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NOTES ON NORMS OF CIRCULANT MATRICES WITH LUCAS NUMBER

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Abstract. In this note, we first construct the so-called circulant matrix with the Lucas number and then present lower and upper bounds for the Euclidean and spectral norms of the matrix as a function of n and L_n, where L_n is the nth Lucas number. Also, we give lower and upper bounds for the Euclidean and spectral norm of the Hadamard inverse of this matrix.

Key Words. Circulant matrix, Lucas number, matrix norm.

1. Introduction

A Toeplitz matrix is an $n \times n$ matrix $T_n = [t_{k,j}; k, j = 0, 1, \ldots, n-1]$, where $t_{k,j} = t_{k-j}$. This structure is rather interesting for the rich theoretical properties and applications. Many problems involve Toeplitz-like matrices (i.e., Cauchy-Toeplitz, Cauchy-Hankel, Hilbert) or matrices having a displacement structure.

A Cauchy-Toeplitz matrix is a matrix that is both Cauchy (i.e. $[1/(x_i - y_j)]_{i,j=1}^n$, $x_i \neq y_j$) and Toeplitz (i.e. $[z_{i-j}]_{i,j=1}^n$) such that

$$T_n = \left[\frac{1}{g + (i-j)h}\right]_{i,j=1}^n \tag{1}$$

where g and $h \neq 0$ are arbitrary numbers and g/h is not an integer.

A Cauchy-Hankel matrix is a Cauchy matrix that is also Hankel (i.e. $[h_{i+j}]_{i,j=1}^n$) such that

$$H_n = \left[\frac{1}{g + (i+j)h}\right]_{i,j=1}^n \tag{2}$$

where g and $h \neq 0$ are arbitrary numbers and g/h is not an integer. The matrix

$$A_n = \left[\frac{1}{i+j-1}\right]_{i,j=1}^n \tag{3}$$

is known as Hilbert matrix.

Recently, there have been several papers on the norms of Cauchy-Toeplitz and Cauchy-Hankel matrices [5, 6, 7, 8]. Turkmen and Bozkurt [2] established bounds for the spectral norms of Cauchy-Toeplitz and Cauchy-Hankel matrices in the forms (1) and (2) by taking $g = 1/k$ and $h = 1$, respectively. Solak and Bozkurt [3] obtained lower and upper bounds for the spectral and Euclidean norms of T_n and H_n that have given in (1) and (2), respectively. Güngör [9] obtained lower bounds for the spectral norm and Euclidean norm of Cauchy-Toeplitz matrix in the form

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(1) by taking $g = 1/2$ and $h = 1$. Also, Güngör [10] established lower bounds for the spectral and Euclidean norms of Hilbert matrix in the form (3).

For any given $a_0,\ a_1, \ldots, a_{n-1} \in \mathbb{C}$, the circulant matrix $B = (b_{ij})_{n\times n}$ is defined by $b_{ij} = a_{j-i(\operatorname{mod} n)}$, that is,

$$B = \begin{bmatrix} a_0 & a_1 & a_2 & \ldots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \ldots & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 & \ldots & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_1 & a_2 & \ldots & a_0 \end{bmatrix}$$

A circulant matrix is a special Toeplitz matrix where each row vector is rotated one element to the right relative to the preceding row vector. In numerical analysis circulant matrices are important because they can be quickly solved using the discrete Fourier transform.

Solak and Bozkurt [11] defined almost circulant matrix as follows:

$$D_n = \begin{cases} a, & i = j \\ \frac{1}{k}(j-i) \equiv k(\operatorname{mod} n), & \text{otherwise} \end{cases}$$

where $a \in \mathbb{R} - \{0\}$ and $k = 1, 2, \ldots, n-1$, and then established upper bound for the l_p matrix and l_p operator norms of the matrix D_n.

The Lucas numbers are the integer sequence $\{L_n\}_{n=1}^{\infty}$ defined by the linear recurrence equation, with $L_1 = 1$ and $L_2 = 3$, and

$$L_{n+2} = L_{n+1} + L_n$$

If we start with zero, then the sequence is given by

n	0	1	2	3	4	5	6	7	$\ldots$
L_n	2	1	3	4	7	11	18	29	$\ldots$

Lucas numbers and their generalization have many interesting properties and applications to almost every field of science and art. For the beauty and rich applications of these numbers and their relatives one may see Vajda's book [13].

Solak [12] defined the circulant matrix with the Lucas number as

$$C = [L_{(\operatorname{mod}(j-i,n))}]_{i,j=1}^{n} \tag{4}$$

where L_n is nth Lucas number, and then showed lower and upper bounds with the Fibonacci numbers for the Euclidean and spectral norms of the matrix C.

In this note, we first construct the circulant matrix with the Lucas number and then present lower and upper bounds for the Euclidean and spectral norms of this matrix as a function of n and L_n, where L_n is nth Lucas number. We shall also study the norm bounds for the Hadamard inverse of this matrix.

We begin with some background necessary to understand this note. For more comprehensive treatments on matrices we refer readers to [1].

Let A be any $n \times n$ matrix. The Euclidean norm of the matrix A is defined as

$$\|A\|_E = \left(\sum_{i,j=1}^{n} |a_{ij}|^2 \right)^{1/2}$$

The spectral norm of the matrix A is

$$\|A\|_2 = \sqrt{\max_{1 \le i \le n} \lambda_i}$$

where λ_i is an eigenvalue of $A^H A$ and A^H is conjugate transpose of the matrix A. It is well known that

$$\|A\|_2 \leq \|A\|_E \leq \sqrt{n}\,\|A\|_2 \tag{5}$$

Define the maximum column length norm $c_1(\cdot)$ and the maximum row length norm $r_1(\cdot)$ of any matrix A respectively by

$$c_1(A) = \max_j \left(\sum_i |a_{ij}|^2\right)^{1/2}$$

and

$$r_1(A) = \max_i \left(\sum_j |a_{ij}|^2\right)^{1/2}$$

Let $A = (a_{ij})$ and $B = (b_{ij})$ be matrices of the same size, not necessarily square. Then their Hadamard product (also called Schur produt) $A \circ B$ is defined by entrywise multiplication: $A \circ B = [a_{ij}b_{ij}]$. The Hadamard unit matrix is the matrix U all of whose entries are 1 (the size of U is understood in context). A matrix A is Hadamard invertible if all its entries are non-zero, and $A^{\circ -1} = [a_{ij}^{-1}]$ is then called the Hadamard inverse of A.

Let A, B and C be $m \times n$ matrices. If $A \circ B = C$. Then

$$\|A\|_2 \leq r_1(B)c_1(C) \tag{6}$$

2. Main Results

Definition 1. *Let*

$$v = (L_1, L_2, \ldots, L_n)$$

be a row vector in $\mathbb{C}^n$ *and define the shift operator* $\Delta : \mathbb{C}^n \to \mathbb{C}^n$ *by*

$$\Delta(L_1, L_2, ..., L_n) = (L_n, L_1, ..., L_{n-1})$$

where L_n *is nth Lucas number. The circulant matrix* C *with Lucas number is the matrix whose kth row is given by* $\Delta^{k-1}v, k = 1, 2, \ldots, n,$ *i.e., a matrix of the form*

$$C = \begin{bmatrix} L_1 & L_2 & L_3 & \ldots & L_n \\ L_n & L_1 & L_2 & \ldots & L_{n-1} \\ L_{n-1} & L_n & L_1 & \ldots & L_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ L_2 & L_3 & L_4 & \ldots & L_1 \end{bmatrix} \tag{7}$$

Theorem 1. *Let* C *be the matrix defined in* (7)*. Then*

$$\|C\|_E = \sqrt{nL_nL_{n+1} - 2n}$$

and

$$\sqrt{L_nL_{n+1} - 2} \leq \|C\|_2 \leq L_nL_{n+1} - 2$$

where L_n *is the nth Lucas number.*

Proof. Let matrices K and M be defined as

$$K = (k_{ij}) = \begin{cases} k_{ij} = L_{(\mathrm{mod}(j-i,n))+1}, & i \geq j \\ k_{ij} = 1, & i < j \end{cases}$$

and

$$M = (m_{ij}) = \begin{cases} m_{ij} = L_{(\mathrm{mod}(j-i,n))+1}, & i < j \\ m_{ij} = 1, & i \geq j \end{cases}$$

such that $C = K \circ M$. Then

$$r_1(K) = \sqrt{\max_i \left(\sum_j |k_{ij}|^2 \right)}$$

$$= \sqrt{\sum_{s=1}^{n} L_s^2} \tag{8}$$

and

$$c_1(M) = \sqrt{\max_j \left(\sum_i |m_{ij}|^2 \right)}$$

$$= \sqrt{\sum_{s=1}^{n} L_s^2} \tag{9}$$

From (6), (8) and (9), we obtain an upper bound for the spectral norm:

$$\|C\|_2 \leq \sum_{s=1}^{n} L_s^2 = L_n L_{n+1} - 2.$$

For the Euclidean norm of the matrix C, we have

$$\|C\|_E = \sqrt{n \sum_{s=1}^{n} L_s^2}$$

$$= \sqrt{n L_n L_{n+1} - 2n}. \tag{10}$$

From (5) and (10), we get immediately

$$\sqrt{L_n L_{n+1} - 2} \leq \|C\|_2$$

This completes the proof. □

Theorem 2. *Let $C^{\circ -1}$ be the Hadamard inverse of matrix C defined in* (7). *Then*

$$n \sqrt{\frac{n}{L_n L_{n+1} - 2}} \leq \left\| C^{\circ -1} \right\|_E \leq \frac{5}{2} \sqrt{n}$$

and

$$n \sqrt{\frac{1}{L_n L_{n+1} - 2}} \leq \left\| C^{\circ -1} \right\|_2$$

Proof. The Euclidean norm of the matrix $C^{\circ -1}$ is

$$\left\| C^{\circ -1} \right\|_E^2 = n \sum_{s=1}^{n} \frac{1}{L_s^2} \tag{11}$$

Since $\frac{1}{L_n} \geq \frac{1}{L_n^2}$, we have

$$\left(\frac{1}{L_1} + \frac{1}{L_2} + ... + \frac{1}{L_n} + 1 \right)^2 \geq \left(\frac{1}{L_1^2} + \frac{1}{L_2^2} + ... + \frac{1}{L_n^2} + 1 \right)^2 \tag{12}$$

Also, we know that

$$\left(\sum_{s=1}^{n} \frac{1}{L_s^2} - 1 \right)^2 \geq 0 \Leftrightarrow \left(\sum_{s=1}^{n} \frac{1}{L_s^2} + 1 \right)^2 \geq 4 \sum_{s=1}^{n} \frac{1}{L_s^2} \tag{13}$$

Hence, from (11), (12) and (13) we get

$$\left\| C^{\circ -1} \right\|_E^2 \leq \frac{n}{4} \left(\frac{1}{L_1} + \frac{1}{L_2} + ... + \frac{1}{L_n} + 1 \right)^2 \tag{14}$$

Now, for $n \geq 2$ we will prove the inequality

$$L_n \geq \left(\frac{3}{2}\right)^{n-2}$$

by the mathematical induction. The statement is true for $n = 2$. Suppose $L_k \geq \left(\frac{3}{2}\right)^{k-2}$ holds when $2 \leq k \leq n$, we show it also holds when $k = n+1$:

$$\begin{aligned} L_{n+1} &= L_n + L_{n-1} \\ &\geq \left(\frac{3}{2}\right)^{n-2} + \left(\frac{3}{2}\right)^{n-3} \\ &= \left(\frac{3}{2}\right)^{n-3}\left(1+\frac{3}{2}\right) \\ &> \left(\frac{3}{2}\right)^{n-3}\left(\frac{3}{2}\right)^{2} \\ &= \left(\frac{3}{2}\right)^{n-1} \end{aligned}$$

Thus, we write

$$L_n \geq \left(\frac{3}{2}\right)^{n-2} \tag{15}$$

with $n \geq 2$. From (14) and (15),

$$\begin{aligned} \left\|C^{\circ -1}\right\|_E^2 &\leq \frac{n}{4}\left(\frac{1}{L_1}+\frac{1}{L_2}+...+\frac{1}{L_n}+1\right)^2 \\ &\leq \frac{n}{4}\left[1+\left(\frac{2}{3}\right)^{2-2}+\left(\frac{2}{3}\right)^{3-2}+...+\left(\frac{2}{3}\right)^{n-2}+1\right]^2 \\ &= \frac{n}{4}\left[3+\frac{2}{3}\left(1+\left(\frac{2}{3}\right)+...+\left(\frac{2}{3}\right)^{n-3}\right)\right]^2 \\ &= \frac{n}{4}\left[3+\frac{2}{3}\frac{1-\left(\frac{2}{3}\right)^{n-2}}{1-\left(\frac{2}{3}\right)}\right]^2 \\ &< \frac{25}{4}n \end{aligned} \tag{16}$$

So

$$\left\|C^{\circ -1}\right\|_E \leq \frac{5}{2}\sqrt{n}$$

Therefore, by a simple computation we see that

$$\begin{aligned} n^2 &\leq \left(L_1^2+L_2^2+...+L_n^2\right)\left(\frac{1}{L_1^2}+\frac{1}{L_2^2}+...+\frac{1}{L_n^2}\right) \\ &= \left(\sum_{i=1}^{n} L_i^2\right)\left(\sum_{i=1}^{n}\frac{1}{L_i^2}\right) \\ &= \frac{1}{n^2}\left\|C\right\|_E^2\left\|C^{\circ -1}\right\|_E^2 \end{aligned} \tag{17}$$

From (10) and (17) it follows that

$$n^3 \leq \left(L_nL_{n+1}-2\right)\left\|C^{\circ -1}\right\|_E^2$$

or

$$n\sqrt{\frac{n}{L_nL_{n+1}-2}} \leq \left\|C^{\circ -1}\right\|_E$$

Thus, if we consider the inequality (5), then we have

$$n\sqrt{\frac{1}{L_nL_{n+1}-2}} \leq \left\|C^{\circ -1}\right\|_2$$

This completes the proof. □

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