

## A SURVEY ON $q$ -POLYNOMIALS AND THEIR ORTHOGONALITY PROPERTIES

ROBERTO S. COSTAS-SANTOS AND JOAQUIN F. SÁNCHEZ-LARA

ABSTRACT. In this paper we study the orthogonality conditions satisfied by the classical  $q$ -orthogonal polynomials that are located at the top of the  $q$ -Hahn tableau (big  $q$ -jacobí polynomials (bqJ)) and the Nikiforov–Uvarov tableau (Askey–Wilson polynomials (AW)) for almost any complex value of the parameters and for all non-negative integers degrees.

We state the degenerate version of Favard’s theorem, which is one of the keys of the paper, that allow us to extend the orthogonality properties valid up to some integer degree  $N$  to Sobolev type orthogonality properties.

We also present, following an analogous process that applied in [16], tables with the factorization and the discrete Sobolev-type orthogonality property for those families which satisfy a finite orthogonality property, i.e. it consists in sum of finite number of masspoints, such as  $q$ -Racah ( $qR$ ),  $q$ -Hahn ( $qH$ ), dual  $q$ -Hahn ( $dqH$ ), and  $q$ -Krawtchouk polynomials ( $qK$ ), among others.

### 1. INTRODUCTION

The classical orthogonal polynomials constitute a very important and interesting set of special functions and more specifically of orthogonal polynomials. They are very interesting mathematical objects that have attracted the attention not only of mathematicians since their appearance at the end of the XVIII century connected with some physical problems. They are used in several branches of mathematical and physical sciences and they have a lot of useful properties: they satisfy a three-term recurrence relation (TTRR), they are the solution of a second order linear differential (or difference) equation, their derivatives (or finite differences) also constitute an orthogonal family, among others (for a recent review see e.g. [6]).

In this survey we are going to focus on classical  $q$ -orthogonal polynomials – also called  $q$ -polynomials– which are polynomial eigenfunctions of the second order hypergeometric-type homogeneous linear difference operator

$$(1.1) \quad \mathcal{H} = \sigma(s) \frac{\Delta}{\Delta x(s - \frac{1}{2})} \frac{\nabla}{\nabla x(s)} + \tau(x(s)) \frac{\Delta}{\Delta x(s)},$$

where  $\widehat{\sigma}(x(s)) \stackrel{\text{def}}{=} \sigma(s) + \frac{1}{2}\tau(x(s))\Delta x(s - \frac{1}{2})$  and  $\tau(x(s))$  are polynomials on  $x(s)$  with  $\deg \widehat{\sigma} \leq 2$  and  $\deg \tau = 1$ .

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In fact they appear in several branches of the natural sciences, e.g., quantum groups and algebras [23, 24, 33], quantum optics, continued fractions, theta functions, elliptic functions, . . . ; among others [8, 19, 20, 29].

$q$ -polynomials have been intensively studied by the American School starting from the works of G. E. Andrews and R. Askey [9] arising the  $q$ -Askey tableau, and the Russian (former Soviet) school, starting from the works in [30] and further developed by N. M. Atakishiyev and S. K. Suslov (see [13, 12, 29, 31] and references therein) arising the Nikiforov–Uvarov tableau.

It is known that any family of polynomial eigenfunctions  $(p_n)$  of (1.1) satisfies a TTRR [30], i.e. there exist two sequences of complex numbers,  $(\beta_n)$  and  $(\gamma_n)$ , such that for  $n \geq 1$

$$(1.2) \quad p_{n+1}(x) = (x - \beta_n)p_n(x) - \gamma_n p_{n-1}(x),$$

with initial conditions  $p_0(x) = 1$ ,  $p_1(x) = x - \beta_0$ .

On the other side, if a sequence of monic polynomials  $(p_n)$  satisfies the initial conditions  $p_0(x) = 1$ ,  $p_1(x) = x - \beta_0$ , and the TTRR (1.2) then these polynomials are orthogonal with respect to the moment functional  $\mathcal{L}_0$  [15, §1], defined by  $\mathcal{L}_0(p_n) = \delta_{n,0}$ ,  $n \geq 0$ , i.e., for  $n \neq m$

$$(1.3) \quad \mathcal{L}_0(p_n p_m) = 0.$$

If  $\gamma_n \neq 0$  for  $n \geq 1$  then the polynomials defined by (1.2), are the unique normal and monic polynomials satisfying the orthogonality property (1.3). Moreover, if  $\beta_n, \gamma_n$  are real,  $\gamma_n > 0$ , then there exists a positive Borel measure  $\mu$  such that

$$\mathcal{L}_0(p) = \int_{\mathbb{R}} p d\mu.$$

This result is known as Favard’s theorem (see [18], [15, p. 21]), although this result was also discovered (independent of J. Favard) by I. P. Natanson in 1935 [28] and was presented by himself in a seminar led by S. N. Bernstein. He then did not publish the result since the work of J. Favard appeared in the meantime.

Our main aims here are to study the orthogonality conditions satisfied by Askey–Wilson and big  $q$ -Jacobi polynomials for almost any complex value of the parameters, any complex value of  $q$  and all non-negative integer degrees. In all the cases, the proposed orthogonality conditions characterizes such polynomials. When there exists a  $\gamma_N = 0$  in (1.2), an extension of the Favard’s result is used in order to establish a Sobolev-type orthogonality. In such a case we also give the factorization

$$p_{n+N} = p_N p_n^{(N)}, \quad n \geq 0,$$

where  $p_n^{(N)}$  is the associated polynomial of order  $N$  and degree  $n$ , which also belongs to the Nikiforov–Ubarov and/or  $q$ -Askey tableaux. We present a table with the Sobolev-type orthogonality and the factorization for all the  $q$ -polynomials considered in Section 2.2 for whose TTRR there exists, at least, one  $N$  such that  $\gamma_N = 0$ .

The structure of the paper is as follows. The preliminaries which will be used throughout the paper as well as the extension of the Favard’s theorem are given in Section 2. In Sections 3 and 4 we study the orthogonality conditions for Askey–Wilson and big  $q$ -Jacobi polynomials respectively for almost any value of the complex parameters. In section 5 we study the orthogonality conditions for Askey–Wilson for  $|q| = 1$ . In Section 6 we give a table for all the families of  $q$ -polynomials

which satisfy a discrete orthogonality with a finite number of masses like  $q$ -Racah,  $q$ -Hahn, dual  $q$ -Hahn, and  $q$ -Krawtchouk polynomials, among others; and we finish this paper giving some conclusions and outlooks. An appendix is also included.

## 2. PRELIMINARIES

In this subsection we summarize some definitions and preliminary results that will be useful throughout the work. Most of them can be found in [15].

**Definition 2.1.** Let  $(\mu_n)$  be a sequence of complex numbers (moment sequence) and  $\mathcal{L}$  a functional acting on the linear space of polynomials  $\mathbb{P}$  with complex coefficients. We say that  $\mathcal{L}$  is a **moment functional associated with  $(\mu_n)$**  if  $\mathcal{L}$  is linear and  $\mathcal{L}(x^n) = \mu_n$ ,  $n \geq 0$ .

**Definition 2.2.** The polynomial sequence  $(p_n)$  is a **orthogonal polynomials system (OPS)** with respect to a moment functional  $\mathcal{L}$  if the following conditions hold:

- (1)  $p_n$  is a polynomial of exact degree  $n$ , i.e. the polynomial sequence  $(p_n)$  is **normal**.
- (2)  $\mathcal{L}(p_n p_m) = 0$ ,  $m \neq n$ .
- (3)  $\mathcal{L}(p_n^2) \neq 0$ .

This third condition is imposed in order to have a unique OPS: if  $\mathcal{L}(p_N^2) = 0$  then

$$\mathcal{L}((p_{N+1} + \alpha p_N)x^m) = 0, \quad m = 0, \dots, N, \quad \forall \alpha \in \mathbb{C}.$$

The next result is a direct consequence of the previous definition [15, §1.2, §1.3, pp. 8-17].

**Theorem 2.3.** *Let  $\mathcal{L}$  be a moment functional and  $(p_n)$  a polynomial sequence. The following statements are equivalent:*

- (1)  $(p_n)$  is an OPS with respect to  $\mathcal{L}$ .
- (2)  $\mathcal{L}(\pi p_n) = 0$  for all polynomials  $\pi$ ,  $\deg \pi < n$ , while  $\mathcal{L}(\pi p_n) \neq 0$  if the  $\deg \pi = n$ .
- (3)  $\mathcal{L}(x^m p_n(x)) = K_n \delta_{n,m}$ , where  $K_n \neq 0$ , for  $m = 0, 1, \dots, n$ .

It is well-known that a monic OPS  $(p_n)$  satisfies a TTRR of the form (1.2) where the coefficients  $\gamma_n$  do not vanish. The converse is also true.

**Theorem 2.4.** (*J. Favard*) *Let  $(p_n)$  be a polynomial sequence satisfying the initial conditions  $p_{-1} = 0$ ,  $p_0 = 1$  and the TTRR (1.2), where  $\gamma_n \neq 0$  for all  $n \geq 1$ . Then  $(p_n)$  is a OPS with respect to the canonical moment functional defined as*

$$\mathcal{L}(p_n) = \delta_{n,0}, \quad n = 0, 1, 2, \dots$$

On the other side, if there exists  $\gamma_N = 0$ , then the sequence  $(p_n)$  can not be a OPS since the identity

$$\gamma_N = \frac{\mathcal{L}(p_N^2)}{\mathcal{L}(p_{N-1}^2)},$$

shows that condition (3) in definition 2.2 does not hold.

Among the generalizations of OPS, one is given by considering symmetric bilinear functionals:

**Definition 2.5.** Given a sequence of polynomials  $(p_n)$ , we say that  $(p_n)$  is a OPS with respect to a symmetric bilinear functional  $\mathcal{B}$  if the following conditions hold:

- (1)  $(p_n)$  is normal.
- (2)  $\mathcal{B}(p_n, p_m) = 0$ ,  $m \neq n$ .
- (3)  $\mathcal{B}(p_n, p_n) \neq 0$ .

It is usual to write  $\langle f, g \rangle$  instead of  $\mathcal{B}(f, g)$  when there is no confusion about the bilinear functional acting. With this definition, the analog of theorem 2.3 is also valid but it is not the TTRR.

Notice that a sufficient condition for the existence of the TTRR is the Hankel's property, i.e.

$$\langle tf, g \rangle = \langle f, tg \rangle,$$

for all polynomials  $f$  and  $g$ , where  $\langle \cdot, \cdot \rangle$  acts on the variable  $t$ .

Among all the bilinear functionals we focus on the following Sobolev type ones:

$$\langle f, g \rangle = \mathcal{L}_0(fg) + \mathcal{L}_1(\mathcal{D}(f)\mathcal{D}(g)),$$

where  $\mathcal{L}_0, \mathcal{L}_1$  are linear functionals and  $\mathcal{D}$  is the derivative, the difference, or the  $q$ -difference operator.

With all this overview we are going to present an extension of Favard's theorem for the case when some  $\gamma$ 's coefficient vanishes.

**2.1. Degenerate generalization of Favard's theorem.** Consider the sequences  $(\beta_n)$  and  $(\gamma_n)$  of complex numbers and the polynomials generated by the following recurrence relation:

$$(2.1) \quad p_{n+1}(x) = (x - \beta_n)p_n(x) - \gamma_n p_{n-1}(x), \quad n = 1, 2, \dots,$$

with initial conditions  $p_0(x) = 1$  and  $p_1(x) = x - \beta_0$ . By Favard's result, we define for  $n \geq 0$  the moment functional as

$$\mathcal{L}_0(p_n) = \delta_{n,0}.$$

Notice that in such a case,  $\mathcal{L}_0(p_n p_m) = 0$  for all  $n \neq m$ .

It is important to point out that if there exists  $N$  so that  $\gamma_N = 0$ , then  $\mathcal{L}(p_N^2) = 0$  and thus the functional  $\mathcal{L}_0$  does not determine the complete polynomials sequence  $(p_n)$ .

In order to give an orthogonality that characterizes the family polynomials  $(p_n)$ , we need to consider a linear operator  $\mathcal{T}_1 : \mathbb{P} \rightarrow \mathbb{P}$ , and polynomial sequence  $(p_{n,1})$  satisfying the following conditions:

- (1)  $\deg(\mathcal{T}_1(p)) = \deg(p) - 1$  for any polynomial  $p$ .
- (2) The polynomial sequence  $(p_{n,1})$  is defined by

$$p_{n-1,1} \stackrel{\text{def}}{=} \frac{\mathcal{T}_1(p_n)}{c_{n,1}}, \quad n \geq 1,$$

where  $c_{n,1}$  is the leading coefficient of  $\mathcal{T}_1(x^n)$ , and it satisfies, for  $n \geq 1$ , the recurrence relation

$$p_{n+1,1}(x) = (x - \beta_{n,1})p_{n,1}(x) - \gamma_{n,1}p_{n-1,1}(x),$$

where the sequence  $(\gamma_{n,1})$  is such that there exists a strictly increasing mapping

$$\lambda : \{n \in \mathbb{N} : \gamma_{n,1} = 0\} \rightarrow \{n \in \mathbb{N} : \gamma_n = 0\},$$

with  $\lambda(n) > n$ .

Observe that this last condition basically means that, after the action of  $\mathcal{T}_1$ , the possible vanishing  $\gamma$ 's are shifted to a lower degree. In fact, for many families of  $q$ -polynomials and their relative natural  $q$ -difference operator the condition about  $\lambda$  writes

$$(2.2) \quad \gamma_{n,1} = 0 \iff \gamma_{n+1} = 0.$$

Under these hypothesis,  $(p_{n,1})$  is also a monic orthogonal polynomials sequence with respect to some moment functional, namely  $\mathcal{L}_1$ . This procedure can be iterated  $j$  times giving a sequence of operators  $\mathcal{T}_j$ , recurrence coefficients,  $(\beta_{n,j})$  and  $(\gamma_{n,j})$ , and moment functionals  $\mathcal{L}_j$  so that the family

$$(\mathcal{T}_k \circ \dots \circ \mathcal{T}_2 \circ \mathcal{T}_1(p_{n+k})),$$

is orthogonal with respect to  $\mathcal{L}_k$ . If we denote by  $\mathcal{T}^{(k)} \stackrel{\text{def}}{=} \mathcal{T}_k \circ \dots \circ \mathcal{T}_1$ , then

$$(2.3) \quad \begin{aligned} p_{n,k} &= C_{n,k} \mathcal{T}^{(k)}(p_{n+k}), & C_{n,k} &\neq 0, \\ p_{n+1,k} &= (x - \beta_{n,k})p_{n,k} - \gamma_{n,k}p_{n-1,k}, & p_{0,k} &= 1, \quad p_{1,k} = x - \beta_{0,k}, \\ \mathcal{L}_k(p_{n,k}p_{m,k}) &= 0, & n &\neq m. \end{aligned}$$

Taking into account this construction we can state the degenerate generalization of Favard's theorem.

**Theorem 2.6.** *Let  $(p_n)$  be a polynomials sequence satisfying the TTRR (2.1), so that there exists a unique  $N \in \mathbb{N}$  so that  $\gamma_N = 0$ , then  $(p_n)$  is the unique (monic) polynomial sequence that fulfills the orthogonality conditions*

$$(2.4) \quad \langle p_n, p_m \rangle = \mathcal{L}_0(p_n p_m) + \mathcal{L}_N(\mathcal{T}^{(N)}(p_n) \mathcal{T}^{(N)}(p_m)) = 0, \quad n \neq m.$$

The choice of  $\mathcal{T}^{(N)}$  and its link with  $\mathcal{L}_N$  guarantees that  $\langle p_n, p_m \rangle = 0$  for all  $n \neq m$ . Hence, we only need to check the orthogonality conditions  $\langle p_n, p_n \rangle \neq 0 \forall n \geq 0$  in order to prove that  $(p_n)$  is a MOPS (thus the family  $(p_n)$  is characterized by the orthogonality property).

If  $n < N$  then, by hypothesis,

$$\langle p_n, p_n \rangle = \mathcal{L}_0(p_n^2) = \gamma_n \cdots \gamma_1 \neq 0,$$

and if  $n \geq N$  then, taking into account (2.3),

$$\begin{aligned} \langle p_n, p_n \rangle &= \mathcal{L}_N(\mathcal{T}^{(N)}(p_n) \mathcal{T}^{(N)}(p_n)) = \frac{1}{C_{N,n-N}^2} \mathcal{L}_N(p_{n-N,N} p_{n-N,N}) \\ &= \frac{\gamma_{n-N,N} \gamma_{n-N-1,N} \cdots \gamma_{1,N}}{C_{n-N,N}^2} \neq 0. \quad \square \end{aligned}$$

*Remark 2.7.* Notice that if there exists  $N' < N$  such that  $\gamma_{n,N'} > 0$ , for all  $n$ , and the  $\gamma$ 's coefficients satisfy  $\gamma_1, \dots, \gamma_{N-1} > 0$  then, the value  $N$  in formula (2.4) can be replaced by  $N'$ , and in such a case the proof of the statement is similar. Now  $\langle p_n, p_n \rangle$  depends on the operators  $\mathcal{L}_0$  and  $\mathcal{L}_{N'}$  and, in this case, it is the sum of two positive terms which do not vanish simultaneously.

**Corollary 2.8.** Let  $(p_n)$  be a polynomial sequence satisfying the TTRR (2.1), and let  $\Lambda \stackrel{\text{def}}{=} \{n \in \mathbb{N} : \gamma_n = 0\}$ . Then  $(p_n)$  is the unique (monic) polynomial sequence that fulfills the orthogonality conditions

$$(2.5) \quad \langle p_n, p_m \rangle = \mathcal{L}_0(p_n p_m) + \sum_{k \in \mathcal{A}} \mathcal{L}_k(\mathcal{T}^{(k)}(p_n) \mathcal{T}^{(k)}(p_m)) = 0, \quad n \neq m,$$

being  $\mathcal{A} = \{N_0, N_1, \dots\}$  with  $N_{j+1} = N_j + \min\{n : \gamma_{n,N_j} = 0\}$ .

The proof is straightforward taking into account the proof of theorem 2.6.

*Remark 2.9.* Observe that if  $\Lambda$  is a finite set, then  $\mathcal{A}$  is also finite as well. Moreover, in (2.5) for any two polynomials there is always a finite number of non vanishing terms, so the result of corollary 2.8 remains even if the set  $\mathcal{A}$  is a infinite set.

Among the operators  $\mathcal{T}$  satisfying the imposed conditions one of the most natural ones is as follows:

$$\mathcal{T}_1(p)(x) \stackrel{\text{def}}{=} \mathcal{L}_0 \left( \frac{p(t) - p(x)}{t - x} \right), \quad p_{n,1} = \mathcal{T}_1(p_n) = p_n^{(1)},$$

here  $\mathcal{L}_0$  acts on the variable  $t$ . This is due the fact that this operator commutes with the multiplication operator by  $x$ , thus the coefficients of the TTRR are shifted by 1, i.e., for  $n \geq 1$ ,

$$p_{n+1}^{(1)}(x) = (x - \beta_{n+1})p_n^{(1)}(x) - \gamma_{n+1}p_{n-1}^{(1)}(x).$$

Notice that, although theorem 2.6 seems to be new, it has been used implicitly in [16] where the operator  $\mathcal{T}_j$  is the forward difference operator  $\Delta$  and it is applied to Racah, Hahn dual Hahn, and Krawtchouk polynomials. Orthogonality conditions of this type were also used in [1, 3, 4, 25] for Laguerre and Jacobi polynomials where the operator  $\mathcal{T}_j \equiv \mathcal{T}$  is the standard derivative, providing a Sobolev type orthogonality to these families. We are going to focus throughout this paper on the orthogonality properties of  $q$ -polynomials where the operator  $\mathcal{T}_j$  is a  $q$ -difference type operator.

**2.2. The  $q$ -Askey and Nikiforov-Uvarov tableaux.** In this section we summarize the data for the classical  $q$ -orthogonal families of the  $q$ -Hahn tableau assuming that  $\sigma(x)$  is a monic polynomial (see, e.g., [29, 21, 26, 22, 5]); We also include the two families of  $q$ -polynomials found by R. Álvarez-Nodarse and J. C. Medem in [7], namely the “0-Jacobi/Bessel”  $q$ -polynomials (0JB), and the “0-Laguerre/Bessel”  $q$ -polynomials (0LB) (see also cf. [5, pp. 214–217]).

Family	Hyp. Repres.	Family	Hyp. Repres.
cdqH	${}_3\varphi_2(ae^{i\theta}, ae^{-i\theta}; ab, ac q)$	bqJ	${}_3\varphi_2(abq^{n+1}, x; aq, cq q)$
qH	${}_3\varphi_2(\alpha\beta q^{n+1}, x; \alpha q, q^{-N} q)$	dqH	${}_3\varphi_2(q^{-x}, \gamma\delta q^{x+1}; \gamma q, q^{-N} q)$
0JB	${}_2\varphi_1(aq^n; - xa^{-1}b^{-1})$	bqL	${}_3\varphi_2(0, x; aq, bq q)$
lqJ	${}_2\varphi_1(abq^{n+1}; aq qx)$	qM	${}_2\varphi_1(x; bq  - q^{n+1}c^{-1})$
QqK	${}_2\varphi_1(x; q^{-N} pq^{n+1})$	AqK	${}_2\varphi_1(q^{-N}x; q^{-N} xp^{-1})$
qK	${}_2\varphi_1(x; xq^{N-n+1}  - pq^{n+N+1})$	dqK	${}_2\varphi_1(x; xq^{N-n+1} cqx^{-1})$
0LB	${}_2\varphi_1(0; - xa^{-1})$	lqL	${}_2\varphi_1(0; aq qx)$
qL	${}_2\varphi_1(-x; 0 q^{n+\alpha+1})$	AqC	${}_2\varphi_1(-aq^n; 0 qx)$
qC	${}_2\varphi_1(x; 0  - q^{n+1}a^{-1})$	ACI	${}_2\varphi_1(x^{-1}; 0 qxa^{-1})$
ACII	${}_2\varphi_1(x; - q^na^{-1})$	SW	${}_1\varphi_1(; 0  - q^{n+1}x)$

TABLE 1. Basic hypergeometric series representation of some  $q$ -polynomials

Notice that the families for which  $\sigma$  is not a polynomial on  $x(s)$  are not included in some of those tables; moreover to simplify the notation, we use for table 1 the following reduction:

$${}_r\varphi_s(\vec{a}; \vec{b}|z) \equiv {}_r\varphi_s \left( \begin{matrix} q^{-n} \vec{a} \\ \vec{b} \end{matrix} \middle| q; z \right).$$

$\sigma(x)$	$(q-1)\tau(x)$	$p_n(x)$
$(x-aq)(x-cq)$	$(abq^2-1)x+q(a+c-abq-acq)$	$p_n(x; a, b, c; q)$
$(x-1)(x-\alpha q^N)$	$(\alpha\beta q^2-1)x+1-\alpha(q-q^N+\beta q^{1+N})$	$h_n^{(\alpha, \beta)}(x, N; q)$
$(x-aq)(x-bq)$	$-x+q(a+b-abq)$	$p_n(x; a, b; q)$
$(x-q^{-N})(x-pq)$	$-x+q(q^{-N-1}+q-q^{-N+1})$	$k_n^{Aff}(x; p, N; q)$
$(x-1)(x-a)$	$-x+1+a$	$u_n^{(\alpha)}(x; q)$
$x(x-1)$	$(abq^2-1)x+1-aq$	$p_n(x; a, b q)$
$x(x-q^{-N})$	$-(1+pq)x+(pq-q^{-N})$	$k_n(x; p, N; q)$
$x(x-1)$	$-(aq+1)x+1$	$k_n(x; a; q)$
$x(x-1)$	$-x+1-aq$	$p_n(x; a q)$
$x^2$	$(aq-1)x-abq$	$j_n(x; a, b)$
$x^2$	$-x+aq$	$l_n(x; a)$
$x-bq$	$qc^{-1}x-qc^{-1}-1+bq$	$m_n(x; b, c; q)$
$x-1$	$-pq^{2-N}x+q1-N-1+pq$	$k_n^{qtm}(x; p, N; q)$
$x$	$qx-1$	$s_n(x; q)$
$x$	$\alpha qx-\alpha q-1$	$l_n^{(\alpha)}(x; q)$
$x$	$qa^{-1}x-qa^{-1}-1$	$c_n(x; a; q)$
$1$	$a^{-1}x-a^{-1}-1$	$v_n^{(a)}(x; q)$

TABLE 2. Basic data of some monic  $q$ -polynomials of the  $q$ -Hahn tableau

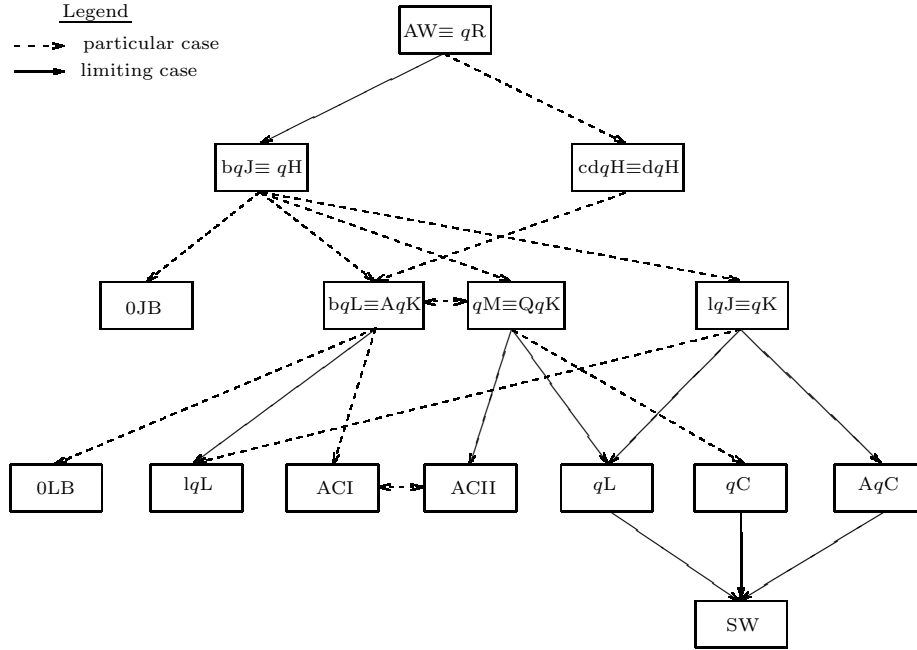


FIGURE 1. Most of the known links between  $q$ -polynomials

On the other hand, since the characterization theorems characterize the  $q$ -polynomials (see e.g. [2, 6, 17]) then it is a direct calculation by using table 2 to check

the following identities:

$$\begin{aligned} p_n(x + aq; a, a; q) &= l_n(x; a - a^2q), \\ p_n(x + aq; a, b, a; q) &= j_n(x; abq, 1 + ab^{-1} - b^{-1}q^{-1} - aq). \end{aligned}$$

Although most of the identities we present here are already known (see [21, §4]) we believe it is a good idea to show them here (see table 5).

### 3. THE ASKEY-WILSON POLYNOMIALS

This family of  $q$ -polynomials, which were introduced by R. Askey and J. Wilson in [10], are located at the top of the  $q$ -Askey tableau. The monic Askey-Wilson polynomials can be written as a basic hypergeometric series [21, p. 63]

$$p_n(x; a, b, c, d|q) = \frac{(ab; q)_n (ac; q)_n (ad; q)_n}{(2a)^n (abcdq^{n-1}; q)_n} {}_4\phi_3 \left( \begin{matrix} q^{-n}, abcdq^{n-1}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix} \middle| q; q \right),$$

with  $x = \cos \theta$ . Moreover they fulfill, for  $n \geq 0$ , the TTRR

$$(3.1) \quad xp_n(x) = p_{n+1}(x) + \beta_n p_n(x) + \gamma_n p_{n-1}(x),$$

where  $\beta_n = (a + a^{-1} - A_n - C_n)/2$ , and  $\gamma_n = A_{n-1}C_n/4$  being

$$\begin{aligned} A_n &= \frac{(1 - abq^n)(1 - acq^n)(1 - adq^n)(1 - abcdq^{n-1})}{a(1 - abcdq^{2n-1})(1 - abcdq^{2n})}, \\ C_n &= \frac{a(1 - q^n)(1 - bcq^{n-1})(1 - bdq^{n-1})(1 - cdq^{n-1})}{(1 - abcdq^{2n-2})(1 - abcdq^{2n-1})}. \end{aligned}$$

Observe that, since  $\Lambda = \{n \in \mathbb{N} : \gamma_n = 0\}$ , then

$$\Lambda = \emptyset \iff ab, ac, ad, bc, bd, cd \notin \Omega(q) \stackrel{\text{def}}{=} \{q^{-k} : k \in \mathbb{N}_0\}.$$

In the forthcoming sections we only consider normal polynomials sequences therefore  $abcd \notin \Omega(q)$ .

**3.1. The orthogonality conditions for  $|q| < 1$ .** It is known that if the parameters  $a, b, c$ , and  $d$  are real, or occur in complex conjugate pairs if complex,  $\max\{|a|, |b|, |c|, |d|\} < 1$ , the family fulfills the orthogonality conditions [11]

$$(3.2) \quad \frac{1}{2\pi} \int_{-1}^1 p_m(x)p_n(x) \frac{\omega(x)}{\sqrt{1-x^2}} dx = d_n^{2(AW)} \delta_{n,m}, \quad n, m \geq 0,$$

where  $d_n^{2(AW)}$  is the squared norm of the monic Askey-Wilson polynomial of degree  $n$

$$(3.3) \quad d_n^{2(AW)} = \frac{(abcdq^{2n}; q)_\infty}{4^n (abcdq^{n-1}; q)_n (q^{n+1}, abq^n, acq^n, adq^n, bcq^n, bdq^n, cdq^n; q)_\infty},$$

and

$$\omega(x) = \left| \frac{(e^{2i\theta}; q)_\infty}{(ae^{i\theta}, be^{i\theta}, ce^{i\theta}, de^{i\theta}; q)_\infty} \right|^2 = \frac{h(x, 1)h(x, -1)h(x, q^{\frac{1}{2}})h(x, -q^{\frac{1}{2}})}{h(x, a)h(x, b)h(x, c)h(x, d)},$$

with

$$h(x, \alpha) \stackrel{\text{def}}{=} \prod_{k=0}^{\infty} (1 - 2\alpha xq^k + \alpha^2 q^{2k}) = (\alpha e^{i\theta}, \alpha e^{-i\theta}; q)_\infty, \quad x = \cos \theta.$$



Observe that the orthogonality conditions given in (3.2) are a particular case of the non-hermitian complex orthogonality conditions

$$(3.4) \quad \int_{\Gamma} p_n \left( \frac{z+z^{-1}}{2} \right) p_m \left( \frac{z+z^{-1}}{2} \right) W(z) dz = d_n^{2(AW)} \delta_{n,m}, \quad n \neq m,$$

which were obtained by Askey and Wilson (see [10]), being

$$W(z) = \frac{1}{z} w \left( \frac{z+z^{-1}}{2} \right).$$

The poles of  $w$  are

$$\frac{\alpha q^k + (\alpha q^k)^{-1}}{2}, \quad \alpha = a, b, c, d, \quad k \in \mathbb{N}_0,$$

therefore  $W$  has convergent poles, since  $|q| < 1$ , at

$$a q^k, \quad b q^k, \quad c q^k, \quad d q^k, \quad k \in \mathbb{N}_0,$$

and divergent poles at

$$a^{-1} q^{-k}, \quad b^{-1} q^{-k}, \quad c^{-1} q^{-k}, \quad d^{-1} q^{-k}, \quad k \in \mathbb{N}_0.$$

The contour  $\Gamma$  is a curve separating the divergent poles from the convergent poles, encircling them only once. In fact, if the parameters satisfies  $\max\{|a|, |b|, |c|, |d|\} < 1$  then  $\Gamma$  can be taken as the unit circle, otherwise it is a deformation of the unit circle.

The poles can be separated only if

$$a^2, b^2, c^2, d^2, ab, ac, ad, bc, bd, cd \notin \Omega(q),$$

so in the following we focus our attention when this does not occur. Looking at the expression of the coefficient  $\gamma_n$ , it vanishes only if

$$ab, ac, ad, bc, bd, cd \notin \Omega(q),$$

and since any rearrangement of the parameters does not change the polynomial, it is enough to study the following three key cases:

- only  $a^2 \in \Omega(q)$ ,
- only  $ab \in \Omega(q)$ ,
- or only  $a^2 = q^{-M}$  and  $ab = q^{-N}$ , with  $M < N - 1$ , belong to  $\Omega(q)$ .

3.1.1.  $a^2 \in \Omega(q)$  and  $ab, ac, ad, bc, bd, cd \notin \Omega(q)$ . Although the poles can not be separated, there is no  $\gamma_n$  vanishing in the TTRR, so we look for a simple reformulation of (3.4). Let us assume  $a^2 = q^{-M}$  with  $M \in \mathbb{N}_0$ , then the poles that can not be separated are

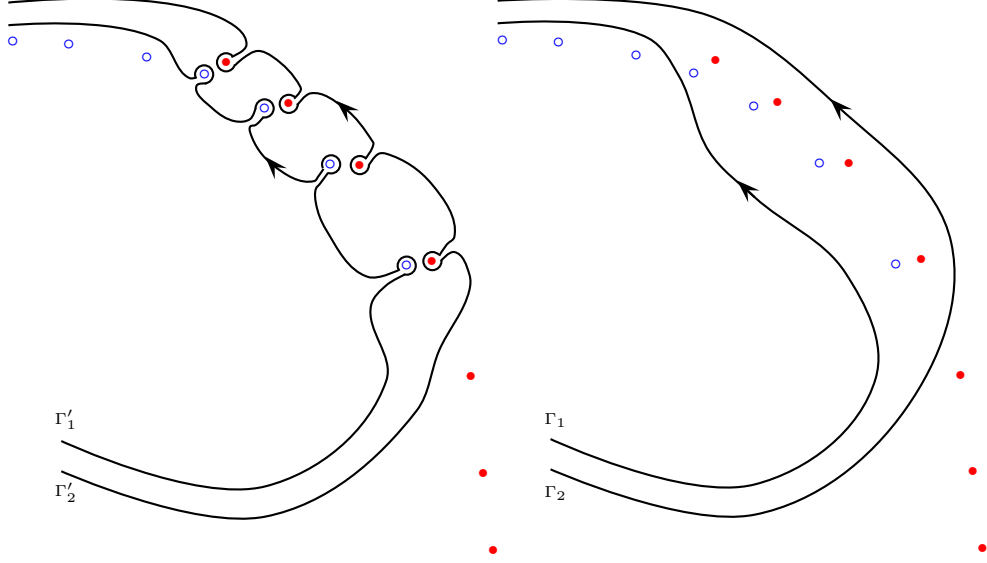
$$Z = \{q^{-M/2}, q^{1-M/2}, \dots, q^{M/2}\}, \quad \text{or} \quad Z = \{-q^{-M/2}, -q^{1-M/2}, \dots, -q^{M/2}\}.$$

Notice that if some of these poles coincide with the generated by  $b$ , then  $ab \in \Omega(q)$  which is not possible in this case. Hence  $Z$  has empty intersection with the rest of the poles of  $W$ .

We consider this case as the limit for  $p_n(\bullet; \alpha, b, c, d; q)$  with  $\alpha \rightarrow a$ , so the poles of  $W(\bullet; \alpha, b, c, d; q)$  can be separated adequately. Thus the orthogonality conditions (3.4) are valid and can be expressed as

$$0 = \int_{\Gamma'_1 \cup \Gamma'_2} p_n \left( \frac{z+z^{-1}}{2} \right) p_m \left( \frac{z+z^{-1}}{2} \right) W(z) dz,$$

where the curves  $\Gamma'_1$  and  $\Gamma'_2$  separate the poles. Therefore these curves can be deformed in order to obtain the integral through two curves,  $\Gamma_1$  and  $\Gamma_2$ , such that they separate the convergent poles from the divergent ones, but the poles in  $Z$  which stand between the two curves, with several residues added (see next figures).



When  $\alpha \rightarrow a$ , the poles  $\alpha q^k$  with  $k \leq M$  and  $\alpha^{-1}q^{-(M-k)}$  converges to  $aq^k$  and it can be seen that the sum of the two residues at this points tends to zero. So the limit  $\alpha \rightarrow a$  yields

$$\int_{\Gamma_1 \cup \Gamma_2} p_n \left( \frac{z+z^{-1}}{2} \right) p_m \left( \frac{z+z^{-1}}{2} \right) W(z) dz = d_n^2 \delta_{n,m},$$

with  $d_n^2$  the normalizing factor given by (3.3).

3.1.2.  $ab = q^{-N+1}$  and  $a^2, b^2 \notin \{q^0, \dots, q^{-N+2}\}$ , i.e  $\gamma_N = 0$ , so  $N \in \Lambda$ . The orthogonality conditions depend on the size of  $\Lambda$  (see corollary 2.8), so we show how it is in the simplest case  $\Lambda = \{N\}$ , i.e.,  $ac, ad, bc, bd, cd \notin \Omega(q) \setminus \{q^{-N}\}$ .

Since monic  $q$ -Racah polynomials can be written in terms of the basic hypergeometric functions as [21, (3.2.1)]

$$r_n(\mu(x); \alpha, \beta, \gamma, \delta | q) = {}_4\phi_3 \left( \begin{matrix} q^{-n}, \alpha\beta q^{n+1}, q^{-x}, \gamma\delta q^{x+1} \\ \alpha q, \beta\delta q, \gamma q \end{matrix} \middle| q; q \right),$$

with  $\mu(x) = q^{-x} + \gamma\delta q^{x+1}$ , the following identity linking Askey-Wilson and  $q$ -Racah polynomials holds

$$p_n(x; a, b, c, d; q) = r_n(2ax; q^{-N}, cdq^{-1}, adq^{-1}, ad^{-1}; q),$$

and it yields the moment functional  $\mathcal{L}_0$  in theorem 2.6 which is the one known for  $q$ -Racah polynomials

$$(3.5) \quad \mathcal{L}_0(p) = \sum_{j=0}^{N-1} \frac{(q^{-N+1}, ac, ad, a^2; q)_j}{(q, a^2 q^N, ac^{-1}q, ad^{-1}q; q)_j} \frac{(1 - a^2 q^{2j})}{(cdq^{-N})^j (1 - a^2)^j} \left( \frac{q^{-j} + a^2 q^j}{2a} \right)^p.$$

Notice that the assumptions on  $a^2$  and  $b^2$  guarantees the definition of  $\mathcal{L}_0$ .

Furthermore, since

$$\mathcal{D}_q p_n(x; a, b, c, d; q) = \frac{q^n - 1}{q - 1} p_{n-1}(x; aq^{1/2}, bq^{1/2}, cq^{1/2}, dq^{1/2}; q),$$

where the  $q$ -difference operator, also called the Hahn's operator, is

$$\mathcal{D}_q(f)(z) \stackrel{\text{def}}{=} \begin{cases} \frac{f(z) - f(qz)}{(1-q)z}, & z \neq 0 \wedge q \neq 1, \\ f'(z), & z = 0 \vee q = 1, \end{cases}$$

the operator  $\mathcal{T}$  can be chosen as  $\mathcal{D}_q$  and the condition (2.2) holds. Hence, for  $n \geq N$ ,

$$\mathcal{D}_q^N p_n(x; a, b, c, d; q) = \frac{(q^{n-N+1}; q)_N}{(1-q)^N} p_{n-N}(x; aq^{N/2}, bq^{N/2}, cq^{N/2}, dq^{N/2}; q),$$

so  $\mathcal{L}_N$  is the moment functional associated with the Askey-Wilson polynomials with parameters  $aq^{N/2}$ ,  $bq^{N/2}$ ,  $cq^{N/2}$ , and  $dq^{N/2}$ , i.e.

$$\mathcal{L}_N(p) = \int_{\Gamma} p \left( \frac{z + z^{-1}}{2} \right) \frac{1}{z} w \left( \frac{z + z^{-1}}{2} \right) dz,$$

where

$$w(z) = w(z; aq^{N/2}, bq^{N/2}, cq^{N/2}, dq^{N/2}; q),$$

and  $\Gamma$  is a contour which separates the poles. Then, by theorem 2.6, the polynomial sequence  $(p_n(x; a, b, c, d))$  is uniquely determined, up to a constant, by the orthogonality conditions, for  $n \neq m$ ,

$$\langle p_n(\bullet; a, b, c, d; q), p_m(\bullet; a, b, c, d; q) \rangle = \mathcal{L}_0(p_n p_m) + \mathcal{L}_N(\mathcal{D}_q^N(p_n) \mathcal{D}_q^N(p_m)) = 0.$$

3.1.3.  $ab = q^{-N+1}$  and  $a^2 = q^{-M}$ , with  $M \in \{0, \dots, N-2\}$ . Also the form of the orthogonality depends on the numbers of elements of  $\Lambda$ . For simplicity, we see only the case when the cardinal of  $\Lambda$  is one, and when  $\Lambda$  is greater, the orthogonality is given by corollary 2.8.

The orthogonality is basically the same that in the case 3.1.2, but now  $\mathcal{L}_0$  is not valid since it has lost several orthogonality conditions. The adequate form of  $\mathcal{L}_0$  is obtained as a limit case. Let us consider the linear functional

$$\mathcal{L}_0^\alpha(p) = \sum_{j=0}^{N-1} A_j(\alpha) p(\mu_j(\alpha)),$$

with  $\mu_j(x; \alpha) = (\alpha q^j + \alpha^{-1} q^{-j})/2$ , and

$$A_j(\alpha) = \frac{(q^{-N+1}, \alpha c, \alpha d, \alpha^2; q)_j}{(q, \alpha^2 q^N, \alpha c^{-1} q, \alpha d^{-1} q; q)_j} \frac{(1 - \alpha^2 q^{2j})}{(cdq^{-N})^j (1 - \alpha^2)}.$$

Straightforward computations yields

$$A_j(a) = 0,$$

for  $j \in \{M+1, \dots, N-1\}$  and  $j = M/2$  if  $M$  is even, and

$$A_j(a) + A_{M-j}(a) = 0, \quad \mu_j(a) = \mu_{M-j}(a),$$

for  $j \in \{0, \dots, M\}$  but  $j = M/2$  if  $M$  is even. Thus  $\mathcal{L}_0^\alpha$  tends to the null functional. But since it is possible to consider any normalization, we remove the common factor  $(\alpha - a)$ ,

$$\lim_{\alpha \rightarrow a} \frac{A_j(\alpha)}{\alpha - a} p(\mu_j(\alpha)) = A'_j(a) p(\mu(a)),$$

for  $j = M + 1, \dots, N$  and if  $M$  is even  $j = M/2$ , and also

$$\begin{aligned} \lim_{\alpha \rightarrow a} \frac{A_j(\alpha)p(\mu_j(\alpha)) + A_{M-j}(\alpha)p(\mu_{M-j}(\alpha))}{\alpha - a} \\ = (A'_j(a) + A'_{M-j}(a))p(\mu_j(a)) + A_j(a)(q^j - q^{M-j})p'(\mu_j(a)) \end{aligned}$$

for  $j = 0, \dots, M$  but if  $M$  is even  $j \neq M/2$ .

Hence we define  $\mathcal{L}_0$  as

$$\begin{aligned} \mathcal{L}_0(p) &= \sum_{j=0}^{(M-1)/2} (A'_j(a) + A'_{M-j}(a))p(\mu_j(a)) + A_j(a)(q^j - q^{M-j})p'(\mu_j(a)) \\ &+ \sum_{j=M+1}^{N-1} A'_j(a)p(\mu_j(a)) \end{aligned}$$

if  $M$  is odd, and

$$\begin{aligned} \mathcal{L}_0(p) &= \sum_{j=0}^{M/2-1} (A'_j(a) + A'_{M-j}(a))p(\mu_j(a)) + A_j(a)(q^j - q^{M-j})p'(\mu_j(a)) \\ &+ \sum_{j=M+1}^{N-1} A'_j(a)p(\mu_j(a)) + A'_{M/2}(a)p(\mu_{M/2}(a)) \end{aligned}$$

if  $M$  is even. The Askey-Wilson polynomials of degree at most  $N$  with  $ab = q^{-N+1}$ ,  $a^2 = q^{-M}$  and  $M = 0, \dots, N - 2$  are uniquely determined by the orthogonality property:

$$\mathcal{L}_0(p_n p_m) = 0, \quad 0 \leq m < n \leq N.$$

In particular,  $p_N$  has simple roots on  $\mu_j(a)$ ,  $j = M + 1, \dots, N$  and on  $\mu_{M/2}(a)$  if  $M$  is even; the rest of the roots,  $\mu_j(a)$ ,  $j = 0, \dots, [(M - 1)/2]$  are double.

The moment functional  $\mathcal{L}_N$  in theorem 2.6 is the same that the one given in section 3.1.2.

**3.2. The orthogonality conditions for  $|q| \geq 1$ .** Taking into account the relation between basic hypergeometric series [21, p. 9]

$$(3.6) \quad {}_4\varphi_3 \left( \begin{matrix} q^{-n}, a, b, c \\ d, e, f \end{matrix} \middle| q; q \right) = {}_4\varphi_3 \left( \begin{matrix} q^n, a^{-1}, b^{-1}, c^{-1} \\ d^{-1}, e^{-1}, f^{-1} \end{matrix} \middle| q^{-1}; \frac{abcq^{-n}}{def} \right).$$

We can relate each family of  $q$ -polynomials on the parameter  $q$  into another family of  $q$ -polynomials on the parameter  $q^{-1}$ . In fact in this case it provides

$$p_n(x; a, b, c, d|q^{-1}) = p_n(x; a^{-1}, b^{-1}, c^{-1}, d^{-1}|q).$$

Therefore if  $|q| > 1$  we can get analogous orthogonality conditions just using this relation and the orthogonality conditions given in Section 3.1 for  $|q| < 1$ .

If  $q$  is a primitive root of unity, i.e.  $q = e^{2\pi i M/N}$  with  $\gcd(N, M) = 1$  then  $\{kN : k \in \mathbb{N}\} \subseteq \Lambda$ , so, by corollary 2.8, for  $k = 1$  we need to construct the following orthogonality property for the Askey-Wilson polynomials up to degree  $N$  [14], i.e.

$$(3.7) \quad \sum_{s=0}^{N-1} p_n(x_s)p_m(x_s)\omega_s = \gamma_1 \cdots \gamma_n \delta_{n,m},$$

where  $n, m = 0, 1, \dots, N-1$ ,  $\{x_s\}_{s=0}^{N-1}$  are the zeroes of  $p_N$ , and the weight function is

$$\omega_s = \frac{\gamma_1 \cdots \gamma_{N-1}}{p_{N-1}(x_s)p'_N(x_s)}.$$

Observe that the only requirement to be added is that all zeros  $x_s$  must be simple.

Since the method considered in [32] to obtain  $\omega_s$  can be applied to obtain such weights functions to other families of  $q$ -polynomials, next we give a brief outline of it.

It is known that Askey-Wilson polynomials are polynomial eigenfunctions of the second order homogeneous linear difference operator:

$$\sigma(-s) \frac{\Delta p_n(x(s))}{\Delta x(s)} + \sigma(s) \frac{\nabla p_n(x(s))}{\nabla x(s)} - \lambda_n \Delta x(s - \frac{1}{2}) p_n(x(s)) = 0,$$

being  $\sigma(s) = -(q^{1/2} - q^{-1/2})^2 q^{-2s+1/2} (q^s - a)(q^s - b)(q^s - c)(q^s - d)$ , and their corresponding eigenvalues

$$\lambda_n = -4q^{-n+1}(1 - q^n)(1 - abcdq^{n-1}).$$

Notice that such difference operator can be rewritten, by using the definition of the difference operators  $\Delta$  and  $\nabla$ , as [21, Eq. (3.1.7)], [32, Eq. (3.6)]

$$A(z^{-1})p_n(q^{-1}z) - (A(z) + A(z^{-1}))p_n(z) + A(z)p_n(qz) = \lambda_n p_n(z),$$

where  $A(z) = (1 - az)(1 - bz)(1 - cz)(1 - dz)/((1 - z^2)(1 - qz^2))$ . Therefore, multiplying the previous equation by a function  $\rho(s)$  satisfying the only requirement of periodicity  $\rho(s + N) = \rho(s)$ , and combining it with a similar equation for the polynomials  $p_m(x_s)$ , one can get a bilinear relation:

$$\begin{aligned} & A_s \sigma(s) (p_n(x_{s-1})p_m(x_s) - p_n(x_s)p_m(x_{s-1})) \\ & + C_s \sigma(s) (p_n(x_{s+1})p_m(x_s) - p_n(x_s)p_m(x_{s+1})) \\ & = (\lambda_n - \lambda_m) \sigma(s) p_n(x_s) p_m(x_s). \end{aligned}$$

Choose  $\rho(s)$  in such a way that

$$(3.8) \quad A_{s+1} \rho(s+1) = C_s \rho(s),$$

summing from  $s = 0$  to  $s = N - 1$  and using the obvious periodicity property of  $\rho(s)$  we get the orthogonality property:

$$(\lambda_n - \lambda_m) \sum_{s=0}^{N-1} p_n(x_s) p_m(x_s) \rho(s) = 0, \quad n \neq m.$$

Hence  $\omega_s = \omega_0 \rho(s)$ , with  $\omega_0$  is the normalization constant, is determined from the relation (3.8).

Spiridonov and Zhedanov found that the polynomials  $(p_n(\bullet; a, b, c, d; e^{2\pi i M/N}))$ , with  $0 \leq n \leq N$ , under the assumptions

$$abcd, ab, ac, ad, bc, bd, cd \neq q^k, \quad k = 0, \dots, N-1,$$

are uniquely determined by the orthogonality conditions

$$\mathcal{L}_0(p_n p_m) = d_n^2 \delta_{n,m}, \quad d_n^2 \neq 0,$$

being

$$\mathcal{L}_0(p) = \sum_{j=0}^{N-1} \left( \frac{q}{abcd} \right)^j \frac{(1 - rq^{2j})(ar, br, cr, dr; q)_j}{(1 - r^2)(qr/a, qr/b, qr/c, qr/d; q)_j} p(rq^j + r^{-1}q^{-j}),$$

and  $r$  the root with minimal argument of the equation

$$r^N = E_N/2 + \sqrt{E_N^2/4 - 1},$$

being

$$E_N = \frac{a^N + b^N + c^N + d^N - (abc)^N - (abd)^N - (acd)^N - (bcd)^N}{1 - (abcd)^N}.$$

*Remark 3.1.* A straightforward computation shows that

$$\rho(s) \stackrel{\text{def}}{=} \left( \frac{q}{abcd} \right)^s \frac{(1 - rq^{2s})(ar, br, cr, dr; q)_s}{(1 - r^2)(qr/a, qr/b, qr/c, qr/d; q)_s},$$

satisfies the condition (3.8). A hint for such calculation can be found in [17, Lemma 5.1].

Due to the cyclic behavior of the TTRR coefficients and since  $\gamma_N = 0$ , these polynomials satisfy the identity

$$p_n = p_N^\ell p_m, \quad n = \ell N + m, \quad 0 \leq m < N,$$

which explains the behavior of the polynomial for greater degrees. However corollary 2.8 is applicable. For  $n \geq N$

$$\mathcal{D}_q^N p_n(x; a, b, c, d; q) = \frac{(q^{n-N+1}; q)_N}{(1-q)^N} p_{n-N}((-1)^M x; a, b, c, d; q),$$

so the orthogonality conditions that characterizes all polynomials are the following:

- If  $M$  is even:

$$\langle p_n, p_m \rangle = \sum_{j=0}^{\infty} \mathcal{L}_0(\mathcal{D}_q^{Nj}(p_n) \mathcal{D}_q^{Nj}(p_m)).$$

- If  $M$  is odd:

$$\langle p_n, p_m \rangle = \sum_{j=0}^{\infty} \mathcal{L}_0(\mathcal{D}_q^{2jN}(p_n) \mathcal{D}_q^{2jN}(p_m)) + \mathcal{L}_N(\mathcal{D}_q^{(2j+1)N}(p_n) \mathcal{D}_q^{(2j+1)N}(p_m)),$$

being

$$\mathcal{L}_N(p) = \sum_{j=0}^{N-1} \left( \frac{q}{abcd} \right)^j \frac{(1 - rq^{2j})(ar, br, cr, dr; q)_j}{(1 - r^2)(qr/a, qr/b, qr/c, qr/d; q)_j} p(-rq^j - r^{-1}q^{-j}).$$

#### 4. THE BIG $q$ -JACOBI POLYNOMIALS

The big  $q$ -Jacobi polynomials, which were introduced by Hahn in 1949, are located at the top of the  $q$ -Hahn tableau. The monic big  $q$ -Jacobi polynomials can be written in terms of basic hypergeometric series as [21, p. 73]

$$(4.1) \quad p_n(x; a, b, c; q) = \frac{(aq, cq; q)_n}{(abq^{n+1}; q)_n} {}_3\varphi_2 \left( \begin{matrix} q^{-n}, abq^{n+1}, x \\ aq, cq \end{matrix} \middle| q; q \right).$$

In fact they are the most general family of  $q$ -polynomials on the  $q$ -exponential lattice, also called  $q$ -linear lattice; and they appear, among others branches of physics, in the representation theory of the quantum algebras [33]. The monic big  $q$ -Jacobi polynomials fulfill, for  $n \geq 1$ , the following TTRR:

$$(4.2) \quad xp_n(x) = p_{n+1}(x) + \beta_n p_n(x) + \gamma_n p_{n-1}(x),$$

with  $\beta_n = 1 - \hat{A}_n - \hat{C}_n$ , and  $\gamma_n = \hat{A}_{n-1}\hat{C}_n$  being

$$(4.3) \quad \begin{aligned} \hat{A}_n &= \frac{(1 - aq^{n+1})(1 - abq^{n+1})(1 - cq^{n+1})}{(1 - abq^{2n+1})(1 - abq^{2n+2})}, \\ \hat{C}_n &= -acq^{n+1} \frac{(1 - q^n)(1 - abc^{-1}q^n)(1 - bq^n)}{(1 - abq^{2n})(1 - abq^{2n+1})}. \end{aligned}$$

*Remark 4.1.* Observe that if  $a = 0$  then the coefficients  $\gamma_n = 0$  for all  $n \in \mathbb{N}_0$ , and if  $b = 0$ , or  $c = 0$ , then the big  $q$ -Jacobi polynomials become the big  $q$ -Laguerre or the little  $q$ -Jacobi polynomials respectively, which are located below in the  $q$ -Askey tableau thus we omit these cases.

A slightly less detailed study on orthogonality conditions for the big  $q$ -Jacobi can be found in [27].

**4.1. The orthogonality conditions for  $|q| < 1$ .** It is known that if  $0 < q < 1$ ,  $0 < a, b < q^{-1}$ , and  $c < 0$  the family of big  $q$ -Jacobi polynomials fulfills the orthogonality conditions [21, p. 73]

$$(4.4) \quad \int_{cq}^{aq} \frac{(a^{-1}x, c^{-1}x; q)_\infty}{(x, bc^{-1}x; q)_\infty} p_n(x; a, b, c; q) p_m(x; a, b, c; q) d_q x = d_n^{2(BqJ)} \delta_{n,m},$$

where the Jackson  $q$ -integral (see [20, 21]) is defined as follows

$$\int_a^b f(t) d_q t = a(q-1) \sum_{s=0}^{\infty} f(aq^s) q^s - b(q-1) \sum_{s=0}^{\infty} f(bq^s) q^s.$$

The aim of this section is to give orthogonality conditions for the big  $q$ -Jacobi polynomials for general complex parameters, including complex  $|q| < 1$ , except for those for which the family is not normal, i.e.  $ab \in \Omega(q)$ .

In fact, notice that if the parameters belong to compact sets where the integrand in (4.4) is bounded, hence such series converges uniformly. Thus we can apply the Weierstrass theorem and analytic prolongation in order to asserts that (4.4) is valid for

$$a, b, c, abc^{-1} \notin \Omega(q),$$

which it is equivalent to  $\Lambda = \emptyset$ , therefore in the following we focus our attention in the case  $\Lambda \neq \emptyset$ . More precisely, we study the cases for which this set has exactly one element, namely  $N$ . If this set is greater we refer the reader to corollary 2.8.

**4.1.1.  $c = q^{-N}$  and  $a, b, abc^{-1} \notin \Omega(q) \setminus \{q^{-N}\}$ .** Taking into account that the big  $q$ -Jacobi and  $q$ -Hahn polynomials are linked through the relation

$$p_n(x; a, b, q^{-N}; q) = h_n^{(a,b)}(x; N-1; q),$$

the moment functional  $\mathcal{L}_0$  in theorem 2.6 is the one known for the  $q$ -Hahn polynomials [21] with parameters  $a, b$  and  $N-1$ .

$$(4.5) \quad \mathcal{L}_0(p) = \sum_{x=0}^{N-1} \frac{(aq, q^{-N+1}; q)_x}{(q, b^{-1}q^{-N+1}; q)_x} (abq)^{-x} p(q^{-x}).$$

Moreover, since

$$\mathcal{D}_{q^{-1}} h_n^{(\alpha, \beta)}(x; M; q) = \frac{q^{-n} - 1}{q^{-1} - 1} h_{n-1}^{(\alpha q, \beta q)}(x; M-1; q),$$

the operator  $\mathcal{F}$  in theorem 2.6 can be chosen as  $\mathcal{D}_{q^{-1}}$  and the condition (2.2) holds (see the relation between  $q$ -Hahn and big  $q$ -Jacobi polynomials and the expression (4.3) for the coefficients  $\gamma_n$ ). Also, for  $n \geq N$ ,

$$\mathcal{D}_{q^{-1}}^N p_n(x; a, b, q^{-N}; q) = \frac{(q^{-n}; q)_N}{(1 - q^{-1})^N} p_{n-N}(x; aq^N, bq^N, 1; q).$$

Accordingly with these expressions and the weight function for the big  $q$ -Jacobi polynomials with parameters  $aq^N$ ,  $bq^N$  and 1, if we define

$$\mathcal{L}_N(p) = \int_q^{aq^{N+1}} \frac{(a^{-1}q^{-N}x; q)_\infty}{(bq^N x; q)_\infty} p(x) d_q x,$$

then, by theorem 2.6, the orthogonality conditions for  $n \neq m$ ,

$$\langle p_n(\bullet; a, b, q^{-N}; q), p_m(\bullet; a, b, q^{-N}; q) \rangle = \mathcal{L}_0(p_n p_m) + \mathcal{L}_N(\mathcal{D}_{q^{-1}}^N(p_n) \mathcal{D}_{q^{-1}}^N(p_m)) = 0,$$

determine uniquely the big  $q$ -Jacobi polynomials for all non-negative integer degrees up to a constant factor.

4.1.2.  $a = q^{-N}$ ,  $b, c, abc^{-1} \notin \Omega(q) \setminus \{q^{-N}\}$ . By using the identity

$$(4.6) \quad p_n(x; a, b, c; q) = p_n(x; c, abc^{-1}, a; q),$$

which can be obtained easily from the hypergeometric representation (4.1) or from the TTRR (4.2), this case is reduced to subsection 4.1.1.

4.1.3.  $b = q^{-N}$  and  $a, c, abc^{-1} \notin \Omega(q) \setminus \{q^{-N}\}$ . The orthogonality in this case can be obtained taking the limit  $b \rightarrow q^{-N}$ .

Multiplying relation (4.4) by the factor  $(b - q^{-N})$ , taking limit  $b \rightarrow q^{-N}$ , and removing some non-vanishing constants one gets, for  $n \neq m$ ,

$$(4.7) \quad \sum_{s'=0}^{N-1} \frac{(a^{-1}cq^{s'+1}, q^{s'+1}; q)_\infty q^{-N} q^{s'}}{(cq^{s'+1}; q)_\infty (q^{-N+s'+1}; q)_{N-s'-1}} p_n(cq^{s'+1}; a, q^{-N}, c; q) p_m(cq^{s'+1}; a, q^{-N}, c; q) = 0.$$

The others terms of the two series in the series representation for the Jackson  $q$ -integral, vanish after taking the limit since these series converges uniformly for  $b$  in a compact neighborhood of  $q^{-N}$ .

Reversing the summation and using the identity

$$(\alpha; q)_s = (\alpha^{-1}q^{1-s}; q)_s (-\alpha)^s q^{\binom{s}{2}},$$

orthogonality property (4.7) can be rewritten as

$$(4.8) \quad \sum_{s=0}^{N-1} \frac{(ac^{-1}q^{-N+1}, q^{-N+1}; q)_s q^{(N-1)s}}{(c^{-1}q^{-N+1}, q; q)_s} \frac{q^{(N-1)s}}{a^s} p_n(cq^N q^{-s}; a, q^{-N}, c; q) p_m(cq^N q^{-s}; a, q^{-N}, c; q) = 0.$$

Comparing (4.5) and (4.8), we get

$$(4.9) \quad \begin{aligned} p_n(x; a, q^{-N}, c; q) &= c^n q^{nN} h_n^{(ac^{-1}q^{-N}, c)}(c^{-1}q^{-N}x; N-1; q) \\ &= c^n q^{nN} p_n(c^{-1}q^{-N}x; ac^{-1}q^{-N}, c, q^{-N}; q). \end{aligned}$$

The used identities are not valid for several configurations of the parameters, however (4.9) is also valid for these configurations by using analytic continuation. Thus the case treated in this subsection can be reduced to the case considered in subsection §4.1.1 by setting  $x \mapsto c^{-1}q^{-N}x$ .



It is curious that identity (4.9) has the hypergeometric form

$$\begin{aligned} & {}_3\varphi_2 \left( \begin{matrix} q^{-n}, aq^{n-N+1}, x \\ aq, cq \end{matrix} \middle| q; q \right) \\ &= \frac{c^n q^{nN} (ac^{-1}q^{-N+1}, q^{-N+1}; q)_n} {(aq, cq; q)_n} {}_3\varphi_2 \left( \begin{matrix} q^{-n}, aq^{n-N+1}, c^{-1}q^{-N}x \\ ac^{-1}q^{-N+1}, q^{-N+1} \end{matrix} \middle| q; q \right), \end{aligned}$$

which coincides with [20, (3.2.6)]

$${}_3\varphi_2 \left( \begin{matrix} q^{-n}, \widehat{a}q^n, \widehat{b} \\ \widehat{d}, \widehat{e} \end{matrix} \middle| q; \frac{\widehat{d}\widehat{e}}{\widehat{a}\widehat{b}} \right) = \frac{(\widehat{a}q/\widehat{d}, \widehat{a}q/\widehat{e}; q)_n} {(\widehat{d}, \widehat{e}; q)_n} \left( \frac{\widehat{d}\widehat{e}}{\widehat{a}q} \right)^n {}_3\varphi_2 \left( \begin{matrix} q^{-n}, \widehat{a}q^n, \widehat{a}\widehat{b}q/\widehat{d}\widehat{e} \\ \widehat{a}q/\widehat{d}, \widehat{a}q/\widehat{e} \end{matrix} \middle| q; \frac{q}{\widehat{b}} \right)$$

in the parameters but it does not in the arguments if one sets  $\widehat{a} = aq^{-N+1}$ ,  $\widehat{b} = x$ ,  $\widehat{d} = aq$ , and  $\widehat{e} = cq$ .

4.1.4.  $abc^{-1} = q^{-N}$  and  $a, b, c \notin \Omega(q) \setminus \{q^{-N}\}$ . Once again, by (4.6), this case can be reduced to the case in subsection §4.1.3.

4.2. **The orthogonality conditions for  $|q| \geq 1$ .** Identities (3.6) and (3.2.2) in [20]

$${}_3\varphi_2 \left( \begin{matrix} q^{-n}, a, b \\ d, e \end{matrix} \middle| q; q \right) = \frac{(e/a; q)_n}{(e; q)_n} a^n {}_3\varphi_2 \left( \begin{matrix} q^{-n}, a, d/b \\ d, aq^{1-n}/e \end{matrix} \middle| q; \frac{bq}{e} \right),$$

yield

$$\begin{aligned} & {}_3\varphi_2 \left( \begin{matrix} q^{-n}, abq^{n+1}, x \\ aq, cq \end{matrix} \middle| q; q \right) \\ &= \frac{(c(abq)^{-1}; q^{-1})_n}{(cq^n; q^{-1})_n} (ab)^n q^{n^2+n} {}_3\varphi_2 \left( \begin{matrix} q^n, (ab)^{-1}q^{-n-1}, (aq)^{-1}x \\ (aq)^{-1}, c(abq)^{-1} \end{matrix} \middle| q^{-1}; q^{-1} \right). \end{aligned}$$

which in terms of big  $q$ -Jacobi polynomials writes as

$$p_n(x; a, b, c; q) = \frac{1}{(a^{-1}q^{-1})^n} p_n(a^{-1}q^{-1}x; a^{-1}, b^{-1}, ca^{-1}b^{-1}; q^{-1}).$$

Hence the orthogonality conditions for big  $q$ -Jacobi polynomials with  $|q| > 1$  follow from section 4.1.

If  $q$  is a primitive root of unity, i.e.  $q = e^{2\pi i M/N}$  with  $\gcd(N, M) = 1$  then  $\{kN : k \in \mathbb{N}\} \subseteq \Lambda$ , and as we did for the Askey-Wilson polynomials, the set of big  $q$ -Jacobi polynomials  $(p_n(x; a, b, c; q))_{n=0}^N$  under the assumptions

$$a, b, c, ab, abc^{-1} \neq q^k, \quad k = 0, \dots, N-1,$$

are uniquely determined by the orthogonality conditions

$$\mathcal{L}_0(p_n p_m) = d_n^2 \delta_{n,m}, \quad d_n^2 \neq 0,$$

being

$$\mathcal{L}_0(p) = \sum_{j=0}^{N-1} \omega_0 \frac{(ra^{-1}q^j, rc^{-1}q^j; q)_\infty}{(rq^j, rbc^{-1}q^j; q)_\infty} q^s p(rq^j),$$

with initial condition  $\mathcal{L}_0(1) = 1$ , and  $r$  the root with minimal argument of the equation

$$r^N = \frac{a^N + c^N - (ab)^N - (ac)^N}{1 - (ab)^N}.$$

Moreover, since for  $n \geq N$

$$\mathcal{D}_q^N p_n(x; a, b, c; q) = \frac{(q^{n-N+1}; q)_N}{(1-q)^N} p_{n-N}(x; a, b, c; q),$$

the orthogonality conditions that characterizes big  $q$ -Jacobi polynomials in such case are

$$\langle p_n, p_m \rangle = \sum_{j=0}^{\infty} \mathcal{L}_0(\mathcal{D}_q^{Nj}(p_n) \mathcal{D}_q^{Nj}(p_m)).$$

*Remark 4.2.* In [32] the particular case  $c = 1$  is considered and, in such a case, they got

$$\omega_s = \frac{(1-a^N)(1-abq)(b; q)_s}{aq(b-1)(1-a^N b^N)(a^{-1}; q)_s} q^s.$$

## 5. EXTENDING ORTHOGONALITY PROPERTIES VALID UP TO DEGREE $N$

The aims of this section are for one side to present the factorization for those  $q$ -polynomials for which there exists an  $N$  such that  $\gamma_N = 0$ , and hence an orthogonality until degree  $N$  takes place, and for the other we extend that orthogonality properties for all non-negative degrees obtaining a Sobolev type orthogonality properties.

Taking into account the basic idea about how the factorization process works is already known (see e.g. [16]) we only show the sketch regarding the factorization for the  $q$ -polynomials.

Since  $q$ -polynomials fulfill, for  $n \geq 0$ , the TTRR

$$p_{n+1}(x) = (x - \beta_n)p_n(x) - \gamma_n p_{n-1}(x),$$

with  $p_{-1} \equiv 0$ ,  $p_0(x) = 1$ , observe that if there exists some integer  $N > 0$  so that  $\gamma_N = 0$ , then it is straightforward to check that, for  $n \geq N$ , the following relation holds:

$$(5.1) \quad p_n = p_N p_{n-N}^{(N)},$$

where  $(p_n^{(N)})$  is the family of  $N$ th associated polynomials which fulfills, for  $n \geq 0$ , the recurrence relation:

$$p_{n+1}^{(N)}(x) = (x - \beta_{n+N})p_n^{(N)}(x) - \gamma_{n+N}p_{n-1}^{(N)}(x),$$

with initial conditions  $p_{-1}^{(N)}(x) \equiv 0$ ,  $p_0^{(N)}(x) = 1$ .

Notice that in the case of  $q$ -polynomials the existence of an integer  $N$  so that  $\gamma_N = 0$  is directly related with the fact that there is a term of the form  $q^{-N+1}$  in the denominator parameters of one of the hypergeometric representations (see (3.1) and (4.2)). In such a case the hypergeometric function  ${}_p\varphi_{p-1}$  with a suitable normalization factorizes as follows:

Let  $a = \{a_1, \dots, a_{p-1}\}$  and  $b = \{b_1, \dots, b_{p-2}\}$ , then

$$\begin{aligned} & (q^{-N+1}; q)_{n+N} p^{\varphi_{p-1}} \left( \begin{matrix} q^{-n-N}, a \\ q^{-N+1}, b \end{matrix} \middle| q; z \right) \\ &= \frac{(q^{-n-N}; q)_N (a; q)_N z^N}{(b; q)_N (q; q)_N} \sum_{k=0}^n \frac{(q^{-n}; q)_k (aq^N; q)_k z^k}{(bq^N; q)_k (q^{N+1}; q)_k} \frac{(q; q)_n}{(q; q)_k} \\ &= \frac{(q^{N+1}; q)_n (a; q)_N z^N}{(b; q)_N} (-1)^N q^{N(-n-N)+N(N-1)/2} p^{\varphi_{p-1}} \left( \begin{matrix} q^{-n}, aq^N \\ q^{N+1}, bq^N \end{matrix} \middle| q; z \right). \end{aligned}$$

Hence it is straightforward combining to obtain the following factorization:

$$(5.2) \quad \begin{aligned} (q^{-N+1}; q)_{n+N} p^{\varphi_{p-1}} \left( \begin{matrix} q^{-n-N}, a \\ q^{-N+1}, b \end{matrix} \middle| q; z \right) &= q^{-nN} (q^{N+1}; q)_n (q^{-N+1}; q)_N \\ &\quad \times p^{\varphi_{p-1}} \left( \begin{matrix} q^{-N}, a \\ q^{-N+1}, b \end{matrix} \middle| q; z \right) p^{\varphi_{p-1}} \left( \begin{matrix} q^{-n}, aq^N \\ q^{N+1}, bq^N \end{matrix} \middle| q; z \right). \end{aligned}$$

Notice that the first hypergeometric function of the right-hand side of (5.2), with its corresponding normalization coefficient, is the polynomial of degree  $N$  and the second one is the  $N$ th associated polynomial in the factorization (5.1) for  $n \rightarrow n+N$ . Table 3 shows the  $N$ th associated polynomial.

In the sequel we are going to assume that no element of  $b$  belongs to  $\Omega(q)$ , therefore theorem 2.6 is applicable in such a case. Let us go on to describe how to obtain functionals  $\mathcal{L}_0, \mathcal{L}_N$  and the linear operator  $\mathcal{F}^{(N)} = \mathcal{F}^N$  in (2.4).

Obviously  $\mathcal{L}_0(p) \stackrel{\text{def}}{=} \langle \mathbf{u}, p \rangle$  where  $\mathbf{u}$  is the linear form with respect to the corresponding family of  $q$ -polynomials  $(p_n)_{n=1}^N$  is orthogonal. Moreover, due the difference properties of such families  $\mathcal{F}$  is going to be a difference operator and  $\mathcal{L}_N(p) \stackrel{\text{def}}{=} \langle \mathbf{v}, p \rangle$  where  $\mathbf{v}$  is the linear form with respect to the polynomial sequence  $(\mathcal{F}^{(N)} p_{n+N})$  is orthogonal [17].

Let us describe briefly the most complicated case: the  $q$ -Racah polynomials.

Notice that setting  $\alpha = q^{-N}$  then the  $N$ th  $\gamma$ 's coefficient for  $q$ -Racah polynomial vanishes [21, (3.2.3)], i.e.  $\gamma_N = 0$ , and therefore we can apply theorem 2.6, obtaining:

$$r_n^{(N)}(x; q^N, \beta, \gamma, \delta; q) = p_n(\mu(x)/(2\sqrt{\gamma\delta q}); q^N \sqrt{\gamma\delta q}, \sqrt{q/\gamma\delta}, \beta\sqrt{\delta q/\gamma}, \sqrt{\gamma q/\delta}; q),$$

Moreover, taking into account that for these polynomials

$$\frac{\Delta}{\Delta\mu(x)} r_n(\mu(x); \alpha, \beta, \gamma, \delta|q) = \frac{q^{-n} - 1}{q^{-1} - 1} r_{n-1}(\mu(x); \alpha q \beta q, \gamma q, \delta|q),$$

and their connection with the Askey-Wilson polynomials (see table 5) it is clear that the operator  $\mathcal{F} = \Delta/\Delta\mu(x)$  for which we obtain that  $\mathcal{F}^N(r_n(x; \alpha, \beta, \gamma, \delta; q))$  is, up to a constant, equal to

$$p_n(\mu(x)/(2\sqrt{\gamma\delta q^{N+1}}); \sqrt{\gamma\delta q^{N+1}}, \sqrt{q/\gamma\delta q^N}, \beta\sqrt{\delta q^{N+1}/\gamma}, \sqrt{\gamma q^{N+1}/\delta}; q).$$

Thus the linear functional  $\mathbf{v}$  is related with the linear operator of Askey-Wilson polynomials with parameters  $\sqrt{\gamma\delta q^{N+1}}, \sqrt{q/\gamma\delta q^N}, \beta\sqrt{\delta q^{N+1}/\gamma}$ , and  $\sqrt{\gamma q^{N+1}/\delta}$ .

Family and condition		Nth associated polynomial	
qH	$N \mapsto N - 1$	bqJ	$p_n(xq^N; \alpha q^N, \beta q^N, q^N; q)$
dqH	$N \mapsto N - 1$	cdqH	$p_n(\mu(x)/(2\sqrt{\gamma\delta q}); q^N \sqrt{\gamma\delta q}, \sqrt{q/\gamma\delta}, \sqrt{\gamma q/\delta} q)$
qK	$N \mapsto N - 1$	bqJ	$p_n(xq^N; q^N, -pq^{N-1}, 0; q)$
QqK	$N \mapsto N - 1$	qM	$m_n(xq^N; q^N, -p^{-1}q^{-N}; q)$
AqK	$N \mapsto N - 1$	bqL	$p_n(xq^N; pq^N, q^N; q)$
dqK	$N \mapsto N - 1$	cdqH	$p_n(\lambda(x)/(2\sqrt{cq^{1-N}}); \sqrt{cq^{N+1}}, \sqrt{q^{N+1}/c}, 0 q)$

TABLE 3. Nth associated polynomials involved in the factorization (5.1)

$p_n$	$\mathcal{F}$	$\mathcal{F}^N(p_{n+N})$
qH	$\frac{\Delta}{\Delta q^{-x}}$	$p_n(x; \alpha q^N, \beta q^N, 1; q)$
dqH	$\frac{\Delta}{\Delta \mu(x)}$	$p_n(\mu(x)/(2\sqrt{\gamma\delta q^{N+1}}); \sqrt{\gamma\delta q^{N+1}}, \sqrt{q/\gamma\delta q^N}, \sqrt{\gamma q^{N+1}/\delta} q)$
qK	$\frac{\Delta}{\Delta q^{-x}}$	$p_n(x; 1, -pq^{2N-1}, 0; q)$
QqK	$\frac{\Delta}{\Delta q^{-x}}$	$m_n(x; 1, -p^{-1}q^{-N}; q)$
AqK	$\frac{\Delta}{\Delta q^{-x}}$	$p_n(x; pq^N, 1; q)$
dqK	$\frac{\Delta}{\Delta \lambda(x)}$	$p_n(\lambda(x)/(2\sqrt{cq}); \sqrt{cq}, \sqrt{q/c}, 0 q)$

TABLE 4. Unnormalized  $\mathcal{F}^{(N)}(p_{n+N})$  involved in factorization (5.1)

qR $\rightarrow$ AW	$r_n(x; \alpha, \beta, \gamma, \delta; q)$	$p_n\left(\frac{x}{2\sqrt{\gamma\delta q}}; \sqrt{\gamma\delta q}, \alpha\sqrt{\frac{q}{\gamma\delta}}, \beta\sqrt{\frac{\delta q}{\gamma}}, \sqrt{\frac{\gamma q}{\delta}}; q\right)$
AW $\rightarrow$ qR	$p_n(x; a, b, c, d; q)$	$r_n\left(2ax; \frac{ab}{q}, \frac{cd}{q}, \frac{ad}{q}, \frac{a}{d}; q\right)$
bqJ $\rightarrow$ qH	$p_n(x; a, b, c; q)$	$h_n(x; a, b, -1 - \log_q c; q)$
qH $\rightarrow$ bqJ	$h_n(x; a, b, N; q)$	$p_n(x; a, b, q^{-N-1}; q)$
dqH $\rightarrow$ cdqH	$r_n(x; \gamma, \delta, N; q)$	$p_n\left(\frac{x}{2\sqrt{\gamma\delta q}}; \sqrt{\gamma\delta q}, \sqrt{\frac{\gamma q}{\delta}}, \frac{1}{q^N \sqrt{\gamma\delta q}} q\right)$
cdqH $\rightarrow$ dqH	$p_n(x; a, b, c q)$	$r_n\left(2ax; \frac{ab}{q}, \frac{a}{b}, -\log_q(ac); q\right)$
QqK $\rightarrow$ qM	$k_n^{qtm}(x; p, N; q)$	$m_n\left(x; q^{-N-1}, -\frac{1}{p}; q\right)$
qM $\rightarrow$ QqK	$m_n(x; b, c; q)$	$k_n^{qtm}\left(x; -1 - \log_q b, -\frac{1}{c}; q\right)$
QqK $\rightarrow$ AqK	$k_n^{qtm}(x; p, N; q)$	$k_n^{aff}(xq^N; p^{-1}, N; q^{-1})$
AqK $\rightarrow$ QqK	$k_n^{aff}(x; p, N; q)$	$k_n^{qtm}(xq^{-N}; p^{-1}, N; q^{-1})$
qK $\rightarrow$ lqJ	$k_n(x; p, N; q)$	$p_n(xq^N; -pq^N, q^{-N-1}; q)$
lqJ $\rightarrow$ qK	$p_n(x; a, b; q)$	$k_n(bqx; -abq, -1 - \log_q b; q)$
AqK $\rightarrow$ bqL	$k_n^{aff}(x; p, N; q)$	$p_n(x; p, q^{-N-1}; q)$
bqL $\rightarrow$ AqK	$p_n(x; a, b; q)$	$k_n^{aff}(x; a, -1 - \log_q N; q)$
lqJ $\rightarrow$ bqJ	$p_n(x; a, b; q)$	$p_n(bqx; b, a, 0; q)$
qK $\rightarrow$ bqJ	$k_n(x; p, N; q)$	$p_n(x; q^{-N-1}, -pq^N, 0; q)$

TABLE 5. Some unnormalized identities between  $q$ -polynomials.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, SANTA BARBARA, CALIFORNIA 93106, US

*E-mail address:* [rscosa@gmail.com](mailto:rscosa@gmail.com)

DEPARTAMENTO DE MATEMÁTICA APLICADA, FACULTAD DE CC. ECONÓMICAS Y EMPRESARIALES, UNIVERSIDAD DE GRANADA, CAMPUS DE LA CARTUJA, s/n. 18071 GRANADA, SPAIN

*E-mail address:* [jslara@ugr.es](mailto:jslara@ugr.es)