Applied Mathematical Sciences, Vol. 5, 2011, no. 64, 3173 - 3181

On the Pell, Pell-Lucas and Modified Pell Numbers By Matrix Method

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Abstract

In this paper, we consider the usual Pell, Pell-Lucas and Modified Pell sequences, and we define some new matrices, which are based on Modified Pell and Pell-Lucas numbers, with determinants equal to 2^n or 2^{3n}.This unique properties provide very important formulas like *the Simpson Formula* of Modified Pell and Pell-Lucas numbers. Also we show that the Modified Pell and Pell-Lucas numbers have *the generating matrices*.Further, we present some elementary identities between them by these matrices.

Mathematics Subject Classification: 11B37; 15A15

Keywords: Pell Number, Modified Pell Number, Pell-Lucas Number, Generating Matrix, Simpson Formula

1 Introduction

In the literature, especially in mathematics and physics, there are a lot of integer sequences, which are used in almost every field modern sciences. Admittedly, the Fibonacci sequence is one of the most famous and curious numerical sequences in mathematics and have been widely studied from both algebraic and combinatorial prospectives.Also, there is the Pell sequence, which is as important as the Fibonacci sequence.The Pell sequence $\{P_n\}$ are defined by recurrence $P_n = 2P_{n-1} + P_{n-2}$, $n \geq 2$ with $P_0 = 0$ and $P_1 = 1$ and the Pell-Lucas sequence $\{Q_n\}$ and Modified Pell sequence $\{q_n\}$ by the same recurrence but with initial conditions $Q_0 = Q_1 = 2$ and $q_0 = 1$ and $q_1 = 1$, respectively.Infact,

$$Q_n = 2q_n. \tag{1}$$

Consequently, the known properties of $\{Q_n\}$ can be easily written for $\{q_n\}$. Thereby, a study of $\{q_n\}$ involves inevitably familiarity with $\{Q_n\}$.Further details about Pell, Pell-Lucas and Modified Pell sequences can be found in [3, 5, 9, 13].For instance, in [3], Horadam gave some results as follows:

$$q_n = P_{n+1} - P_n, \tag{2}$$

$$q_{n+1} = P_{n+1} + P_n \tag{3}$$

and

$$P_{n+1} = \frac{q_{n+1} + q_n}{2}. \tag{4}$$

Explicit Binet forms for $\{P_n\}$, $\{Q_n\}$ and $\{q_n\}$ are

$$P_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \tag{5}$$

$$Q_n = \alpha^n + \beta^n, \tag{6}$$

and

$$q_n = \frac{\alpha^n + \beta^n}{\alpha + \beta} \tag{7}$$

where α and β are the roots of the characteristic equation $x^2 - 2x - 1 = 0$.[1, 11]

The some of elements of the Pell, Pell-Lucas and Modified Pell sequences are given by the following table;

n	0	1	2	3	4	5	6	7	8	9	$\cdots$
P_n	0	1	2	5	12	29	70	169	408	985	$\cdots$
Q_n	2	2	6	14	34	82	198	478	1154	2786	$\cdots$
q_n	1	1	3	7	17	41	99	239	577	1393	$\cdots$

Table 1. The values of $\{P_n\}$, $\{Q_n\}$ and $\{q_n\}$.

In the last decades, some mathematicians have studied to find miscellaneous affinities between the matrices and linear recurrences.They want to obtain always the terms of linear recurrences by matrices, which are said to be *the Generating Matrices*.Comparable matrix generators have been considered previously for the Fibonacci and Pell sequences[7, 8, 12]. In [12], the following property of the nth power of the Q-matrix is found

$$Q^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}, \tag{8}$$

where $Q = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.Similarly, in [8] author showed that the members of the Pell sequence can be derived by a matrix representation:

$$M^n = \begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix}, \tag{9}$$

where $M = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$.

The matrix method has also played an important and effective role stemming from Number Theory.As an example of the usage of the matrix approach, we can exemplify to obtain *the Simpson Formula* for the Pell numbers[1], namely,

$$P_{n+1}P_{n-1} - P_n^2 = (-1)^n, \tag{10}$$

which may, of course, be established by means of the Binet form (5).

The permanent of an n-square matrix $A = (a_{ij})$ is defined by

$$per A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}, \tag{11}$$

where the summation extends over all permutations σ of the symmetric group S_n.The most important applications of permanents are in the areas of physics and chemistry.One can find more applications of permanents in [11].The permanent of a matrix is analogous to the determinant, where all of the signs used in the Laplace expansion of minors are positive.

The purpose of this article is to urge a greater use of the properties between the matrices and linear recurrences composed of the Pell, Pell-Lucas and Modified Pell sequences.Emphasis in this paper will be considered a few square 2×2 matrices.Then we investigate the identities of these matrices.Even, we shall introduce different matrices to obtain new results.Also we will show that the identities presented before can be produce by the aid of them.

2 The Pell and Modified Pell Numbers

In this section, we introduce a new matrix whose rows are [3,1] and [1,1]. Clearly,

$$N = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}. \tag{12}$$

Considering the matrix N and the equations (2-3), we can easily see

$$\begin{bmatrix} q_{n+2} \\ q_{n+1} \end{bmatrix} = N \begin{bmatrix} P_{n+1} \\ P_n \end{bmatrix} \ or \begin{bmatrix} P_{n+2} \\ P_{n+1} \end{bmatrix} = \frac{1}{2} N \begin{bmatrix} q_{n+1} \\ q_n \end{bmatrix} \tag{13}$$

and

$$\begin{cases} P_n = \frac{1}{2^{\frac{n}{2}}} \begin{bmatrix} 1 & 0 \end{bmatrix} N^n \begin{bmatrix} 0 \\ 1 \end{bmatrix}, & if\, n\, even, \\ q_n = \frac{1}{2^{\frac{n-1}{2}}} \begin{bmatrix} 1 & 0 \end{bmatrix} N^n \begin{bmatrix} 0 \\ 1 \end{bmatrix}, & if\, n\, odd. \end{cases} \tag{14}$$

We also define a matrix E by

$$E=\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}. \tag{15}$$

Moreover from the equations (2-4), we can feasible write

$$\begin{bmatrix}q_{n+1}\\ q_n\end{bmatrix}=E\begin{bmatrix}P_{n+1}\\ P_n\end{bmatrix}\ and \begin{bmatrix}P_{n+1}\\ P_n\end{bmatrix}=\frac{1}{2}E\begin{bmatrix}q_{n+1}\\ q_n\end{bmatrix}. \tag{16}$$

Note that the matrix E acts as the transformer (or reflector) mission between the Pell and Modified Pell numbers.Furher there is a very interesting property of the matrix E. Its powers are consecutively equal to multiplication of itself by $2^{\frac{n-1}{2}}$ and multiplication of the identity matrix by $2^{\frac{n}{2}}$. Clearly,

$$E^n=\begin{cases}2^{\frac{n-1}{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} & if\, n\ odd,\\ \\ 2^{\frac{n}{2}}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} & if\ n\ even.\end{cases} \tag{17}$$

First we start with the following theorem in connection with the matrix N.

Theorem 2.1 *Let the N be a matrix as in (12). For all positive integers n and m, the following determinantal equalities are held:*

i. $N^n=\begin{cases}2^{\frac{n}{2}}\begin{bmatrix}P_{n+1} & P_n\\ P_n & P_{n-1}\end{bmatrix} & if\, n\, even,\\ \\ 2^{\frac{n-1}{2}}\begin{bmatrix}q_{n+1} & q_n\\ q_n & q_{n-1}\end{bmatrix} & if\, n\, odd,\end{cases}$

ii. $\det(N^n)=2^n$,

iii. $per\,(N^n)=2^{n-1}(q_{2n}+1)$,

iv. $N^{n+m}=N^nN^m$,

v. $N^{n-m}=N^nN^{-m}$,

Proof.

(i)(Induction on n) We will use the induction method on odd n and even n, separately.Let n be an odd integer.The case for $n=1$ is clear.Now we suppose that it is correct for odd n integer.Then we show that it is correct for $n+2$.By the definition of Modified Pell sequence, we can write

$$N^{n+2}=N^nN^2=2^{\frac{n+1}{2}}\begin{bmatrix}q_{n+3} & q_{n+2}\\ q_{n+2} & q_{n+1}\end{bmatrix}. \tag{18}$$

Thus, this part of the assertion for all odd n integers is to be true.Now let n be an even integer.Then the case for $n = 2$ is clear.We assume that it is correct for even n.Now we show that it is to be true for $n + 2$.Then, by the definition of Pell sequence, we compute

$$N^{n+2} = N^n N^2 = 2^{\frac{n+2}{2}} \begin{bmatrix} P_{n+3} & P_{n+2} \\ P_{n+2} & P_{n+1} \end{bmatrix} \tag{19}$$

(ii) (Induction on n) When $n = 1$, (ii) is seen to be true.We assume that it is to be true for n.Then we show that the equation holds for $n + 1$.

$$\det\left(N^{n+1}\right) = (\det N^n)\det(N) = 2^n 2 = 2^{n+1}, \tag{20}$$

and the proof of (ii) is completed.

(iii) Using the Binet formulæ of the Pell and Modified Pell sequences and from F, the proof of (ii) can be easily seen.

$(iv - v)$ If we use a property of matrix multiplication, the proof of (iii) and (iv) are readily seen. ■

We now zoom in the properties of the matrix N. Its characteristic equation is

$$x^2 - 4x + 2 = 0 \tag{21}$$

and the eigenvalues of the nth power of N are

$$(\lambda, \mu) = \begin{cases} 2^{\frac{n}{2}}\left(q_n + P_n\sqrt{2}, q_n - P_n\sqrt{2}\right), & if\, n\, even, \\ \\ 2^{\frac{n-1}{2}}\left(q_n + P_n\sqrt{2}, q_n - P_n\sqrt{2}\right), & if\, n\, odd. \end{cases} \tag{22}$$

To put it more simply, we write

$$(\lambda, \mu) = \begin{cases} 2^{\frac{n}{2}}\left(\alpha^n, \beta^n\right), & if\, n\, even, \\ \\ 2^{\frac{n-1}{2}}\left(\alpha^n, \beta^n\right), & if\, n\, odd, \end{cases} \tag{23}$$

where α and β are the roots of the characteristic equation $x^2 - 2x - 1 = 0$ (so-called Pell equation).

Since $\det N \neq 0$, it is a invertible matrix. Thus, we can easily obtained

$$N^{-n} = \begin{cases} \frac{1}{2^{\frac{n}{2}}} \begin{bmatrix} P_{n-1} & -P_n \\ -P_n & P_{n+1} \end{bmatrix} & if\ n\ even, \\ \\ \frac{1}{2^{\frac{n+1}{2}}} \begin{bmatrix} q_{n-1} & -q_n \\ -q_n & q_{n+1} \end{bmatrix} & if\ n\ odd. \end{cases} \tag{24}$$

As an example of the usage of elementary identities of matrices, we can write the following theorem.

Theorem 2.2 *For all positive integers n and m, the following determinantal equalities are held:*

i. $P_{n+1}P_{n-1} - P_n^2 = (-1)^n$,

ii. $P_{n+1}P_{n-1} + P_n^2 = \frac{1}{2}\left(q_{2n} + (-1)^n\right)$,

iii. $q_{n+1}q_{n-1} - q_n^2 = 2(-1)^{n+1}$,

iv. $q_{n+1}q_{n-1} + q_n^2 = q_{2n} + (-1)^{n+1}$,

v. $P_{m+n} = P_m P_{n+1} + P_{m-1}P_n$,

vi. $2P_{m+n} = q_m q_{n+1} + q_{m-1}q_n$,

vii. $q_{m+n} = P_m q_{n+1} + P_{m-1}q_n$,

viii. $(-1)^{m+1}P_{n-m} = P_n q_{m-1} - q_{n-1}P_m$,

ix. $(-1)^{m+1}q_{n-m} = P_n q_{m+1} - P_{n+1}q_m$,

x. $2(-1)^{m+1}P_{n-m} = q_n q_{m+1} - q_{n+1}q_m$,

xi. $P_{2n} = 2P_n q_n$.

Proof. The proof of theorem is easily shown from Theorem 2.1.iv-v. ■

Note that the matrix N gives unlimited opportunities for us because it allows to get an infinite numbers of fundamental identities involved the Pell and Modified Pell numbers between themselves. We can also conclude that Theorem 2.2.*i-iii* are *the Simpson Formula* for the Pell and Modified Pell numbers, respectively. But, they are found by Horadam beforehand, in [1, 3]. Here we only show that they can be obtained with the more different way from method used in the papers of Horadam.

Theorem 2.3 *Let E be a matrix as in (15). Then*

$$EM^n = \begin{bmatrix} q_{n+1} & q_n \\ q_n & q_{n-1} \end{bmatrix} \text{ and } E\begin{bmatrix} q_{n+1} & q_n \\ q_n & q_{n-1} \end{bmatrix} = 2M^n \tag{25}$$

where the matrix M is defined as before.

Proof. By the equations (2-3), we can easily see that these relationships between the matrices E and M.Thus, the proof is completed. ■

Now we define a new matrix as follows:

$$L = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}. \tag{26}$$

In mathematics, an involutary matrix is a matrix that is its own inverse.That is, matrix A is an involution if $A^2 = \mathrm{I}$. One of the tree classes of elementary matrix, namely the row-interchange matrix, that which represents multiplication of a row or column by -1, is also involutory, it is in fact a trivial example of a signature matrix, all of which are involutory. Involutary matrices are all square roots of the identity matrix. This is simply a consequence of the fact that any nonsingular matrix multiplied by its inverse is the identity.

Considering the above information, we can write that the matrix L is a involutory matrix. Thus, we can write

$$L^2 = \mathrm{I} \tag{27}$$

Let us define a 2×2 matrix H_n as follows:

$$H_n = \begin{bmatrix} P_{n+1} & q_n \\ P_n & q_{n-1} \end{bmatrix}. \tag{28}$$

Then we have the following theorem.

Theorem 2.4 *Let L and H_n be respectively as in (26) and (28). Then*

$$M^n = H_n \cdot L \ \ and\ H_n = M^n \cdot L \tag{29}$$

where M is defined as before.

Proof. Considering the equation 2, the proof of the first equation is easily seen. Also, from the fact that L is an involutory matrix, the proof of the second equation is easily seen.Thus, the proof is completed. ■

We can in fact extent the above theorem, namely for $n, m \in Z^+$

$$H_{n+m} = M_n \cdot H_m. \tag{30}$$

Its proof can easily be seen by the basic matrix operations.

3 The Pell and Pell-Lucas Numbers

In this section we will consider on the findings of Section 2.Then we will predispose them to the Pell-Lucas numbers. First recall that aforementioned equation[3]:

$$Q_n = 2q_n. \tag{31}$$

So the known properties of the Modified Pell numbers all are easily transferable to Pell-Lucas numbers.Then we can rewrite the above everything.To do this, we shall define two new matrices.Define the 2×2 matrices $R\ and\ F$ as follows:

$$R = \begin{bmatrix} 6 & 2 \\ 2 & 2 \end{bmatrix} \tag{32}$$

and

$$F=\begin{bmatrix} 2 & 2 \\ 2 & -2 \end{bmatrix}. \tag{33}$$

Frankly speaking, these matrices can be obtained with multiplication of matrices N *and* E by 2, respectively.

The proof of the next theorem is analogous to the proof of Theorem (2.1), so it will be omitted.

Theorem 3.1 *Let the R be a matrix as in (32). For all positive integers n and m, the following determinantal equalities are held:*

i. $R^n=\begin{cases} 2^{\frac{3}{2}n}\begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix} & if\ n\ even, \\ \\ 2^{\frac{3}{2}(n-1)}\begin{bmatrix} Q_{n+1} & Q_n \\ Q_n & Q_{n-1} \end{bmatrix} & if\ n\ odd, \end{cases}$

ii. $\det(R^n)=2^{3n}$,

iii. $per(R^n)=2^{3n-2}(Q_{2n}+2)$,

iv. $R^{n+m}=R^nR^m$,

v. $R^{n-m}=R^nR^{-m}$.

Similarly, employing the formula (31) or using the matrices R and F, we can write as follows:

$$Q_{n+1}Q_{n-1}-Q_n^2=8(-1)^{n+1}, \tag{34}$$

$$Q_{n+1}Q_{n-1}+Q_n^2=2(Q_{2n}+2(-1)^{n+1}), \tag{35}$$

$$8P_{m+n}=Q_mQ_{n+1}+Q_{m-1}Q_n, \tag{36}$$

$$Q_{m+n}=P_mQ_{n+1}+P_{m-1}Q_n, \tag{37}$$

$$2(-1)^{m+1}P_{n-m}=P_nQ_{m-1}-Q_{n-1}P_m, \tag{38}$$

$$(-1)^{m+1}Q_{n-m}=P_nQ_{m+1}-P_{n+1}Q_m, \tag{39}$$

and

$$8(-1)^{m+1}P_{n-m}=Q_nQ_{m+1}-Q_{n+1}Q_m \tag{40}$$

It is well-known that the equation (34) was found by Horadam in [2] beforehand. However, we show solely that they can be obtained by using R-matrix.

Finally, we write relationships between matrices M^n and F as follows:

$$FM^n=\begin{bmatrix} Q_{n+1} & Q_n \\ Q_n & Q_{n-1} \end{bmatrix} \tag{41}$$

and

$$8M^n = F \begin{bmatrix} Q_{n+1} & Q_n \\ Q_n & Q_{n-1} \end{bmatrix}. \tag{42}$$

ACKNOWLEDGEMENTS. The author would like to thank the referees for constructive comments, which improved the presentation of the paper.

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Received: June 30, 2011