The characterization theorems and the Rodrigues operator. A general approach

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Menu

1. Overview of Classical orthogonal polynomials
2. Connection with Operator Theory. The Rodrigues operator
3. More general Hahn Theorem
4. New characterization Theorem for q-polynomials
5. Overview of Semiclassical orthogonal polynomials
6. A generalization of semiclassical OP. The quasi-semiclassical OP
7. An example: The N-Askey-Wilson Polynomials

PART I

First ingredients: The Classical Orthogonal Polynomials

The definition

Definition 1 The sequence $\{P_n\}_{n\geq 0}$ is said to be a q**-classical OPS** on the lattice $x(s)$ if satisfies the orthogonality conditions

$$\sum_{s=a}^{b-1} P_n(s)P_m(s)\rho(s)\nabla x_1(s) = d_n^2\delta_{n,m}, \quad \Delta s = 1, \quad n, m = 0, 1, \ldots$$

where

(i) $\rho(s)$ is a solution of the q-Pearson equation $\Delta[\sigma(s)\rho(s)] = \tau(s)\rho(s)\nabla x_1(s)$.

(ii) $\sigma(s) + \frac{1}{2}\tau(s)\nabla x_1(s)$ is a polynomial on $x(s)$ of degree, at most, 2.

(iii) $\tau(s)$ is a polynomial on $x(s)$ of degree 1.

q-numbers

$$[s]_q := \frac{q^{\frac{s}{2}} - q^{-\frac{s}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}, \quad q \in \mathbb{C}, |q| \neq 1.$$

q-Polynomials

1. q-classical orthogonal polynomials (or q-Polynomials)

> $\mathfrak{H}_q = \sigma(s)\frac{\Delta}{\nabla x_1(s)}\frac{\nabla}{\nabla x(s)} + \tau(s)\frac{\Delta}{\Delta x(s)}, x_k(s) = x(s+\frac{k}{2}),$

> $\sigma(s) := \tilde{\sigma}(x(s)) - \frac{1}{2}\tilde{\tau}(x(s))\nabla x_1(s), \tau(s) = \tilde{\tau}(x(s)),$

> $\Delta[\sigma(s)\rho(s)] = \tau(s)\rho(s)\nabla x_1(s),$

> $x(s) = c_1 q^s + c_2 q^{-s} + c_3.$

Polynomial eigenfunctions of $\mathfrak{H}_q$

$$P_n(s)_q := \left[\frac{B_n \nabla\rho_1(s)}{\rho_0(s)\nabla x_1(s)}\right]\left[\frac{\nabla\rho_2(s)}{\rho_1(s)\nabla x_2(s)}\right]\cdots\left[\frac{\nabla\rho_n(s)}{\rho_{n-1}(s)\nabla x_{n-1}(s)}\right],$$

Symmetric form of $\mathfrak{H}_q$

$$\mathfrak{H}_q = \left[\frac{1}{\rho(s)}\frac{\nabla}{\nabla x_1(s)}\rho_1(s)\right]\frac{\Delta}{\Delta x(s)}.$$

Standard and Δ

1. Standard classical orthogonal polynomials (Hermite, Laguerre, Jacobi)

   > $\mathfrak{H} := \tilde{\sigma}(x)\dfrac{d^2}{dx^2} + \tilde{\tau}(x)\dfrac{d}{dx}, \qquad \lambda_n = n\tilde{\tau}' + n(n-1)\frac{\tilde{\sigma}''}{2}.$

   > $\frac{d}{dx}[\tilde{\sigma}(x)\rho(x)] = \tilde{\tau}(x)\rho(x).$

2. Δ-classical orthogonal polynomials (Hahn, Meixner, Kravchuk, Charlier, etc)

   > $\mathfrak{H}_\Delta := \sigma(s)\Delta\nabla + \tau(s)\Delta, \qquad \lambda_n = n\tilde{\tau}' + n(n-1)\frac{\tilde{\sigma}''}{2}.$

   > $\sigma(x) := \tilde{\sigma}(x) - \frac{1}{2}\tilde{\tau}(x), \tau(s) = \tilde{\tau}(x),$

   > $\Delta[\sigma(s)\rho(s)] = \tau(s)\rho(s),$

   > $\Delta f(s) = f(s+1) - f(s), \nabla f(s) = f(s) - f(s-1),$

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$$> \quad \sigma(x) := \tilde{\sigma}(x) - \tfrac{1}{2}\tilde{\tau}(x),\ \tau(s) = \tilde{\tau}(x),$$

$$> \quad \Delta[\sigma(s)\rho(s)] = \tau(s)\rho(s),$$

$$> \quad \Delta f(s) = f(s+1) - f(s),\ \nabla f(s) = f(s) - f(s-1),$$

One foretaste: Connection with Operator Theory

The Rodrigues Operator

Definition 2 Given functions σ and ρ, where ρ is supported on Ω, and a lattice $x(s)$, we define the **k-th Rodrigues operator associated with** $(\sigma(s), \rho(s), x(s))$ as

$$R_0(\sigma, \rho, x) := I, \; R_1(\sigma, \rho, x) := \frac{\nabla}{\rho(s)\nabla x_1(s)} \rho_1(s),$$

$$R_k(\sigma, \rho, x) := R_1(\sigma(s), \rho(s), x(s)) \circ R_{k-1}(\sigma(s), \rho_1(s), x_1(s)),$$

where $\rho_1(s) = \sigma(s+1)\rho(s+1)$ and I is the identity operator.

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where $\rho_1(s) = \sigma(s+1)\rho(s+1)$ and I is the identity operator.

Standard COP: $R_1(\sigma,\rho) := \dfrac{1}{\rho(x)}\dfrac{d}{dx}\rho_1(x),\ \rho_1(x) := \rho(x)\tilde{\sigma}(x).$

Δ-COP: $R_1(\sigma,\rho) := \dfrac{\nabla}{\rho(s)}\rho_1(s),\ \rho_1(s) := \rho(s+1)\sigma(s+1).$

A very important result

Theorem 1 If (σ, ρ, x) is a q-classical tern, then for every integer k, if $\pi \in \mathbb{P}_m[x_{k+1}]$,

$$R_1(\sigma, \rho_k, x_k)[\pi] = \widetilde{\pi} \in \mathbb{P}_{m+1}[x_k].$$

If π monic, the leading coefficient of $\widetilde{\pi}$ is

$$-\lambda_{m+1+2k}/[m+1+2k].$$

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$$\{P_n\}_{n\geq 0} \text{ is } q\text{-classical} \iff \left\{\frac{\Delta P_{n+1}}{\Delta x(s)}\right\}_{n\geq 0} \text{ is } q\text{-classical.}$$

Theorem 3 $\{P_n\}_{n\geq 0}$ is q-classical $\iff$ $\{R_n(\sigma, \rho_{-1}, x_{-1})[1]\}_{n\geq 0}$ is q-classical.

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$$R_{n+1}(\sigma, \rho_{-1}, x_{-1}) = R_1(\sigma, \rho_{-1}, x_{-1}) \circ R_n(\sigma, \rho, x).$$

Some well-known result

1. $R_1(\sigma,\rho,x)[1]=\tau(s)$. q-Pearson equation.
2. For every integers, $n,\ k$, $n\geq\max\{k,0\}$, there exists a constant, $C_{n,k}$ such that

$$\Delta^{(k)}P_n(s)_q=C_{n,k}R_{n-k}(\sigma,\rho_k,x_k)[1].$$

Where $x_k(s):=x(s+\frac{k}{2})$, $\rho_k(s):=\rho_{k-1}(s+1)\sigma(s+1)$, being $\rho_0\equiv\rho$, and

$$\Delta^{(k)}:=\begin{cases}\dfrac{\Delta}{\Delta x_{k-1}}\dfrac{\Delta}{\Delta x_{k-2}}\cdots\dfrac{\Delta}{\Delta x}, & \text{if } k\geq 1,\\ R_k(\sigma,\rho_k,x_k), & \text{if } k\leq 0.\end{cases}$$

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$$\Delta^{(k)}P_n(s)_q=C_{n,k}R_{n-k}(\sigma,\rho_k,x_k)[1].$$

Where $x_k(s):=x(s+\frac{k}{2})$, $\rho_k(s):=\rho_{k-1}(s+1)\sigma(s+1)$, being $\rho_0\equiv\rho$, and

$$\Delta^{(k)}:=\begin{cases}\dfrac{\Delta}{\Delta x_{k-1}}\dfrac{\Delta}{\Delta x_{k-2}}\cdots\dfrac{\Delta}{\Delta x}, & \text{if } k\geq 1,\\ R_k(\sigma,\rho_k,x_k), & \text{if } k\leq 0.\end{cases}$$

The Chef's Special: The Main Results

New Hahn's Theorem

Theorem 4 Let $\{P_n\}_{n\geq 0}$ be an OPS with respect to $\rho(s)$ such that is complete as orthonormal set in $\ell^2([a,b],\langle\centerdot,\centerdot\rangle_\rho)$. The following statements are equivalent.

(i) $\{P_n\}_{n\geq 0}$ is q-classical and the following boundary conditions hold

$$x^k(s)x_{-1}(s)^l\sigma(s)\rho(s)\Big|_{s=a}^{s=b}=0,\quad k,l=0,1,\dots(*)$$

(ii) $\{\Delta^{(1)}P_{n+1}\}_{n\geq 0}$ is an OPS with respect to $\widetilde{\rho}(s)$ and the following boundary conditions hold

$$x^k(s)x_{-1}(s)^l\widetilde{\rho}(s-1)\Big|_{s=a}^{s=b}=0,\quad k,l=0,1,\dots$$

New characterization Theorem for q-polynomials

Theorem 5 Let $\{P_n\}_{n\geq 0}$ be an OPS with respect to $\rho(s)$ on the lattice $x(s)$ and let $\sigma(s)$ be such that boundary condition (*) holds. Then the following statements are equivalent:

1. $\{P_n\}_{n\geq 0}$ is a q-classical OPS.
2. The sequence $\{\Delta^{(1)}P_n\}_{n\geq 0}$ is an OPS with respect to $\rho_1(s)$ where ρ satisfies the last q-Pearson equation.
3. For every integer k, the sequence $\{R_n(\rho_k(s), x_k(s))[1]\}_{n\geq 0}$ is an OPS with respect to the weight function $\rho_k(s)$ where $\rho_0(s) = \rho(s)$, $\rho_k(s) = \rho_{k-1}(s+1)\sigma(s+1)$ and ρ satisfies last q-Pearson equation.
4. (Second order difference equation):

$$\sigma(s)\frac{\Delta}{\nabla x_1(s)}\frac{\nabla P_n(s)}{\nabla x(s)} + \tau(s)\frac{\Delta P_n(s)}{\Delta x(s)} + \lambda_n P_n(s) = 0,$$

New characterization Theorem for q-polynomials (cont.)

5. $\{P_n\}_{n\geq 0}$ can be expressed in terms of the Rodrigues operator

$$P_n(s) = B_n R_n(\rho(s), x(s))[1] = \frac{B_n}{\rho(s)} \frac{\nabla}{\nabla x_1(s)} \cdots \frac{\nabla[\rho_n(s)]}{\nabla x_n(s)},$$

6. (First structure relation):

$$\phi(x_1(s)) \frac{\Delta P_n(s)}{\Delta x(s)} = a_n \mathcal{M}P_{n+1}(s) + b_n \mathcal{M}P_n(s) + c_n \mathcal{M}P_{n-1}(s) + j_n x_1(s) \mathcal{M}P_n(s).$$

7. (Second structure relation):

$$\mathcal{M}P_n(s) := \frac{P_n(s+1) + P_n(s)}{2} = e_n \frac{\Delta P_{n+1}(s)}{\Delta x(s)} + f_n \frac{\Delta P_n(s)}{\Delta x(s)} + g_n \frac{\Delta P_{n-1}(s)}{\Delta x(s)},$$

where $e_n \neq 0$, $g_n \neq \gamma_n$ for all $n \geq 0$, and γ_n is the corresponding coefficient of the following three-term recurrence relation $x(s)P_n(s) = \alpha_n P_{n+1}(s) + \beta_n P_n(s) + \gamma_n P_{n-1}(s)$.

PART II

The late ingredients: The Semiclassical Orthogonal Polynomials

The definitions

Definition 3 The sequence $\{P_n\}_{n\geq 0}$ is said to be a q-**semiclassical OPS** on the lattice $x(s)$ if there exists $\mathbf{u} \in \mathbb{P}'$ such that

$$\langle \mathbf{u}, P_n P_m \rangle = d_n^2 \delta_{n,m}, \quad n, m = 0, 1, \ldots$$

where

(i) $\mathbf{u}$ is a solution of the distributional equation $\frac{\Delta}{\nabla x_1(s)}[\phi \mathbf{x}] = \psi \mathbf{x}$.

(ii) $\widehat{\phi}(s) := \phi(s) + \frac{1}{2}\psi(s)\nabla x_1(s)$ is a polynomial on $x(s)$ of degree $p \geq 0$.

(iii) ψ is a polynomial on $x(s)$ of degree, $t \geq 1$.

$(\widehat{\phi}, \psi)$ is an Admisible pair if $t \neq p - 1$, or if $t = p - 1$ and

$$\frac{q^{\frac{m}{2}} + q^{-\frac{m}{2}}}{2} b_t + [m]_q a_p \neq 0, \quad m \in \mathbb{N}_0.$$

n_0-singularity, order and class of $\mathbf{u}$.

q-semiclassical orthogonal polynomials

$$> \quad \phi(s)\frac{\Delta}{\nabla x_1(s)}\frac{\nabla P_n(s)}{\nabla x(s)} + \psi(s)\frac{\Delta P_n(s)}{\Delta x(s)} = \sum_{j=n-\sigma}^{n+t} \lambda_{n,j} P_j(s),$$

$$> \quad \frac{\Delta}{\nabla x_1(s)}[\phi \mathbf{u}] = \psi \mathbf{u},$$

$$> \quad x(s) = c_1 q^s + c_2 q^{-s} + c_3,$$

where

$$\sigma := \max\{t-1, p-2\} \quad \text{order of } \mathbf{u}.$$

Another foretaste: Connection with Operator Theory...again

The Rodrigues Operator

Definition 4 Given a function ϕ and $\mathbf{u} \in \mathbb{P}'$, and a lattice $x(s)$, we define the k**-th Rodrigues operator associated with** $(\phi(s), \mathbf{u}, x(s))$ as follows: $R_k : \mathbb{P}[x_{k+1}] \mapsto \mathbb{P}[x]$

$$R_1(\phi, \rho, x)[f] := g \iff \frac{\Delta}{\nabla x_1(s)}[f\phi\mathbf{u}] = g\mathbf{u},$$

$$R_k(\phi, \mathbf{u}, x) := R_1(\phi, \mathbf{u}, x) \circ R_{k-1}(\phi, \mathbf{u}_1, x_1),$$

where

$$\langle \mathbf{u}_k, P(s)\rangle := \langle \mathbf{u}_{k-1}, \phi P(s-1)\rangle, \quad k = 1, 2, \ldots$$

A very important result

Theorem 6 If (σ, ρ, x) is a q-semiclassical tern, then for every integer k, if $\pi \in \mathbb{P}_m[x_{k+1}]$,

$$R_1(\sigma, \rho_k, x_k)[\pi] = \widetilde{\pi} \in \mathbb{P}_{m+1}[x_k].$$

If π monic, the leading coefficient of $\widetilde{\pi}$ is

$$\frac{q^{m+(\sigma+2)k} + q^{-m-(\sigma+2)k}}{2} b_r + [m + (\sigma + 2)k]_q a_p.$$

A very important result

Theorem 6 If (σ, ρ, x) is a q-semiclassical tern, then for every integer k, if $\pi \in \mathbb{P}_m[x_{k+1}]$,

$$R_1(\sigma, \rho_k, x_k)[\pi] = \widetilde{\pi} \in \mathbb{P}_{m+1}[x_k].$$

If π monic, the leading coefficient of $\widetilde{\pi}$ is

$$\frac{q^{m+(\sigma+2)k} + q^{-m-(\sigma+2)k}}{2} b_r + [m + (\sigma+2)k]_q a_p.$$

Remark 1

$\{P_n\}_{n\geq 0}$ is q-semiclassical $\not\Leftrightarrow \left\{ \dfrac{\Delta P_{n+1}}{\Delta x(s)} \right\}_{n\geq 0}$ is q-semiclassical.

Remark 2

$\{P_n\}_{n\geq 0}$ is q-semiclassical $\not\Leftrightarrow \{R_n(\phi, \mathbf{u}, x_{-1})[1]\}_{n\geq 0}$ is q-semiclassical.

The q-semiclassical quasi-orthogonal polynomials

The definition

Definition 5 Given $\mathbf{u} \in \mathbb{P}'$, the sequence of polynomials $\{P_n\}_{n\geq 0}$ is said to be quasi-orthogonal with respect $\mathbf{u}$ of order σ if

$$\langle \mathbf{u}, P_n P_m \rangle = 0, \quad |n-m| \geq \sigma + 1,$$
$$\langle \mathbf{u}, P_n P_m \rangle \neq 0, \quad |n-m| = \sigma.$$

Definition 6 A sequence of polynomials $\{P_n\}_{n\geq 0}$ is a sequence of q-semiclassical quasi-orthogonal polynomials (SQOPS) with respect $\mathbf{u}$ of order σ if $\mathbf{u}$ is a q-linear semiclassical functional and satisfies the last quasi-orthogonality relations.

Theorem 7 Let $\{P_n\}_{n\geq 0}$ be a SQOPS orthogonal with respect to $\mathbf{u} \in \mathbb{P}'$ such that is complete as orthonormal set in $\ell^2([a,b],\mathbf{u})$. The following statements are equivalent.

(i) $\{P_n\}_{n\geq 0}$ is q-quasi-orthogonal semiclassical.

(ii) $\{\Delta^{(1)} P_{n+1}\}_{n\geq 0}$ is a QOPS.

New characterization Theorem for q-semiclassical quasi-orthogonal polynomials

Theorem 8 Let $\{P_n\}_{n\geq 0}$ be a PS quasi-orthogonal with respect to $\mathbf{u} \in \mathbb{P}'$ on the lattice $x(s)$ and let ϕ be such some boundary condition hold. Then the following statements are equivalent:

1. $\{P_n\}_{n\geq 0}$ is a q-SQOPS.
2. The sequence $\{\Delta^{(1)} P_n\}_{n\geq 0}$ is a SQOPS with respect to $\mathbf{u}_1$ where $\mathbf{u}$ satisfies the last distributional equation.
3. For every integer k, the sequence $\{R_n(\phi, \mathbf{u}_k, x_k)[1]\}_{n\geq 0}$ is a SQOPS with respect to $\mathbf{u}_k$ where $\mathbf{u}$ satisfies last distributional equation.

New characterization Theorem for q-SQOP (cont.)

4. (Second order difference equation):

$$\phi(s)\frac{\Delta}{\nabla x_1(s)}\frac{\nabla P_n(s)}{\nabla x(s)}+\psi(s)\frac{\Delta P_n(s)}{\Delta x(s)}=\sum_{j=n-\sigma_1}^{n+\sigma_0}\Lambda_{j,n}P_j(s),$$

where σ_i, $i=0,1$, is the order of quasi-orthogonality of P_n and $\Delta P_n(s)/\Delta x(s)$, respectively.

5. $\{P_n\}_{n\geq 0}$ can be expressed in terms of the Rodrigues operator.

The dessert

The N-Askey-Wilson polynomials

See attached file.

The coffee: Relevant references

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Finally ....

THANKS FOR YOUR ATTENTION !!