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Generalized polynomials and associated operational identities

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Abstract

We use operational identities to introduce multivariable Laguerre polynomials. We explore the wealth of differential equations they satisfy. We analyze their properties and the link with Legendre-type polynomials. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

The use of operational methods of the Lie type in the theory of ordinary [8] and generalized [5] special functions, offers a powerful tool to treat the relevant generating functions and the differential equations they satisfy.

In the case of multivariable generalized special functions, the use of operational techniques, combined with the principle of monomiality [4,5] has provided new means of analysis for the derivation of the solution of large classes of partial differential equations often encountered in physical problems [3].

The two variable Laguerre polynomials [6]

$$L_n(x, y) = n! \sum_{k=0}^{n} \frac{(-1)^k y^{n-k} x^k}{(n-k)!(k!)^2} \tag{1}$$

offer an efficient example to illustrate the above concepts.

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They are indeed quasi-monomials (q.m.) under the action of the operators [1]

$$\hat{M} = y - \hat{D}_x^{-1}, \qquad \hat{P} = -\hat{D}_x x \hat{D}_x; \tag{2a}$$

in fact,

$$\hat{M} L_n(x,y) = L_{n+1}(x,y), \qquad \hat{P} L_n(x,y) = n L_{n-1}(x,y). \tag{2b}$$

In addition, since

$$\frac{\partial}{\partial y} L_n(x,y) = n L_{n-1}(x,y), \tag{2c}$$

it follows that the $L_n(x,y)$ are the natural solution of

$$\frac{\partial}{\partial y} L_n(x,y) = -\hat{D}_x x \hat{D}_x L_n(x,y) \tag{3}$$

which is a kind of heat diffusion equation. By setting indeed $x = \rho^2$, the differential operator on the r.h.s. of Eq. (3) becomes the transverse Laplacian in cylindrical coordinates. The identities given in Eq. (2) can be further handled to get the new relations

$$\begin{aligned} &\left[yx \frac{\partial^2}{\partial x^2} - (x-y)\frac{\partial}{\partial x} + n \right] L_n(x,y) = 0, \\ &\left[y \frac{\partial^2}{\partial x \partial y} - \left(\frac{\partial}{\partial y} + n \frac{\partial}{\partial x} \right) \right] L_n(x,y) = 0. \end{aligned} \tag{4}$$

The considerations, just developed for Laguerre-type polynomials, have also been exploited to deal with other classes of polynomials, like Hermite and relevant generalizations, and have provided new and unsuspected possibilities to deal with, seemingly unrelated problems from a unified point of view.

In this paper we will extend the method to a new class of Legendre polynomials. The results we will obtain and discuss are a further contribution along the line developed in [3–6]. We will see that this new class of polynomials yield new and interesting possibilities for dealing with a wide class of partial differential equations.

The layout of the paper is as follows. In Section 2 we introduce a new class of polynomials associated with the two-variable Laguerre polynomials and discuss their properties. In Section 3 we discuss the link between Laguerre-type and Legendre polynomials. We devote Section 4 to concluding remarks, with particular reference to the wealth of differential equations satisfied by the various introduced polynomials.

2. A useful generalizations of the two variable Laguerre polynomials

It is well known that, along with Laguerre polynomials one can introduce the associated Laguerre polynomials whose two variable extension has been proposed in Ref. [7] and reads

$$L_n^{(m)}(x,y) = (1 - y\hat{D}_x)^m (y - \hat{D}_x^{-1})^n. \tag{5}$$

[1] Where $\hat{D}_x$ is the derivative and $\hat{D}_x^{-1}$ its inverse.

The properties of this class of polynomials have been discussed in the reference already quoted but for the purposes of the present note is more convenient to introduce the following new set, which will be defined as associated of second kind,

$$ {}_1L_{n,r}(x,y) = \sum_{k=0}^{n} \frac{n!\, y^{n-k} x^{r+k}}{(n-k)!k!(r+k)!}. \tag{6} $$

The reason for the index "1" will be clarified in the following section.

It is not difficult to infer that they satisfy the q.m. properties

$$ \begin{aligned} & x^r \hat{D}_x x^{-r+1} \hat{D}_x\, {}_1L_{n,r}(x,y) = n\, {}_1L_{n-1,r}(x,y), \\ & (y + \hat{D}_x^{-1})\, {}_1L_{n,r}(x,y) = {}_1L_{n+1,r}(x,y) \end{aligned} \tag{7} $$

which can be combined to prove that they satisfy the following differential equation:

$$ \{ yx\hat{D}_x^r + [x + (1-r)y]\hat{D}_x \}\, {}_1L_{n,r}(x,y) = (n+r)\, {}_1L_{n,r}(x,y). \tag{8} $$

Furthermore, since

$$ \frac{\partial}{\partial y}\, {}_1L_{n,r}(x,y) = n\, {}_1L_{n-1,r}(x,y), \tag{9} $$

it is also proved that this new set of polynomials satisfy the p.d.e.

$$ \begin{aligned} & \frac{\partial}{\partial y}\, {}_1L_{n,r}(x,y) = x^r \hat{D}_x x^{-r+1} \hat{D}_x\, {}_1L_{n,r}(x,y), \\ & {}_1L_{n,r}(x,0) = \frac{x^{n+r}}{(n+r)!}. \end{aligned} \tag{10} $$

The generating function can also be derived quite straightforwardly. From the second expression of Eqs. (7) we get indeed

$$ (y + \hat{D}_x^{-1})^n \frac{x^r}{r!} = {}_1L_{n,r}(x,y). \tag{11} $$

By multiplying both sides of (11) by $t^n/n!$ and then by summing up, we find [2]

$$ \sum_{n=0}^{\infty} \frac{t^n}{n!}\, {}_1L_{n,r}(x,y) = \mathrm{e}^{ty} \mathrm{e}^{t\hat{D}_x^{-1}} \frac{x^r}{r!} = \mathrm{e}^{ty} x^r C_r(-xt), \tag{12} $$

where

$$ C_r(x) = \sum_{s=0}^{\infty} \frac{(-1)^s x^s}{s!(r+s)!} \tag{13} $$

is the Tricomi function of the rth order [5]. A similar procedure yields

$$ \begin{aligned} \sum_{n=0}^{\infty} t^n\, {}_1L_{n,r}(x,y) &= \frac{1}{(1-yt)} \frac{1}{1-(t/(1-yt))\hat{D}_x^{-1}} \frac{x^r}{r!} \\ &= \frac{1}{1-yt} \sum_{s=0}^{\infty} \frac{t^s}{(1-yt)^s} \frac{x^{r+s}}{(r+s)!} = \frac{1}{(1-yt)^{1-r}} \frac{1}{t^r} \left[\mathrm{e}^{xt/(1-yt)} - \mathrm{e}_{r-1}\left(\frac{xt}{1-yt} \right) \right], \end{aligned} \tag{14} $$

[2] Recall that $\hat{D}_x^{-k} = x^k/k!$, for further comments see [6].

where $e_r(x)$ is the truncated exponential ($e_{-1}(x) = 0$). Further comments will be given in the concluding section.

The link between the two variable associated Laguerre polynomials of second kind and the ordinary associated Laguerre polynomials is provided by

$$L_n^{(r)}(x) = \frac{n!}{(n+r)!} x^{-r} \, {}_1L_{n,r}(-x, 1). \tag{15}$$

We must emphasize that the formal properties we have discussed, hold unchanged if r is any real number ρ. In this case,

$${}_1L_{n,\rho}(x, y) = n! \sum_{k=0}^{n} \frac{y^{n-k} x^{\rho+k}}{(n-k)!k!\Gamma(k+\rho+1)} \tag{16}$$

is a function whose properties will be touched on in the final part of the paper.

3. The Legendre generalized polynomials

In this section we will show that a proper generalization of the two-variable Laguerre polynomials may bring to light unsuspected relations with the Legendre polynomials.

According to the point of view developed in this paper, we can write the two-variable Laguerre polynomials as[3]

$${}_1L_n(x, y) = (y + \hat{D}_x^{-1})^n = n! \sum_{k=0}^{n} \frac{y^{n-k} \hat{D}_x^{-k}}{(n-k)!k!}. \tag{17}$$

In this operational form they are nothing but polynomials generated by binomial powers. Within such a framework the most straightforward extension of (17) is

$${}_2L_n(x, y) = H_n(y, \hat{D}_x^{-1}) = n! \sum_{k=0}^{[n/2]} \frac{y^{n-2k} \hat{D}_x^{-k}}{(n-2k)!k!}, \tag{18}$$

where $H_n(y, z)$ denotes Kampé-de Fériet polynomials [2].

According to Eq. (18), the ${}_2L_n(x, y)$ polynomials are q.m. under the action of the operators

$$\hat{M} = y + 2\hat{D}_x^{-1} \frac{\partial}{\partial y}, \qquad \hat{P} = \frac{\partial}{\partial y}. \tag{19}$$

Therefore, the identity

$$\hat{P}\hat{M} \, {}_2L_n(x, y) = (n+1) \, {}_2L_n(x, y) \tag{20}$$

in differential form yields

$$\left[2\frac{\partial^2}{\partial y^2} + y\frac{\partial^2}{\partial x \partial y} - n\frac{\partial}{\partial x} \right] {}_2L_n(x, y) = 0. \tag{21}$$

[3] Note that $L_n(x, y) = {}_1L_n(-x, y)$.

It can also be easily checked that ${}_2L_n(x, y)$ are the natural solutions of

$$\frac{\partial^2}{\partial y^2}\,{}_2L_n(x, y) = (\hat{D}_x x \hat{D}_x)\,{}_2L_n(x, y) \tag{22a}$$

which can also be rewritten in the form

$$\left(\frac{\partial^2}{\partial y^2} - x\frac{\partial^2}{\partial x^2}\right)\,{}_2L_n(x, y) = \frac{\partial}{\partial x}\,{}_2L_n(x, y). \tag{22b}$$

The generating function of the ${}_2L_n(x, y)$ polynomials can be obtained as in the case of the Kampé-de Feriét polynomials [4],[4] namely

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\,{}_2L_n(x, y) = \mathrm{e}^{t(y+2\hat{D}_x^{-1}\partial/\partial y)} = \mathrm{e}^{ty}\mathrm{e}^{t^2\hat{D}_x^{-1}} = \mathrm{e}^{ty}C_0(-xt^2) \tag{23}$$

it is also worth noting that

$$\left(y + 2\hat{D}_x^{-1}\frac{\partial}{\partial y}\right)^n = \sum_{s=0}^{n} 2^s \binom{n}{s} H_{n-s}(y, \hat{D}_x^{-1})\hat{D}_x^{-s}\frac{\partial^s}{\partial y^s}. \tag{24}$$

As a further remark, let us note that we are able to derive from ${}_2L_n(x, y)$ the Legendre polynomials [1], in fact

$${}_2L_n(-\tfrac{1}{4}(1 - y^2), y) = P_n(y). \tag{25}$$

By setting therefore $y = \sin\varphi$, we find from Eqs. (25) and (23), the well-known relation [1]

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\,{}_2L_n\left(-\frac{1}{4}\cos^2\varphi, \sin\varphi\right) = \sum_{n=0}^{\infty}\frac{t^n}{n!}P_n(\sin\varphi) = \mathrm{e}^{t\sin\varphi}J_0(t\cos\varphi). \tag{26}$$

It is evident that the obvious generalization of Eq. (18)

$${}_mL_n(x, y) = H_n^{(m)}(y, \hat{D}_x^{-1}) = n!\sum_{k=0}^{[n/m]}\frac{y^{n-mk}\hat{D}_x^{-k}}{(n-mk)!k!}, \tag{27}$$

satisfies equations of the type

$$\begin{aligned}
&\frac{\partial^m}{\partial y^m}\,{}_mL_n(x, y) = (\hat{D}_x x \hat{D}_x)\,{}_mL_n(x, y),\\
&\left(m\frac{\partial^m}{\partial y^m} + y\frac{\partial^2}{\partial x\partial y} - n\frac{\partial}{\partial x}\right)\,{}_mL_n(x) = 0,
\end{aligned} \tag{28}$$

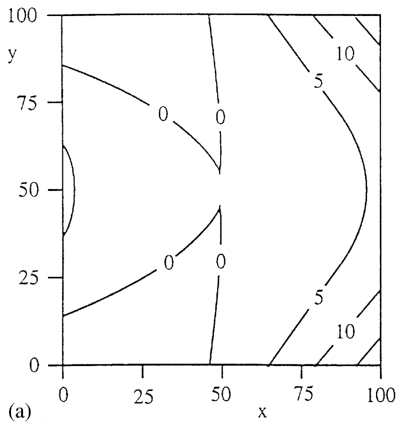
the last of which generalizes Eq. (21). The polynomials (27) are indeed q.m. under the action of the same operator $\hat{P}$ in Eq. (19) and of

$$\hat{M} = y + m\hat{D}_x^{-1}\frac{\partial^{m-1}}{\partial y^{m-1}}. \tag{29}$$

[4] Recall that

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}H_n(x, y) = \mathrm{e}^{xt+yt^2}$$

and it is clear that at least with respect to the y variable the ${}_2L_n(x, y)$ can be viewed as Appell polynomials.



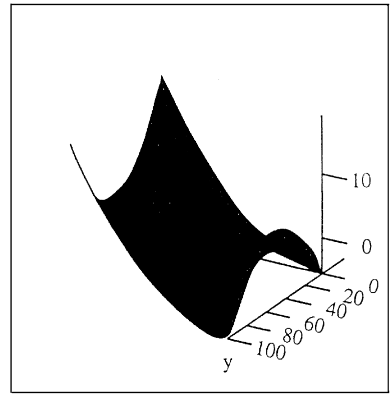


Fig. 1. (a) contour plot of ${}_2L_4(x,y)$; (b) 3-D view of ${}_2L_4(x,y)$.

Furthermore, they are specified by the generating function

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}\, {}_mL_n(x,y) = \mathrm{e}^{yt} C_0(-xt^m). \tag{30}$$

We can also introduce associated Legendre Polynomials linked to ${}_mL_n(x,y)$ by

$${}_mL_n\left(-\frac{1}{4}(1-y^2), y\right) = {}_mP_n(y), \tag{31}$$

obeying the generating function

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}\, {}_mP_n(\sin\varphi) = \mathrm{e}^{t\sin\varphi} J_0(\cos\varphi t^{m/2}). \tag{32}$$

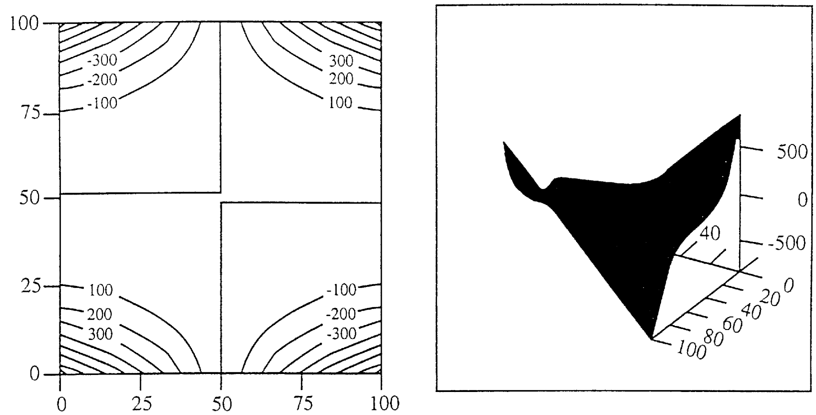
The results of these sections and the implications we have proved, confirm the usefulness of the researches associated to this class of “exotic” polynomials, in particular for the search of exact solutions of various forms of p.d.e.

An idea of the behaviour of these functions is provided by Figs. 1 and 2.

4. Concluding remarks

The use of the above polynomials and of the associated operational calculus is particularly useful to explore the properties of new generalized special functions and also to derive new identities for already known polynomials. Let us, indeed, consider the Rainville-type generating function

$$\sum_{\ell=0}^{\infty} \frac{t^\ell}{\ell!}\, {}_1L_{n+\ell}(x,y) = {}_1S_n(x,y;t). \tag{33}$$



(a) (b)

Fig. 2. Same as Fig. 1 for ${}_7L_4(x,y)$.

By recalling that

$$ {}_1L_n(x,y) = (y + \hat{D}_x^{-1})^n, \tag{34} $$

we can rewrite Eq. (33) in the operator form

$$ {}_1S_n(x,y;t) = \mathrm{e}^{t(y+\hat{D}_x^{-1})}\, {}_1L_n(x,y). \tag{35} $$

By noting also that

$$ \hat{D}_x^{-r}\, {}_1L_n(x,y) = {}_1L_{n,r}(x,y), \tag{36} $$

we eventually end up with

$$ {}_1S_n(x,y;t) = \mathrm{e}^{yt} \sum_{r=0}^{\infty} \frac{t^r}{r!}\, {}_1L_{n,r}(x,y). \tag{37} $$

We can interpret the summation on the r.h.s. of Eq. (37) as an exponential function defined as

$$ {}_1E_n(x,y;t) = \sum_{r=0}^{\infty} \frac{t^r\, {}_1L_{n,r}(x,y)}{r!}. \tag{38} $$

This function ${}_1E_n(x,y;t)$ has a number of interesting properties

(a) It is solution of the partial differential equation

$$ \begin{aligned} &\frac{\partial}{\partial y}\, {}_1E_n(x,y;1) + y\frac{\partial^2}{\partial x \partial y}\, {}_1E_n(x,y;1) = n\frac{\partial}{\partial x}\, {}_1E_n(x,y;1), \\ &{}_1E_n(x,0;1) = x^n C_n(-x). \end{aligned} \tag{39} $$

(b) It can be formally defined as

$$ {}_1E_n(x,y;1) = (y + \hat{D}_x^{-1})^n \mathrm{e}^{\hat{D}_x^{-1}}. \tag{40} $$

(c) It satisfies recurrences of the type

$$\begin{aligned} &(y+\hat{D}_x^{-1})E_n(x,y;1) = {}_1E_{n+1}(x,y;1),\\ &\frac{\partial}{\partial y}\,{}_1E_n(x,y;1) = n\,{}_1E_{n-1}(x,y;1). \end{aligned} \tag{41}$$

A point to be stressed is that we have recognized that the Legendre polynomials are a particular case of ${}_2L_n(x,y)$ it is therefore, worth noting that, along with Eq. (23) we can recover the further generating function

$$\sum_{n=0}^{\infty} t^n\,{}_2L_n(x,y) = \frac{1}{[1+(y^2-4x)t^2-2yt]^{1/2}} \tag{42}$$

which clearly reduces to that of the ordinary Legendre polynomials for $x=-\frac{1}{4}(1-y^2)$.

We have stressed that one of the main motivation of the present work is the analysis of the partial differential equations, associated with the polynomials we have so far introduced. Let us therefore go back to Eq. (10), which can also be rewritten as

$${}_1L_{n,r}(x,y) = \mathrm{e}^{y[(1-r)\hat{D}_x+x\hat{D}_x^2]}\frac{x^{n+r}}{(n+r)!}. \tag{43}$$

This equation offers the possibility of much interesting speculation, which will be fully developed elsewhere. Here we note that in the case of $r=1$, we find

$${}_1L_{n,1}(x,y) = \mathrm{e}^{yx\hat{D}_x^2}\frac{x^{n+1}}{(n+1)!}. \tag{44}$$

By summing up on the index n and by using the generating function (14), we get

$$\mathrm{e}^{x/(1-y)} - 1 = \mathrm{e}^{yx\hat{D}_x^2}(\mathrm{e}^x - 1). \tag{45}$$

It is therefore evident that the polynomials ${}_1L_{n,1}(x,y)$ offer the possibility of solving Fokker and Planck equations of the type

$$\frac{\partial}{\partial y}f(x,y) = x\hat{D}_x^2 f(x,y) \tag{46}$$

which play an important role in the framework of phenomena related to quantum beam life time in Storage Rings [9].

The general case (43) deserves particular interest too. Here we limit ourselves to noting that (see Eq. (12))

$$\mathrm{e}^{y[(1-r)\hat{D}_x+x\hat{D}_x^2]}(x)^r C_r(-x) = \mathrm{e}^{+y}(x)^r C_r(-x). \tag{47}$$

Before closing this paper we want to add a final consideration we believe important. The polynomials

$$\tilde{H}_n^{(m)}(y,x) = H_n^{(m)}(\hat{D}_y^{-1},x) = n!\sum_{k=0}^{[n/m]}\frac{D_y^{-(n-mk)}x^k}{(n-mk)!k!} = n!\sum_{k=0}^{[n/m]}\frac{y^{n-mk}x^k}{[(n-mk)!]^2k!}, \tag{48}$$

can be considered the mirror image of Eq. (27), satisfy the generating function

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}H_n^{(m)}(\hat{D}_y^{-1},x) = \mathrm{e}^{xt^m}C_0(-yt) \tag{49}$$

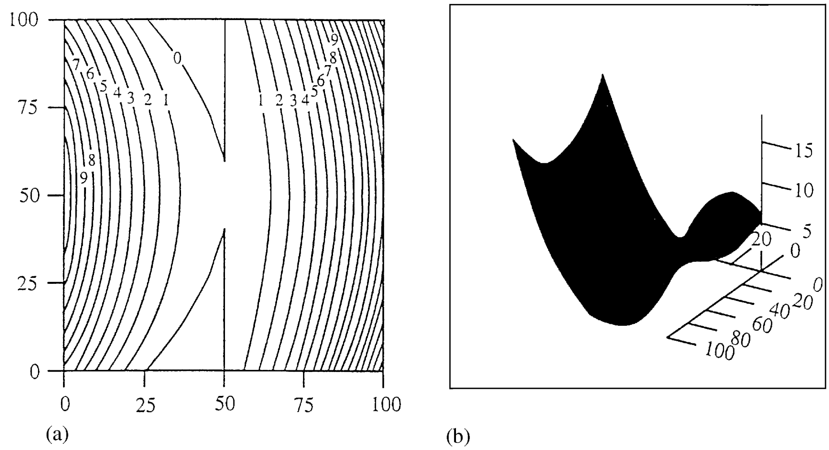


Fig. 3. Same as Fig. 1 for $\tilde{H}_4^{(2)}(x, y)$.

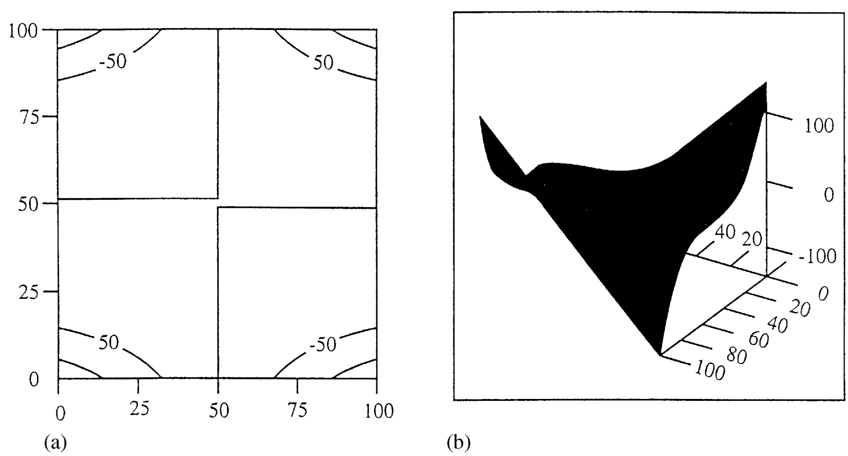


Fig. 4. Same as Fig. 1 for $\tilde{H}_4^{(7)}(x, y)$.

and are solutions of the p.d.e.,

$$\begin{aligned} &\frac{\partial}{\partial x}\tilde{H}_n^{(m)}(y,x) = (\hat{D}_y y \hat{D}_y)^m \tilde{H}_n^{(m)}(y,x), \\ &\tilde{H}_n^{(m)}(y,0) = \frac{y^n}{n!}. \end{aligned} \tag{50}$$

See Figs. 3 and 4 to get an idea of the behaviour of the $\tilde{H}_n^{(m)}$ polynomials.

The topics we have treated in this paper, the variety of problems we have connected, going through the partially unexplored world of generalized polynomials, yield perhaps an idea of the capabilities of the techniques we are developing. Many of the points just touched in the paper will provide the basic elements of forthcoming investigations.

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