLLARY: Under the same assumptions we have

$$a(R)T^n = \sum_{k=0}^{\infty}\binom{n}{k} T^{n-k}\left(\frac{\partial}{\partial R}\right)^k a(R) = \left(T + \frac{\partial}{\partial R}\right)^n a(R) \text{ for } n \in Z.$$

4. SEQUENCES OF BINOMIAL TYPE AND SHEFFER SETS

Consider now the operator $T: F \to F$ defined by $(Tf)(x) = x\frac{1}{Q'}f(x)$ where $Q = \sum_{k=1}^{\infty}\frac{a_k}{k!}D^k$ with $a_1 \neq 0$ is a delta operator and $Q' = \frac{\partial Q}{\partial D}$ its Pincherle derivative with respect to D.

It is obvious that $q_n(x) = T^n 1$, $n \in Z$, is a polynomial of degree n. So $q = (q_n)$ is an admissible set and T the corresponding operator.

Since $Q1=0$ and $Q\left(x\frac{1}{Q'}\right)-\left(x\frac{1}{Q'}\right)Q=(Qx-xQ)\frac{1}{Q'}=I$ we see that Q is the corresponding q-differentiation operator.

Therefore $Qq_n=n\,q_{n-1}$ for each $n\in Z$.

Since for $n>0$ we have $q_n(x)=\left(x\frac{1}{Q'}\right)q_{n-1}(x)$ and thus $q_n(0)=0$ we see that for $n\geqslant 0$ the set $(q_n(x))$ is precisely the set of basic polynomials for the delta operator Q ([4], p. 688).

So we have got an extension of this set of polynomials to negative indices.

EXAMPLE: Let $Q=\Delta=e^D-I$. Then $Q'=e^D=E$ and we get $q_n(x)=$ $=(xE^{-1})^n 1=(x)_n$ for all $n\in Z$. Suppose $n>0$. Then we have

$$(x)_n=x(x-1)\ldots(x-n+1) \text{ and } (x)_{-n}=\frac{1}{(x+1)(x+2)\ldots(x+n)}.$$

It should be noted that the polynomial $q_{-1}(x)$ contains the whole information about the delta operator Q.

For let $Q=\frac{a_1}{1!}D+\frac{a_2}{2!}D^2+\ldots$. Then

$$q_{-1}(x)=\left(Q'\frac{1}{x}\right)1=(a_1I+\frac{a_2}{1!}D+\ldots)\frac{1}{x}=\frac{a_1}{x}-\frac{a_2}{x^2}+\frac{a_3}{x^3}-+\ldots$$

Thus given $q_{-1}(x)$ we can find the coefficients a_k and therefore the delta operator Q.

PROPOSITION: The sequence $(q_n)_{n\in Z}$ has the binomial property

$$(E^a q_n)(x)=q_n(x+a)=\sum_{k=0}^{\infty}\binom{n}{k}q_k(a)\,q_{n-k}(x)$$

for all $n\in Z$.

PROOF: As is shown in [4] we have

$$e^{aD}=\sum_0^{\infty}\frac{a^k}{k!}D^k=\sum_0^{\infty}\frac{q_k(a)}{k!}Q^k.$$

This identity holds on P. But it is easy to see that it holds on F too since

$$\sum\frac{a^k}{k!}D^k=\sum\frac{q_k(a)}{k!}Q^k$$

is an identity in D (or Q) where each power D^k of D occurs on the right side only finitely often.

Therefore we have

$$E^a q_n=(\sum\frac{q_k(a)}{k!}Q^k)q_n=\sum_k\binom{n}{k}q_k(a)\,q_{n-k}$$

for all $n\in Z$.

Another way to show this is by using Theorem 1. Apply this to $e^{aD}T^n$. We get

$$e^{aD}\left(x\frac{1}{Q'}\right)^n = \sum_{k=0}^{\infty}\binom{n}{k}\left(x\frac{1}{Q'}\right)^{n-k}\left(\frac{\partial}{\partial Q}\right)^k e^{aD}.$$

Now from $e^{aD}=\sum\frac{q_k(a)}{k!}Q^k$ we see that $\left(\frac{\partial}{\partial Q}\right)^k e^{aD}1=q_k(a)$ and therefore

$$q_n(x+a)=e^{aD}\,q_n(x)=e^{aD}\left(x\frac{1}{Q'}\right)^n 1=\sum_{k=0}^{\infty}\binom{n}{k}q_k(a)\left(x\frac{1}{Q'}\right)^{n-k}1=$$

$$=\sum_{k=0}^{\infty}\binom{n}{k}q_k(a)\,q_{n-k}(x).$$

EXAMPLE: For $Q=\Delta$ and $n=-1$ we get

$$E^a(x)_{-1}=\frac{1}{x+a+1}=\sum_{k=0}^{\infty}(-1)^k(a)_k\,(x)_{-k-1}.$$

For $a=-1$ this reduces to

$$\frac{1}{x}=\sum_{k=0}^{\infty}\frac{k!}{(x+1)(x+2)\dots(x+k+1)}.$$

In order to get more insight into polynomials of binomial type of negative degree let us first define a composition for admissible sets of polynomials which generalizes umbral composition to arbitrary indices.

Let $p=(p_n)$ and $q=(q_n)$ be two admissible sets. Then we define $r=p\circ q$ by $(p\circ q)_n\,(x)=p_n\,(q(x)$ in the umbral notation of [2]. This means the following: Let $p_n(x)=\sum a_{nk}x^k$. Then

$$(p\circ q)_n(x)=\sum_k a_{nk}\,q_k(x).$$

It is clear that the admissible sets form a group under this operation, the unit element being the set $e=(x^n)_{n\in Z}$.

Let $T(p)$ be the operator corresponding to the admissible set p. Then $T(e)f(x)=xf(x)$, the multiplication operator by x.

We now define $T(p)\circ T(q):=T(p\circ q)$. This defines an operation $\circ$ on all admissible operators. Theorem 7 in [4] proves that for the sets of binomial type p corresponding to the delta operator P and q corresponding to the delta operator Q with $T(p)=x\frac{1}{P'}$ and $T(q)=x\frac{1}{Q'}$ we have

$$T(p)\circ T(q)=T(p\circ q)=x\,\frac{1}{Q'}\,\frac{1}{P'(Q)}=x\frac{1}{(P\circ Q)'}.$$

As a special case we get for the inverse set $p=q^{-1}$ with

$$p \circ q = q \circ p = e$$

that $T(p)=T(q^{-1})=x\dfrac{1}{(G(D))'}$ if $Q=g(D)$ and G is the inverse defined by $G(g(D))=g(G(D))=D$.

EXAMPLE: The inverse set to $((x)_n)$ corresponds to the delta operator $G(D)= \log(1+D)$. Therefore it is given by

$$p_n(x)=(x(1+D))^n 1 \quad \text{for } n \in Z.$$

For $n \geqslant 0$ we get the exponential polynomials.

A direct computation gives us

$$p_{-1}(x)= \frac{1}{1+D}\,\frac{1}{x}=(1-D+D^2-+\dots)\frac{1}{x}=\frac{1}{x}+\frac{1!}{x^2}+\frac{2!}{x^3}+\dots \text{ and}$$

$$p_{-2}(x)=\frac{1}{1+D}\,\frac{1}{x}p_{-1}(x)=\sum_{k=0}^{\infty}(k+1)!\left(1+\tfrac{1}{2}+\dots+\frac{1}{k+1}\right)\frac{1}{x^{k+2}}.$$

Now we can prove a remarkable fact.

THEOREM 2: Let $q=(q_n)$ be the set of binomial type corresponding to the delta operator $Q=g(D)$ and let $p=(p_n)$ denote the inverse set corresponding to the delta operator $P=G(D)$. Then we have

$$\sum_{k=0}^{\infty}(-1)^k q_k(a)\frac{1}{x^k}=\sum_{k=0}^{\infty}(-1)^k a^k p_{-k}(x).$$

PROOF: We start with the formula

$$e^{aD}=\sum\frac{q_k(a)}{k!}(g(D))^k=\sum\frac{a^k}{k!}D^k$$

which implies

$$e^{aG(D)}=\sum\frac{q_k(a)}{k!}D^k=\sum\frac{a^k}{k!}G(D)^k.$$

Now we have

$$\frac{\partial e^{aG(D)}}{\partial D}=e^{aG(D)}x-xe^{aG(D)}=aG'(D)e^{aG(D)}.$$

Applying this to $\frac{1}{x} \in F$ we get

$$e^{aG(D)}1 - xe^{aG(D)}\frac{1}{x} = a\, e^{aG(D)}\left(G'(D)\,\frac{1}{x}\right)$$

$$\Rightarrow \qquad 1 - xe^{aG(D)}\frac{1}{x} = a\, e^{aG(D)}\, p_{-1}(x)$$

$$\Rightarrow \qquad xe^{aG(D)}\frac{1}{x} = 1 - a\sum_{k=0}^{\infty}\frac{a^k}{k!}(G(D))^k\, p_{-1}(x)$$

$$= \sum_{k=0}^{\infty}(-1)^k a^k\, p_{-k}(x).$$

On the other hand we have

$$x\, e^{aG(D)}\frac{1}{x} = x\sum\frac{q_k(a)}{k!}D^k\frac{1}{x} = \sum_{0}^{\infty}(-1)^k q_k(a)\frac{1}{x^k},$$

which proves our theorem.

COROLLARY: Let p and q be as in the theorem and set

$$q_n(x) = \sum_{k=0}^{n} c_{nk}x^k \text{ for } n \geqslant 0.$$

Then

$$p_{-n}(x) = \sum_{k=0}^{\infty}(-1)^k c_{n+k,n}\frac{1}{x^{n+k}}.$$

PROOF: The coefficient of $\frac{a^n}{x^{n+k}}$ in the first sum of the theorem is $(-1)^{n+k}c_{n+k,n}$ and in the second sum $=$ $(-1)^n$ times the coefficient of $\frac{1}{x^{n+k}}$ in $p_{-n}(x)$.

REMARK: This may be considered as a generalization of [4], Theorem 5, Cor. 2.

EXAMPLE: Let $q_n(x) = (x)_n$ and let p be the inverse set. Then

$$(x)_n = \sum_{k=0}^{n} s(n,k)x^k \text{ for } n \geqslant 0.$$

Therefore we get

$$p_{-n}(x) = \sum_{k=0}^{\infty}(-1)^k s(n+k,n)\frac{1}{x^{n+k}} = \sum_{k=0}^{\infty}|s(n+k,n)|\frac{1}{x^{n+k}}$$

where $s(n,k)$ are the Stirling numbers of the first kind.

Now let the delta operator Q be given. Then $T = x\frac{1}{Q'}$ is one operator satisfying $QT - TQ = I$. If $a(Q)$ is any formal power series in Q,

$$a(Q) = a_0 I + \frac{a_1}{1!} Q + \frac{a_2}{2!} Q^2 + \dots,$$

then clearly also $T - a(Q)$ satisfies $Q(T - a(Q)) - (T - a(Q))Q = I$. (On P this would be the most general operator with this property. But unfortunately this is not so on F).

Define now

$$s(Q) = \exp\left(\frac{a_0 Q}{1!} + \frac{a_1 Q^2}{2!} + \frac{a_2 Q^3}{3!} + \dots\right).$$

Then $\frac{\partial s(Q)}{\partial Q} = a(Q)s(Q)$ and therefore we have

$$T - a(Q) = x\frac{1}{Q'} - a(Q) = x\frac{1}{Q'} - \frac{s'(Q)}{s(Q)}, \text{ with } s'(Q) = \frac{\partial s(Q)}{\partial Q}.$$

This may also be written in the form

$$x\frac{1}{Q'} - a(Q) = s^{-1}(Q)\left(x\frac{1}{Q'}\right)s(Q)$$

since

$$\left(x\frac{1}{Q'}\right)s(Q) = s(Q)\,x\frac{1}{Q'} - s'(Q).$$

From this representation it is obvious that each such operator is admissible.

For $n \geqslant 0$ we get

$$(T - a(Q))^n 1 = s^{-1}(Q)\left(x\frac{1}{Q'}\right)^n s(Q)1 = s^{-1}(Q)\left(x\frac{1}{Q'}\right)^n 1$$

since $s(Q)1 = 1$. This means that $s_n(x) = (T - a(Q))^n 1$ is the Sheffer set relative to Q and the invertible operator $s(Q)$ with

$$s^{-1}(Q) = \sum_0^\infty \frac{s_k}{k!} Q^k \text{ and } s_0 = 1.$$

We have thus obtained an extension of each Sheffer set. It is clear that $Qs_n = ns_{n-1}$ holds for all $n \in Z$ and also that

$$s_n(x+a) = \sum_{k=0}^\infty \binom{n}{k} s_k(a)\, q_{n-k}(x)$$

holds for all $n \in Z$.

EXAMPLE: For the Hermite polynomial with variance v we get the operator

$$x - vD = e^{-v(D^2/2)}\, x\, e^{v(D^2/2)}.$$

This means that

$$H_n^v(x) = (x - vD)^n 1 \text{ for all } n \in Z.$$

5. SOME GENERALIZATIONS AND EXAMPLES

Given any admissible sequence of polynomials (q_n) we can construct the pair of operators T and R. Let us call (q_n) a basic set if $q_n(0) = 0$ for $n = 1, 2, 3, \ldots$. Consider now an operator of the form $T - a(R)$ for some formal power series in R. As before there exists an invertible formal power series $s(R)$ such that

$$T - a(R) = s^{-1}(R) T s(R).$$

We then call the corresponding set of polynomials

$$s_n(x) = s^{-1}(R) q_n(x) = (T - a(R))^n 1$$

the Sheffer set relative to T and the invertible operator $s(R)$.

We can then carry over some of the results on basic sets of binomial type to this more general situation. Let e.g. $q = (q_n)$ be basic and T, R the corresponding operators. Then for each formal power series g the operator

$$T \frac{1}{g'(R)}, \text{ where } g'(R) = \frac{\partial g(R)}{\partial R},$$

is admissible and corresponds to the derivation $g(R)$. But in this case the inverse set to q need not be Sheffer for any function $G(R)$ of R.

It depends of course on the special problem, what operators have to be studied. Suppose e.g. that we want to study Laguerre polynomials. We may start from the formula

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n+\alpha)_{n-k}\, x^k = \frac{1}{(\alpha)_{-n}} \sum_{k=0}^{n} (-1)^k \binom{n}{k} x^k (\alpha)_{-k}$$

if $\alpha \notin Z$.

It is natural to consider in this case the admissible set $(x^k(\alpha)_{-k})_{k \in Z}$. In order to include also the case $\alpha = l \in Z$ we are led to redefine $(\alpha)_k$ by replacing each factor $\alpha - l = 0$ by the factor 1. This gives no change in the Laguerre polynomials themselves since they depend only on the quotient $\dfrac{(\alpha)_{-k}}{(\alpha)_{-n}}$ which remains unchanged. For convenience we restrict ourselves to $\alpha > -1$. Let T_α and R_α be the operators corresponding to the sequence $(x^k(\alpha)_{-k})$.

It is easy to verify that for $\alpha=0$ the operator T_0 is given by $T_0\, x^k = \frac{1}{k+1}\, x^{k+1}$ if $k \neq -1$ and $T_0 \frac{1}{x} = 1$, and $R_0 x^k = k^2 x^{k-1} = (DxD)x^k$. Let us denote by T the operator on P defined by $(Tp)(x) = \int\limits_0^x p(t)dt$. Then on P we have $T_0 = T$ and $R_0 = DxD$.

Similary we can convince ourselves that for $\alpha > -1$ we have on P the equations

$$T_\alpha = x^{-\alpha}\, T x^\alpha \text{ and } R_\alpha = D(x+\alpha T)D.$$

Our formula for $L_n^{(\alpha)}(x)$ gives us at once the representation

$$(\alpha)_{-n}\, L_n^{(\alpha)}(x) = (1-T_\alpha)^n 1.$$

From this formula some immediate consequences can be drawn:

1. $(-1)^n L_n^{(\alpha)}(x)(\alpha)_{-n} = e^{-R_\alpha}\, T_\alpha^n 1.$
This follows from the equation $e^{-R_\alpha}\, T_\alpha\, e^{R_\alpha} = T_\alpha - 1$ and shows that $(-1)^n L_n^{(\alpha)}(x)(\alpha)_{-n}$ is a Sheffer set relative to T_α and the invertible operator e^{R_α}.

2. The duplication formula:

$$L_n^{(\alpha)}(ax)(\alpha)_{-n} = (1-aT_\alpha)^n = [1-a+a(1-T_\alpha)]^n 1 =$$

$$= \sum_{k=0}^{n} \binom{n}{k} (1-a)^{n-k}\, a^k (1-T_\alpha)^k\, 1 = \sum_{k=0}^{n} \binom{n}{k} (1-a)^{n-k} a^k\, L_k^{(\alpha)}(x)(\alpha)_{-k}$$

3. $L_n^{(\alpha)}$ is self-inverse:
$(1-(1-T_\alpha))^n = T_\alpha^n$ gives

$$x^n(\alpha)_{-n} = T_\alpha^n 1 = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (1-T_\alpha)^k 1 = \sum_{k=0}^{n} (-1)^k \binom{n}{k} L_k^{(\alpha)}(x)(\alpha)_{-k},$$

i.e. $L_n^{(\alpha)}(L^{(\alpha)}(x)) = x^n$.

These easy proofs show that this is perhaps a more natural view on the Laguerre polynomials than that proposed by Rota ([4]). This feeling is supported by the fact that the classical inner product for Laguerre polynomials

$$(f(x), g(x)) = \frac{1}{\Gamma(\alpha+1)} \int\limits_0^\infty x^\alpha\, e^{-x}\, f(x)g(x)dx$$

has a simple expression in terms of R_α and T_α.

REMARK: It is of course possible to translate our formulation into Rota's and vice versa by working directly with the corresponding operators.

1. For example we know that $(-1)^n L_n^{(0)}(x)=n!\ (T-1)^n 1$. From Rota's theory we get $(-1)^n L_n^{(0)}(x)=(1-D)[x(1-D)^2]^n 1$. That these formulas are equivalent can be seen as follows:

$$(1-D)[x(1-D)^2]^n 1=[(1-D)x(1-D)]^n 1=[(TD-D)x(DT-D)]^n 1$$

since $DT=1$, and $TD=1$ on all polynomials with vanishing constant term. The last term equals

$$[(T-1)DxD(T-1)]^n 1=[(T-1)R(T-1)]^n 1=n!\ (T-1)^n 1$$

since $R(T-1)^n 1=n(T-1)^{n-1}1$.

2. Let us show that $L_n^{(\alpha)}(x)=x^{-\alpha}(D-1)^n x^{\alpha+n}$ for $n=0, 1, 2, \ldots$. Since we suppose $\alpha>-1$ we have $x^{-\alpha}Dx^{\alpha}=D_\alpha=T_\alpha^{-1}$ on P. This implies

$$x^{-\alpha}(D-1)^n x^{\alpha}x^n=(D_\alpha-1)^n x^n=(1-T_\alpha)^n D_\alpha^n x^n=$$

$$=(1+\alpha)(2+\alpha)\ldots(n+\alpha)(1-T_\alpha)^n 1=\frac{1}{(\alpha)_{-n}}(1-T_\alpha)^n 1=L_n^{(\alpha)}(x).$$

6. THE UMBRAL CALCULUS

Let p and q be admissible sets of polynomials. Then there exists a uniquely determined linear operator $U_q^p\colon F\to F$ which maps $q_n(x)$ into $(p\circ q)_n(x)$ for all $n\in Z$. To this operator U_q^p there corresponds a uniquely determined ring isomorphism $\varphi_q^p\colon F(p)\to F(p\circ q)$ such that the diagram

$$\begin{tikzcd} F \arrow[r, "U_q^p"] \arrow[d, "\varepsilon_q"'] & F \arrow[d, "\varepsilon_{p\circ q}"] \\ F(q) \arrow[r, "\varphi_q^p"'] & F(p\circ q) \end{tikzcd}$$

commutes. This ring isomorphism is given by

$$\varphi_q^p\left(\sum a_k\, T(q)^k\right)=\sum a_k\,(T(p\circ q))^k.$$

It satisfies $\varphi_q^p\,\varphi_e^q=\varphi_e^{p\circ q}$ because of

$$\varepsilon_{p\circ q}^{-1}\,\varphi_e^{p\circ q}\,\varepsilon_e(x^n)=(p\circ q)_n(x)=U_q^p\,U_e^q(x^n)=\varepsilon_{p\circ q}^{-1}\,\varphi_q^p\,\varepsilon_q^{-1}\,\varepsilon_q\,\varphi_e^q\,\varepsilon_e^{-1}(x^n).$$

This formulation of the umbral calculus seems to be slightly more suggestive than the original formulation of Rota to which it is logically equivalent.

The only new fact is the observation that the umbral calculus also holds for negative indices.

EXAMPLE 1: Consider the problem of expanding $\frac{1}{x^n}$ into a factorial series, $\frac{1}{x^n} = \sum a_k(x)_{-k}$.

Let $q = ((x)_n)$ and p its inverse set. Then this equation is equivalent with

$$p_{-n}(x) = \sum \frac{a_k}{x^k}.$$

But we know already that

$$p_{-n}(x) = \sum_{k=0}^{\infty} |s(n+k,\, n)| \frac{1}{x^{n+k}}.$$

Therefore we have

$$\frac{1}{x^n} = \sum_{k=0}^{\infty} |s(n+k,\, n)| (x)_{-n-k}.$$

EXAMPLE 2: Let $f \in F$ have the factorial expansion

$$f(x) = \sum_{k=1}^{\infty} \frac{a_k\, k!}{(x+1)\,\ldots\,(x+k+1)}.$$

Determine the expansion of $f' = Df$.

Now

$$D = \log\,(1+\Delta) = \sum_{1}^{\infty} \frac{(-1)^{k-1}}{k} \Delta^k.$$

Therefore we have

$$f'(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}\,\Delta^k}{k} \sum_{l=1}^{\infty} \frac{a_l \cdot l!}{(x+1)\,\ldots\,(x+l+1)} =$$

$$= - \sum_{n=2}^{\infty} n! \left(\frac{a_n}{1} + \frac{a_{n-1}}{2} + \ldots + \frac{a_1}{n}\right) (x)_{-n-1}.$$

All other purely formal results on factorial series (cf. [3] and the papers cited there) can also be easily obtained in the same way.

PROBLEM: Formal manipulation with polynomials of negative degree would be much simplified if we could give a precise meaning to some infinite sums and integrals in F. E.g. starting from the formula

$$\frac{1}{x} = \int_{-\infty}^{0} e^{xs}\, ds$$

we would get

$$f(D)\,\frac{1}{x} = \int f(x)e^{xs}\,ds$$

and finally

$$q_{-1}(x) = g'(D)\,\frac{1}{x} = \int g'(s)\,e^{xs}ds = \int e^{xG(s)}ds$$

and more generally

$$q_{-n}(x) = \frac{(-1)^{n-1}}{(n-1)!}\int s^{n-1}\,e^{xG(s)}ds.$$

We could extend then the evaluation at 0 functional L on P to all elements

$$f = (f(D))\,\frac{1}{x} \text{ of } F \text{ by } L(f) = \int_{-\infty}^{0} f(t)dt,$$

whenever this integral exists. By doing this we could generalize some formulas containing L. E.g. formula 2.4 of [1] would give us

$$(U_G f)(y) = L(e^{yG(D)}f).$$

Is it possible to make these things precise?

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