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ON SOME EXTENSIONS OF THE MEIXNER-WEISNER GENERATING FUNCTIONS

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I. INTRODUCTION

With the aid of group theory, Weisner [10] derived the Bilinear generating function for the ultraspherical polynomial:

$$\sum_{n=0}^{\infty} \frac{n!t^n}{(2\alpha)_n} C_n^{\alpha}(\cos x) C_n^{\alpha}(\cos y) = \{1 - 2t\cos(x + y) + t^2\}^{-\alpha}\,{}_2F_1\left[\begin{matrix}\alpha,\alpha; & \dfrac{4t\sin x \sin y}{1 - 2t\cos(x + y) + t^2}\\ 2\alpha; & \end{matrix}\right] \tag{1.1}$$

See [5] for definition and properties. (1.1) had also been proved by Meixner [6], Ossicini [7], and Watson [8], and was recently investigated by Carlitz [2], [3]. (1.1) is seen to be a special case of Theorem 1 in this paper, as are the formulas (1.2), (1.4), and (1.5), which appear to be new. Note that the expressions given below are generating functions for the ultraspherical polynomial of type $C_{n+\ell}^{\lambda}(x)$. See Cohen [4] for the single Jacobi polynomial.

$$\sum_{n=0}^{\infty} \frac{t^n(n + \ell)!}{(2u + 2\ell + 1)_n} C_n^{u+\ell+\frac{1}{2}}(x) C_{n+\ell}^{u}(y) = \frac{2^{\ell+1}\,\Gamma(u + \ell + 1)\,\ell!}{\Gamma(u)\Gamma(2u + \ell)[t^2(x^2 - 1)]^{u+\frac{1}{2}\ell}} C_\ell^u\left[\left\{\frac{2(y - xt)^2}{2x^2t^2 - 2xyt - t^2 + 1 + \rho}\right\}^{\frac{1}{2}}\right] D_\ell^u\left[\left\{\frac{2x^2t^2 - 2xyt - t^2 + 1 + \rho}{2t^2(x^2 - 1)}\right\}^{\frac{1}{2}}\right] \tag{1.2}$$

where $\rho = [(1 - 2xyt + t^2)^2 - 4t^2(1 - x^2)(1 - y^2)]^{\frac{1}{2}}$, $|t| < 1$, $|xt/y| < 1$, ℓ is a nonnegative integer, and D_ℓ^u is the Gegenbauer function defined by Watson [9, p. 129] as

$$D_\ell^u(z) = \frac{\Gamma(u)\Gamma(2u + \ell)z^{-2u-\ell}}{2^{\ell+1}\Gamma(u + \ell + 1)}\,{}_2F_1\left[\begin{matrix}u + \frac{1}{2}\ell,\ u + \frac{1}{2}\ell + \frac{1}{2}; & \dfrac{1}{z^2}\\ u + \ell + 1; & \end{matrix}\right]. \tag{1.3}$$

A special case of (1.2) is deduced for $x = 0$, $y = \cos\phi$, and t suitably modified:

$$\sum_{n=0}^{\infty} \frac{t^n(2n + \ell)!}{2^{2n}n!(v + \ell + 1)_n} C_{2n+\ell}^{v}(\cos\phi) = \frac{\ell!\Gamma(v + \ell + 1)2^{\ell+1}}{\Gamma(v)\Gamma(2v + \ell)t^{v+\frac{1}{2}\ell}} C_\ell^v\left[\left\{\frac{2\cos^2\phi}{1 + t + \sigma}\right\}^{\frac{1}{2}}\right] D_\ell^v\left[\left\{\frac{1 + t + \sigma}{2t}\right\}^{\frac{1}{2}}\right] \tag{1.4}$$

where $\sigma = (1 - 2t\cos(2\phi) + t^2)^{\frac{1}{2}}$, and $|t| < 1$.

$$\sum_{n=0}^{\infty} \frac{t^n C_n^{u+\frac{1}{2}}(\cos\theta) C_{n+1}^{u}(\cos\phi)}{C_{n+1}^{u}(0)} \tag{1.5}$$
$$= (\cos\phi - t\cos\theta)^{-1-2u}(1-\xi\eta)^{-1}(1-\xi)^{u+\frac{1}{2}}(1-\eta)^{u+1}$$

where $|t\cos\theta/\cos\phi| < 1$, $|t| < 1$,

$$\xi = \{1 - 2t\cos\theta\cos\phi + t^2 - [(1 - 2t\cos(\theta-\phi) + t^2)(1 - 2t\cos(\theta+\phi) + t^2)]^{\frac{1}{2}}\}/2\sin^2\phi, \text{ and}$$

$$\eta = \{1 - 2t\cos\theta\cos\phi + t^2 - [(1 - 2t\cos(\theta-\phi) + t^2)(1 - 2t\cos(\theta+\phi) + t^2)]^{\frac{1}{2}}\}/2t^2\sin^2\theta.$$

Theorem 2 yields the new finite expansions

$$\sum_{n=0}^{\ell} \frac{(-1)^n t^n}{(2v)_n(1-2\ell-2v)_{\ell-n}} C_n^v(x) C_{\ell-n}^{\frac{1}{2}-v-\ell}(y)$$
$$= \frac{\ell!(y^2-1)^{\frac{1}{2}\ell}}{2^\ell (v)_\ell (2v)_\ell} C_\ell^v\left[\left\{\frac{2(y-xt)^2}{2y^2-2xyt+t^2-1+\rho}\right\}^{\frac{1}{2}}\right] C_\ell^v\left[\left\{\frac{2(y-xt)^2}{2y^2-2xyt+t^2-1-\rho}\right\}^{\frac{1}{2}}\right] \tag{1.6}$$

where ρ is defined in equation (1.2).

Equation (1.6) may also be expressed as

$$\sum_{n=0}^{\ell} \frac{t'^n 2^n}{(2v)_n (v)_{\ell-n}} C_n^v(x) C_{\ell-n}^{v+n}(y')$$
$$= \frac{\ell!}{(v)_\ell (2v)_\ell} C_\ell^v\left[\left\{\frac{y'^2+1+2xy't'+t'^2-\rho'}{2}\right\}^{\frac{1}{2}}\right] C_\ell^v\left[\left\{\frac{y'^2+1+2xy't'+t'^2+\rho'}{2}\right\}^{\frac{1}{2}}\right] \tag{1.7}$$

where $\rho' = \{(y'^2 - 1 + 2xy't' + t'^2)^2 + 4t'^2(1-x^2)\}^{\frac{1}{2}}$.

A special case of (1.6) is the relation

$$\sum_{n=0}^{[\ell/2]} \frac{t^n}{2^{2n}(2v)_{\ell-2n}(1-\ell-v)_n n!} C_{\ell-2n}^v(\cos\phi)$$
$$= \frac{\ell!\, t^{\frac{1}{2}\ell}}{2^\ell (v)_\ell (2v)_\ell} C_\ell^v\left[\left\{\frac{2\cos^2\phi}{1+t+\sigma}\right\}^{\frac{1}{2}}\right] C_\ell^v\left[\left\{\frac{2\cos^2\phi}{1+t-\sigma}\right\}^{\frac{1}{2}}\right] \tag{1.8}$$

where σ is defined in equation (1.4).

Equation (1.8) is deduced from (1.6) by putting $x = 0$, and rearranging the parameters. Also, if $y = 1$ in (1.6), one obtains a known expression [5, p. 227, last formula].

SECTION II

Theorem 1: For u and v arbitrary complex numbers and ℓ a nonnegative integer,

$$\sum_{n=0}^{\infty} \frac{t^n(n+\ell)!}{(2v)_n} C_n^v(x) C_{n+\ell}^u(y)$$

$$(2.1)\quad = (2u)_\ell (y - xt)^{-2u-\ell} F_4\left[\frac{1}{2}(2u+\ell), \frac{1}{2}(2u+\ell+1); v+\frac{1}{2}, u+\frac{1}{2}; \frac{t^2(x^2-1)}{(y-xt)^2}, \frac{y^2-1}{(y-xt)^2}\right]$$

where $|xt/y| < 1$, $|t| < 1$, and F_4 denotes the fourth type of Appell's [1, p. 14] hypergeometric function of two variables defined by

$$F_4[a, b; c, d; x_1, y_1] = \sum_{k_1, k_2}^{\infty} \frac{(a)_{k_1+k_2}(b)_{k_1+k_2}}{k_1!k_2!(c)_{k_1}(d)_{k_2}} x_1^{k_1} x_2^{k_2}.$$

Proof: The left-hand side of (2.1) may be expressed as

$$(2.2)\quad \sum_{n=0}^{\infty} \frac{(2u)_{n+\ell} t^n x^n y^{n+\ell}(n+\lambda)}{n!} {}_2F_1\left[\begin{matrix}-\frac{1}{2}n, -\frac{1}{2}n+\frac{1}{2};\\ v+\frac{1}{2};\end{matrix} \frac{x^2-1}{x^2}\right] \cdot {}_2F_1\left[\begin{matrix}-\frac{1}{2}(n+\ell), -\frac{1}{2}(n+\ell)+\frac{1}{2};\\ u+\frac{1}{2};\end{matrix} \frac{y^2-1}{y^2}\right]$$

$$(2.3)\quad = \sum_{n=0}^{\infty} \frac{(2u)_{n+\ell} t^n x^n y^{n+\ell}(n+\lambda)}{n!} {}_2F_1\left[\begin{matrix}-\frac{1}{2}n, -\frac{1}{2}n+\frac{1}{2};\\ v+\frac{1}{2};\end{matrix} \frac{x^2-1}{x^2}\right] y^{-2u-2n-2\ell} \cdot {}_2F_1\left[\begin{matrix}u+\frac{1}{2}(n+\ell), u+\frac{1}{2}(n+\ell)+\frac{1}{2};\\ u+\frac{1}{2};\end{matrix} \frac{y^2-1}{y^2}\right]$$

$$(2.4)\quad = \sum_{k=0}^{\infty}\sum_{p=0}^{\infty}\left[\frac{t^2(x^2-1)}{4y^2}\right]^k \left[\frac{y^2-1}{4y^2}\right]^p \frac{(2u)_\ell(2u+\ell)_{2p+2k}(n+\lambda+2k)}{y^{2u+\ell}p!k!\left(u+\frac{1}{2}\right)_p\left(v+\frac{1}{2}\right)_k} \cdot \sum_{n=0}^{\infty} \frac{(xt/y)^n(2u+\ell+2k+2p)_n}{n!}$$

From (2.2) to (2.3) we have used the Kummer transformation. Going from (2.3) to (2.4) entails the use of the following:

$$(2.5)\quad 2^{2k}\left(-\frac{1}{2}n\right)_k\left(-\frac{1}{2}n+\frac{1}{2}\right)_k = \frac{n!}{(n-2k)!}$$

$$(2.6)\quad \sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]} f(n, k) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} f(n+2k, k)$$

and

$$(2.7)\quad (2u)_{n+\ell+2k}(2u+n+\ell+2k)_{2p} = (2u)_\ell(2u+\ell)_{2p+2k}(2u+\ell+2p+2k)_n.$$

Now

$$(2.8)\quad \sum_{n=0}^{\infty} \frac{(xt/y)^n(2u+\ell+2k+2p)_n}{n!} = [1 - xt/y]^{-2u-\ell-2k-2p}.$$

Hence, (2.4) reduces to

$$(y - xt)^{-2u-\ell}(2u)_\ell \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} \left[\frac{t^2(x^2-1)}{(y-xt)^2}\right]^k \left[\frac{(y^2-1)}{(y-xt)^2}\right]^p \tag{2.9}$$
$$\cdot \frac{\left(\frac{1}{2}(2u+\ell)\right)_{p+k}\left(\frac{1}{2}(2u+\ell+1)\right)_{p+k}}{k!p!\left(v+\frac{1}{2}\right)_k\left(u+\frac{1}{2}\right)_p}$$

By definition, (2.9) is the right-hand side of Theorem 1.

Theorem 2: For u and v arbitrary complex numbers and ℓ a nonnegative integer,

$$\sum_{n=0}^{\ell} \frac{(-1)^n t^n}{(2v)_n(2u)_{\ell-n}} C_n^v(x) C_{\ell-n}^v(y)$$
$$= \frac{(y-xt)^\ell}{\ell!} F_4\left[-\frac{1}{2}\ell, -\frac{1}{2}\ell+\frac{1}{2}; v+\frac{1}{2}, u+\frac{1}{2}; \frac{t^2(x^2-1)}{(y+xt)^2}, \frac{y^2-1}{(y-xt)^2}\right]. \tag{2.10}$$

Proof: The left-hand side of (2.10) is put in the form

$$\sum_{n=0}^{\infty} \frac{(-1)^n t^n x^n y^{\ell-n}}{n!(\ell-n)!} {}_2F_1\left[\begin{matrix}-\frac{1}{2}n, -\frac{1}{2}n+\frac{1}{2}; & \frac{x^2-1}{x^2}\\ v+\frac{1}{2}; & \end{matrix}\right] \tag{2.11}$$
$$\cdot {}_2F_1\left[\begin{matrix}-\frac{1}{2}(\ell-n), -\frac{1}{2}(\ell-n)+\frac{1}{2}; & \frac{y^2-1}{y^2}\\ u+\frac{1}{2}; & \end{matrix}\right].$$

Following a procedure analogous to that in the proof of Theorem 1, with appropriate changes, (2.11) is simplified to yield the right-hand side of (2.10).

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