Journal of Mathematical Analysis and Applications **257,** 308–320 (2001)
doi:10.1006/jmaa.2000.7345, available online at http://www.idealibrary.com on IDEAL®

Multivariable Lagrange Expansion and Generalization of Carlitz–Srivastava Mixed Generating Functions

G. Dattoli

ENEA, Dipartimento Innovazione, Divisione Fisica Applicata, Centro Ricerche Frascati, P.O. Box 67, I-00044 Frascati, Rome, Italy

S. Lorenzutta

ENEA, Dipartimento Innovazione, Divisione Fisica Applicata, Via don Fiammelli 2, I-Bologna, Italy

and

D. Sacchetti

Dipartimento di Statistica, Probabilità e Statistiche Applicate, Università degli Studi di Roma, "La Sapienza" Piazzale Aldo Moro 5, I-00185 Rome, Italy

Submitted by H. M. Srivastava

Received December 3, 1999

Families of mixed generating functions, generalizing those of the Carlitz–Srivastava type, are derived here by applying methods based on the multivariable extension of the Lagrange expansion. It is also shown that the combination with techniques of operational nature offers a wide flexibility to explore a wealth of mixed bilateral generating functions for special functions with many variables.

1. INTRODUCTION AND PRELIMINARIES

The familiar Lagrange expansion (LE) [1]

$$\begin{aligned}\frac{f(z)}{1 - t \cdot \phi'(z)} &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \left\{ \partial_\lambda^n \left[f(\lambda)(\phi(\lambda))^n \right] \right\}_{\lambda = z_0} \qquad \left(\partial_\lambda := \frac{d}{d\lambda} \right) \\ z &= z_0 + t\phi(z)\end{aligned} \tag{1}$$

0022-247X/01 $35.00

can be used to prove that [1]

$$S(x, y) = \sum_{n=0}^{\infty} \frac{t^n}{n!}(x + ny)^n = \frac{e^{xw}}{1 - yw}, \qquad w = te^{wy}. \tag{2}$$

This result and the operational identity [2, 3]

$$H_n(x, z) = e^{z\partial_x^2} x^n, \tag{3}$$

where $H_n(x, z)$ are the two-variable Hermite polynomials defined by

$$H_n(x, z) = n! \sum_{s=0}^{[n/2]} \frac{z^s x^{n-2s}}{s!(n-2s)!} = i^n z^{+n/2} H_n\left(\frac{-ix}{2\sqrt{z}}\right), \tag{4}$$

allow us to conclude that [4]

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x + ny, z) = \frac{e^{xw+zw^2}}{1 - yw}, \qquad w = te^{wy}, \tag{5}$$

$H_n(x)$ being the classical Hermite polynomials.

The identity (5) was derived earlier by Carlitz [5] within a different framework; it offers a genuine example of combination of LE and of operational methods to study generating functions of mixed type. This technique, put forward in [6], has allowed the derivation of a wealth of old and new results from a unified point of view.

To better illustrate the effectiveness of this method, which we are going to generalize in this paper, we will present further, not yet discussed, examples.

The two-variable Laguerre polynomials are defined by the generating function [6]

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \mathscr{L}_n(x, y) = e^{yt} C_0(xt), \tag{6}$$

where

$$\mathscr{L}_n(x, y) = n! \sum_{s=0}^{n} \frac{(-1)^s x^s y^{n-s}}{(s!)^2 (n-s)!} = y^n L_n\left(\frac{x}{y}\right) \tag{7a}$$

and

$$C_0(x) = \sum_{s=0}^{\infty} \frac{(-1)^s x^s}{(s!)^2} = J_0(2\sqrt{x}), \tag{7b}$$

$J_v(x)$ being the relatively more familiar Bessel function.

It is evident that the generating function (6) reduces to the ordinary case for $y = 1$ and it can be rewritten in the operational form

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \mathscr{L}_n(x, y) = C_0(x\partial_y) e^{yt}. \tag{8}$$

We can therefore argue, as a straightforward consequence of (2), that

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \mathscr{L}_n(x, y+nz) = \frac{e^{yw}}{1-zw} C_0(xw), \qquad w = te^{zw}. \tag{9}$$

In view of the relationship in (7a) with the classical Laguerre polynomials $L_n(x)$, a well-known result by Carlitz gives us the mixed generating function [5]

$$\begin{gathered} \sum_{n=0}^{\infty} t^n \mathscr{L}_n(x+nz, y) = \frac{1}{1-y\zeta} \frac{\exp\left(-\frac{x\zeta}{1-y\zeta}\right)}{1+z\zeta(1-y\zeta)^{-2}} \\ \zeta = t\exp\left(-\frac{z\zeta}{1-y\zeta}\right) \end{gathered} \tag{10}$$

which can be extended to the Hermite–Laguerre polynomials ${}_HL_n(x, y; k)$, defined by the relations (see [3])

$$e^{k\partial_x^2} \mathscr{L}_n(x, y) = {}_H\mathscr{L}_n(x, y; k) \tag{11a}$$

and

$${}_H\mathscr{L}_n(x, y; k) = n! \sum_{s=0}^{n} \frac{(-1)^s y^{n-s} H_s(x, y)}{(s!)^2 (n-s)!}. \tag{11b}$$

By applying the identities (3), (10), and (11a) we can, indeed, conclude that

$$\begin{gathered} \sum_{n=0}^{\infty} t^n {}_H\mathscr{L}_n(x+nz, y; k) = \frac{1}{1-y\zeta} \frac{\exp\left(-\frac{x\zeta}{1-y\zeta} + k\left(\frac{\zeta}{1-y\zeta}\right)^2\right)}{1+z\zeta(1-y\zeta)^{-2}} \\ \zeta = t\exp\left(-\frac{z\zeta}{1-y\zeta}\right). \end{gathered} \tag{12}$$

We will reconsider identities of the type (12) in the concluding section of this paper, which is devoted to the extension of the above methods and to the derivation of mixed generating functions with many indices and many varibales. It will be shown that, within such a framework, a central role is played by the multivariable extension of the Lagrange expansion.

Before closing this section, we recall two operational rules which will be used in our investigation:

(a) Weyl decoupling identity [3]

$$e^{\widehat{A}+\widehat{B}} = e^{\widehat{A}} e^{\widehat{B}} e^{-k/2} \tag{13}$$

if the operators satisfy the commutation relation $[\widehat{A}, \widehat{B}] = k \in C$,

(b) Crofton-like identity [3]

$$e^{\tau\partial_x^m \partial_y^n} f(x, y) = f\left(x + m\tau\partial_x^{m-1}\partial_y^n, y + n\tau\partial_x^m \partial_y^{n-1}\right). \tag{14}$$

Further comments on the use of the above identities can be found in Appendix B.

2. TWO-VARIABLE LAGRANGE EXPANSION AND MIXED GENERATING FUNCTIONS

The two-variable (TV) extension of the LE (1) is provided by [7]

$$\frac{h[u(s,t),v(s,t),s,t]}{\Delta[u(s,t),v(s,t),s,t]} = \sum_{m,n=0}^{\infty} \frac{s^m t^n}{m!n!} \left\{ \partial_x^m \partial_y^n \left[h(x,y,s,t) \times \left[(f(x,y))^m (g(x,y))^n \right] \right] \right\}_{x=y=0} \tag{15}$$

$$u(s,t) = s \cdot f(u(s,t), v(s,t))$$
$$v(s,t) = t \cdot g(u(s,t), v(s,t))$$
$$\Delta(x,y,s,t) = (1 - s\partial_x f)(1 - t\partial_y g) - ts(\partial_y f)(\partial_x g).$$

We will illustrate the usefulness of (15) by discussing a fairly complicated example; we consider indeed the three-variable polynomial

$$L_n^{(m)}(x,y,z) = \sum_{q=0}^{m} \sum_{r=0}^{n} \binom{m}{q} \binom{n}{r} (z-1)^q (y-1)^r L_{n-r}^{(m-q)}(x), \tag{16}$$

where $L_n^{(m)}(x)$ are single-variable modified Laguerre polynomials [8]. The bilateral generating function associated with these last polynomials is provided by (see Appendix A)

$$\begin{aligned} G(x,y,z;u,t) &= \sum_{m,n=0}^{\infty} \frac{u^m}{m!} \frac{t^n}{n!} L_n^{(m)}(x,y,z) \\ &= e^{zu+yt} C_0[(x-u)t]. \end{aligned} \tag{17}$$

The use of TVLE allows us to conclude that

$$\begin{aligned} &\sum_{m,n=0}^{\infty} \frac{u^m}{m!} \frac{t^n}{n!} L_n^{(m)}(x, y+\alpha m+\beta n, z+\gamma m+\delta n) \\ &= \sum_{m,n=0}^{\infty} \frac{u^m}{m!} \frac{t^n}{n!} \partial_\lambda^m \partial_\sigma^n \{ e^{y\sigma+z\lambda} C_0[(x-\lambda)\sigma] \\ &\quad \times [(e^{\alpha\sigma+\gamma\lambda})^m (e^{\beta\sigma+\delta\lambda})^n] \}_{\lambda=\sigma=0} \\ &= \frac{e^{z\psi_1+y\psi_2}}{1-\gamma\psi_1-\beta\psi_2+\Delta\psi_1\psi_2} C_0[(x-\psi_1)\psi_2], \end{aligned} \tag{18}$$

with

$$\Delta = \gamma\beta - \alpha\delta, \qquad \psi_1 = ue^{\gamma\psi_1+\alpha\psi_2}, \qquad \psi_2 = te^{\delta\psi_1+\beta\psi_2}. \tag{19}$$

In [4] it was shown that if

$$S_{m,n}(x,y) = (ax+by)^m (bx+cy)^n \tag{20}$$

then

$$\sum_{m,n=0}^{\infty} \frac{u^m}{m!}\frac{t^n}{n!} S_{m,n}(x+mw, y+nz) = \frac{e^{\psi_1(ax+by)+\psi_2(bx+cy)}}{1-aw\psi_1 - cz\psi_2 + \Delta wz\psi_1\psi_2}, \tag{21}$$

where

$$\Delta = ac-b^2, \qquad \psi_1 = ue^{(a\psi_1+b\psi_2)w}, \qquad \psi_2 = te^{(b\psi_1+c\psi_2)z}. \tag{22}$$

Since two-variable two-index Hermite polynomials are linked to $S_{m,n}(x,y)$ by the operational identity (see [9])

$$e^{(-1/2)\partial_\zeta^T \widehat{M}^{-1}\partial_\zeta} S_{m,n}(x,y) = H_{m,n}(x,y), \tag{23}$$

where T denotes the transpose and

$$\partial_\zeta = \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix}, \qquad \widehat{M} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \tag{24}$$

we can apply Crofton and Weyl identities to conclude that (see also Appendix B)

$$\begin{gathered}\sum_{m,n=0}^{\infty} \frac{u^m}{m!}\frac{t^n}{n!} H_{m,n}(x+mw, y+nz) = \frac{e^{\psi^T\widehat{M}\zeta - \frac{1}{2}\psi^T\widehat{M}\psi}}{1-aw\psi_1 - cz\psi_2 + \Delta wz\psi_1\psi_2}, \\ \Delta = ac-b^2, \qquad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \qquad \zeta = \begin{pmatrix} x \\ y \end{pmatrix}.\end{gathered} \tag{25}$$

It is evident that the procedure leading to the identity (25) is the direct generalization of that providing Eq. (5).

The polynomials $H_{m,n}(x,y;\tau)$ can be defined through the identity

$$H_{m,n}(x,y;\tau) = e^{(-\tau/2)\partial_\zeta^T \widehat{M}^{-1}\partial_\zeta} S_{m,n}(x,y) \tag{26}$$

and the generating function

$$\sum_{m,n=0}^{\infty} \frac{u^m}{m!}\frac{t^n}{n!} H_{m,n}(x,y;\tau) = e^{\rho^T\widehat{M}\zeta - (\tau/2)\rho^T\widehat{M}\rho} \qquad \rho = \begin{pmatrix} u \\ t \end{pmatrix}. \tag{27}$$

We can therefore apply the TVLE to derive the mixed bilateral generating function

$$\sum_{m,n=0}^{\infty} \frac{u^m}{m!}\frac{t^n}{n!} H_{m,n}(x+mw, y+nz; \tau+\alpha m+\beta n) = \frac{e^{\psi^T \widehat{M}\zeta-(\tau/2)\psi^T \widehat{M}\psi}}{D(w,z,\alpha,\beta;\tau)}, \tag{28}$$

where

$$\begin{aligned} &D(w, z, \alpha, \beta; \tau) \\ &\quad = 1 - [aw - \alpha(a\psi_1 + b\psi_2)]\psi_1 - [cz - \beta(c\psi_2 + b\psi_1)]\psi_2 \\ &\qquad + \{(aw - \alpha(a\psi_1 + b\psi_2))(cz - \beta(c\psi_2 + b\psi_1)) \\ &\qquad - (bw - \alpha(c\psi_2 + b\psi_1))(bz - \beta(a\psi_1 + b\psi_2))\}\psi_1\psi_2 \end{aligned} \tag{29}$$

with $\psi_{1,2}$ defined by

$$\psi_1 = ue^{w(a\psi_1+b\psi_2)-(\alpha/2)\psi^T \widehat{M}\psi}, \qquad \psi_2 = te^{z(b\psi_1+c\psi_2)-(\beta/2)\psi^T \widehat{M}\psi}. \tag{30}$$

The examples discussed so far offer a first idea of the usefulness of the TVLE to study problems associated with generalized mixed generating functions. A more general treatment will be presented in the following sections.

3. EXTENSION OF THE METHOD AND FURTHER EXAMPLES

The polynomials $h_{m,n}(x, y; \tau)$ are characterized by the following properties [2]

(a) generating function

$$\sum_{m,n=0}^{\infty} \frac{u^m}{m!}\frac{v^n}{n!} h_{m,n}(x, y; \tau) = e^{ux+vy+\tau uv} \tag{31}$$

(b) series definition

$$h_{m,n}(x, y; \tau) = m!n! \sum_{s=0}^{\min(m,n)} \frac{\tau^s x^{m-s} y^{n-s}}{s!(m-s)!(n-s)!} \tag{32}$$

(c) operational identity

$$e^{\tau\partial^2_{x,y}} x^m y^n = h_{m,n}(x, y; \tau). \tag{33}$$

It is therefore clear that

$$\sum_{m,\,n=0}^{\infty} \frac{u^m}{m!}\frac{v^n}{n!} h_{m,\,n}(x+\alpha m+\beta n,\, y+\gamma m+\delta n;\tau) = \frac{e^{\psi_1 x+\psi_2 y+\tau\psi_1\psi_2}}{1-\alpha\psi_1-\delta\psi_2+\Delta\psi_1\psi_2}, \tag{34}$$

where

$$\Delta=\alpha\delta-\beta\gamma, \qquad \psi_1=ue^{\alpha\psi_1+\gamma\psi_2}, \qquad \psi_2=ve^{\beta\psi_1+\delta\psi_2}. \tag{35}$$

This rather trivial result, obtainable as a particular case of the identity (23), has been explicitly quoted for two reasons:

(a) to prove the result

$$\sum_{m,\,n=0}^{\infty} \frac{u^m v^n}{m!n!}(x+\alpha m+\beta n)^{(m+n)/2} h_{m,\,n}\big(\sqrt{x+\alpha m+\beta n},\, \sqrt{x+\alpha m+\beta n};\tau\big) = \frac{e^{(\psi_1+\psi_2+\tau\psi_1\psi_2)x}}{1-\alpha\psi_1-\beta\psi_2-\tau\big((\alpha+\beta)\psi_1\psi_2\big)} \tag{36}$$

which generalizes an identity obtained by Srivastava [10] valid for the ordinary Hermite polynomials involved in Eq. (4), where

$$\psi_1=ue^{\alpha(\psi_1+\psi_2+\tau\psi_1\psi_2)}, \qquad \psi_2=ve^{\alpha(\psi_1+\psi_2+\tau\psi_1\psi_2)}; \tag{37}$$

(b) to introduce the Laguerre polynomials $L_{m,\,n}^{(\alpha,\beta)}(x, y;\tau)$ defined by the generating function

$$\sum_{m,\,n=0}^{\infty} u^m v^n L_{m,\,n}^{(\alpha,\,\beta)}(x, y;\tau) = \frac{1}{(1-u)^{\alpha+1}}\frac{1}{(1-v)^{\beta+1}} \times \exp\Big(-\frac{xu}{1-u}-\frac{yv}{1-v}+\tau\frac{uv}{(1-u)(1-v)}\Big) \tag{38}$$

and by the operational rule

$$e^{\tau\partial^2_{x,\,y}} L_m^{(\alpha)}(x) L_n^{(\beta)}(y) = L_{m,\,n}^{(\alpha,\beta)}(x, y;\tau). \tag{39}$$

This last identity can be employed to state the extension of the Carlitz formula [5],

$$\sum_{m,\,n}^{\infty} u^m v^n L_{m,\,n}^{(\alpha+\lambda m,\beta+\mu n)}(x+mw,\, y+nz;\tau) = \frac{1}{(1-\zeta_1)^{1+\alpha}}\frac{1}{(1-\zeta_2)^{1+\beta}}\frac{\exp\big(-\frac{x\zeta_1}{1-\zeta_1}-\frac{y\zeta_2}{1-\zeta_2}+\tau\frac{\zeta_1\zeta_2}{(1-\zeta_1)(1-\zeta_2)}\big)}{F_1(\zeta_1)F_2(\zeta_2)}, \tag{40}$$

where

$$F_a(\zeta_a) = 1 - \zeta_a(1-\zeta_a)^{-1}\left[\chi_a - \xi_a(1-\zeta_a)^{-1}\right], \qquad a = 1, 2,$$
$$\chi_1 = \lambda, \quad \chi_2 = \mu, \quad \xi_1 = w, \quad \xi_2 = z, \quad \eta_1 = u, \quad \eta_2 = v \tag{41}$$
$$\zeta_a = \eta_a(1-\zeta_a)^{-\chi_a} e^{-\xi_a\zeta_a/(1-\zeta_a)}.$$

It is worth noting that an extension of (41) involving the TVLE can be written as

$$\sum_{m,n=0}^{\infty} u^m v^n L_{m,n}^{(\alpha+\lambda m+\gamma n, \beta+\mu n+\delta m)}(x+mw+ns, y+nz+mr; \tau)$$
$$= \frac{1}{(1-\zeta_1)^{1+\alpha}} \frac{1}{(1-\zeta_2)^{1+\beta}} \frac{\exp\left(-\frac{x\zeta_1}{1-\zeta_1} - \frac{y\zeta_2}{1-\zeta_2} + \tau\frac{\zeta_1\zeta_2}{(1-\zeta_1)(1-\zeta_2)}\right)}{\Phi(\zeta_1,\zeta_2)}, \tag{42}$$

where

$$\Phi(\zeta_1,\zeta_2) = \left[1 - u\partial_{\zeta_1} A_1(\zeta_1\zeta_2)\right]\left[1 - v\delta_{\zeta_2} A_2(\zeta_1,\zeta_2)\right]$$
$$- uv\left\{\delta_{\zeta_2} A_1(\zeta_1,\zeta_2)\right\}\left\{\partial_{\zeta_1} A_2(\zeta_1,\zeta_2)\right\} \tag{43}$$

and

$$A_1(\zeta_1,\zeta_2) = \frac{1}{(1-\zeta_1)^{\lambda}} \frac{1}{(1-\zeta_2)^{\delta}} \exp(-\frac{w\zeta_1}{1-\zeta_1} - \frac{r\zeta_2}{1-\zeta_2}),$$
$$A_2(\zeta_1,\zeta_2) = \frac{1}{(1-\zeta_1)^{\gamma}} \frac{1}{(1-\zeta_2)^{\mu}} \exp\left(-\frac{s\zeta_1}{1-\zeta_1} - \frac{z\zeta_2}{1-\zeta_2}\right), \tag{44}$$
$$\zeta_1 = uA_1(\zeta_1,\zeta_2), \qquad \zeta_2 = vA_2(\zeta_1,\zeta_2).$$

The examples discussed in this section have perhaps provided a more general feeling on the usefulness and flexibility of the method we have proposed. Further examples will be discussed in Section 4 which is also devoted to concluding remarks and observations.

4. THE MULTIVARIABLE LAGRANGE EXPANSION AND CONCLUDING REMARKS

The results of the previous section can be framed within the context of the following general theorem:

THEOREM. *Let* $A(z_1,z_2), B(z_1,z_2), C(z_1,z_2), z_1^{-1}D(z_1,z_2)$, *and* $z_2^{-1}\times E(z_1,z_2)$ *be arbitrary functions which are analytic in a neighborhood of the origin* $(z_1 = z_2 = 0)$ *and assume that*

$$A(0,0) = B(0,0) = C(0,0)$$
$$= \partial_{z_1} D(z_1,z_2)\big|_{z_1=z_2=0} = \partial_{z_2} E(z_1,z_2)\big|_{z_1=z_2=0}. \tag{45}$$

Define the sequence of functions $\{f_{m,n}^{(\alpha,\beta)}(x, y)\}$ *by means of*

$$A(z_1, z_2)\big[B(z_1, z_2)\big]^{\alpha}\big[C(z_1, z_2)\big]^{\beta}\exp[xD(z_1, z_2) + yE(z_1, z_2)]$$
$$= \sum_{m,n=0}^{\infty} \frac{z_1^m}{m!}\frac{z_2^n}{n!} f_{m,n}^{(\alpha,\beta)}(x, y), \tag{46}$$

where, α, β, x, y *are arbitrary complex numbers independent of* $z_{1,2}$. *Then, for* $\lambda, \gamma, \mu, w, s, k, r$ *independent of* z_1 *and* z_2,

$$\sum_{m,n=0}^{\infty} \frac{u^m}{m!}\frac{v^n}{n!} f_{m,n}^{(\alpha+\lambda m+\gamma n,\, \beta+\mu n+\delta m)}(x + mw + ns, y + nk + mr)$$
$$= \frac{A(\zeta_1, \zeta_2)B(\zeta_1, \zeta_2)^{\alpha}C(\zeta_1, \zeta_2)^{\beta}}{\Psi(\zeta_1, \zeta_2)} e^{xD(\zeta_1,\zeta_2)+yE(\zeta_1,\zeta_2)}, \tag{47}$$

where

$$\begin{aligned} \zeta_1 &= u[B(\zeta_1, \zeta_2)]^{\lambda}[C(\zeta_1, \zeta_2)]^{\delta} e^{wD(\zeta_1,\zeta_2)+rE(\zeta_1,\zeta_2)} \\ \zeta_2 &= v[B(\zeta_1, \zeta_2)]^{\gamma}[C(\zeta_1, \zeta_2)]^{\mu} e^{sD(\zeta_1,\zeta_2)+kE(\zeta_1,\zeta_2)} \end{aligned} \tag{48}$$

and

$$\Psi(\zeta_1, \zeta_2) = \big[1 - u\partial_{\zeta_1} A_1(\zeta_1, \zeta_2)\big]\big[1 - v\partial_{\zeta_2} A_2(\zeta_1, \zeta_2)\big]$$
$$- uv\big\{\partial_{\zeta_2} A_1(\zeta_1, \zeta_2)\big\}\big\{\partial_{\zeta_2} A_1(\zeta_1, \zeta_2)\big\}. \tag{49}$$

The above theorem generalizing an analogous theorem due to Carlitz [5] and to Srivastava [10] (see also Srivastava and Manocha [11]) can be applied to derive further mixed generating functions. We consider therefore the polynomials $\mathscr{L}_{m,n}(x, y; z, w \mid \tau)$ defined as

$$e^{\tau\partial^2_{x,z}} \mathscr{L}_m(x, y)\mathscr{L}_n(z, w) = \mathscr{L}_{m,n}(x, y; z, w \mid \tau), \tag{50}$$

and since

$$\sum_{m,n=0}^{\infty} \frac{u^m}{m!}\frac{v^n}{n!} \mathscr{L}_{m,n}(x, y; z, w \mid \tau) = e^{yu+wv} C_{0,0}(xu, zv; \tau uv),$$

$$C_{0,0}(x, z; \tau) = \sum_{r,s=0}^{\infty} \frac{(-1)^{r+s} h_{r,s}(x, y; \tau)}{(r!)^2(s!)^2} \tag{51}$$

we end up with

$$\sum_{m,n=0}^{\infty} \frac{u^m}{m!}\frac{v^n}{n!} \mathscr{L}_{m,n}(x, y + \alpha m + \beta n; z, w + \gamma m + \delta n \mid \tau)$$
$$= \frac{e^{y\psi_1+w\psi_2} C_{0,0}(x\psi_1, z\psi_2; \tau\psi_1\psi_2)}{1 - \alpha\psi_1 - \delta\psi_2 + \Delta\psi_1\psi_2}, \qquad \Delta = \alpha\delta - \beta\gamma, \tag{52}$$
$$\psi_1 = ue^{\alpha\psi_1+\gamma\psi_2}, \qquad \psi_2 = ve^{\beta\psi_1+\delta\psi_2}.$$

A final point we want to touch upon is the possibility of obtaining further generalizations, based on the multivariable extension of the Lagrange expansion, which reads [12]

$$\begin{aligned}\frac{h(\{u_j(\{s_i\})\})}{\Delta(\{u_j(\{s_i\})\},\{s_P\})} &= \sum_{\{m_j\}}\prod_{j=1}^{N}\frac{s_j^{m_j}}{m_j!}\Big[\partial_{x_j}^{m_j}[h(\{x_i\},\{s_p\})] \\ &\quad\times\prod_{k=1}^{N}[f_k(\{x_i\})]^{m_k}\Big]_{x_j=0}, \\ u_j(\{s_i\}) &= s_j f_j\big(\{u_j(\{s_p\})\}\big),\end{aligned} \tag{53}$$

where $\{\alpha_j\} = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ and Δ denotes the determinant of the matrix with elements

$$L_{i,j} = \delta_{i,j} - s_i\partial_{x_j} f_i(\{x_k\}). \tag{54}$$

As a straightforward application of the MVLE, we consider the following example. Given

$$S_{m_1,m_2,m_3}(x_1,x_2,x_3) = \prod_{\alpha=1}^{3}\left(\sum_{\beta=1}^{3} a_{\alpha,\beta}x_\beta\right)^{m_\alpha} \tag{55}$$

then

$$\sum_{\{m_\alpha\}=0}^{\infty}\prod_{\alpha=1}^{3}\frac{u^{m_\alpha}}{m_\alpha!}S_{\{m_\alpha\}}(\{x_\alpha + m_\alpha w_\alpha\}) = \frac{\exp(\sum_{\alpha=1}^{3}(\psi_\alpha(\widehat{\Gamma}x)_\alpha))}{\Delta(\psi_1,\psi_2,\psi_3)}, \tag{56}$$

where

$$\psi_\alpha = u_\alpha \exp\left(w_\alpha\left(\widehat{\Gamma}^T\psi\right)_\alpha\right), \tag{57}$$

the matrix $\widehat{\Gamma}^T$ is the matrix with entries $a_{\alpha,\beta}$, $\underline{x}$ is the vector of components (x_α) and analogously for the vector $\boldsymbol{\psi}$, and finally $\Delta(\psi_1,\psi_2,\psi_3)$ is the determinant of the matrix with elements

$$T_{i,j} = \delta_{i,j} - a_{j,i}w_i\psi_i. \tag{58}$$

In a forthcoming investigation we will discuss more deeply the applications of MVLE.

APPENDIX A

The generalized Laguerre polynomials for m integer can be defined through the operational identity [13]

$$L_n^{(m)}(x) = (1 - D_x)^m (1 - \mathcal{D}_x^{-1})^n \{1\}, \tag{A1}$$

where $\mathcal{D}_x^{-1}$ is the inverse of the derivative operator D_x. By multiplying both sides of (A1) by $u^m/m!$ and $t^n/n!$ and by summing over the integers m and n, we get

$$\sum_{m,\,n=0}^{\infty} \frac{u^m}{m!} \frac{t^n}{n!} L_n^{(m)}(x) = e^{u(1-D_x)} e^{t(1-\mathcal{D}_x^{-1})} \{1\}. \tag{A2}$$

By recalling that

$$\mathcal{D}_x^{-n}\{1\} = \frac{x^n}{n!} \tag{A3}$$

and by using the Crofton identity we easily end up with

$$\sum_{m,\,n=0}^{\infty} \frac{u^m}{m!} \frac{t^n}{n!} L_n^{(m)}(x) = e^{u+t} C_0((x-u)t). \tag{A4}$$

APPENDIX B

In the first section we have introduced the operational definition of the Hermite polynomials (Eq. (3)), the Crofton rule (14), and the Weyl decoupling identity (13). We must, however, clarify how they are used.

From the operational point of view the quantity

$$\widehat{O} = e^{y\partial_x^2} f(x) \tag{B1}$$

is an operator equivalent to

$$\widehat{O} = f(x + 2y\partial_x) e^{y\partial_x^2}. \tag{B2}$$

If we assume that $\widehat{O}$ acts on unity it reduces to

$$\widehat{O}\{1\} = {}_H f(x) \tag{B3}$$

with ${}_H f(x)$ denoting a function having the same Maclaurin expansion as of $f(x)$ but with x^n replaced by $H_n(x, y)$; it is also clear that in this last case $\widehat{O}\{1\}$ is a function of x. It is also evident that sometimes, when exponential forms are involved, the use of the Crofton rule and of the Weyl identity can be useful. We note indeed that

$$\begin{aligned} e^{\tau\partial^2_{x,y}} e^{xu+yv} &= e^{(x+\tau\partial_y)u+(y+\tau\partial_x)v} \\ &= e^{xu+yv+(\tau/2)uv[\partial_y,y]+(\tau/2)uv[\partial_x,x]} e^{u\tau\partial_y} e^{v\tau\partial_x} \end{aligned} \tag{B4}$$

which, if assumed to act on unity, yields Eq. (33). The same procedure, albeit more cumbersome from the algebraic point of view, applies to the case of two-index, two-variable Hermite polynomials. The Crofton identity (14) indeed yields

$$e^{(1/2)\partial_\zeta^T(\widehat{M}^{-1})\partial_\zeta}(ax+by)^m(bx+cy)^n = \widehat{I}_{m,n}, \tag{B5}$$

where

$$\widehat{I}_{m,n} = (ax+by-\partial_x)^m(bx+cy-\partial_y)^n. \tag{B6}$$

By multiplying both sides of (B5) by $u^m/m!$ and $v^n/n!$ and by summing up over the indices m and n, we obtain

$$\begin{aligned} &e^{-(1/2)\partial_\zeta^T(\widehat{M}^{-1})\partial_\zeta} \sum_{m,n=0}^{\infty} \frac{u^m}{m!}\frac{v^n}{n!}(ax+by)^m(bx+cy)^n \\ &\quad = e^{u(ax+by-\partial_x)} e^{v(bx+cy-\partial_y)} \end{aligned} \tag{B7}$$

which, by means of the Crofton and Weyl identities, yields

$$\begin{aligned} &e^{-(1/2)\partial_\zeta^T(\widehat{M}^{-1})\partial_\zeta} \sum_{m,n=0}^{\infty} \frac{u^m}{m!}\frac{v^n}{n!}(ax+by)^m(bx+cy)^n \\ &\quad = e^{\rho^T\widehat{M}\zeta-(1/2)\rho^T\widehat{M}\rho} \end{aligned} \tag{B8}$$

ACKNOWLEDGMENT

The authors express their sincere appreciation to Professor H. M. Srivastava for his interest, help, and suggestions.

REFERENCES

1. G. Pólya and G. Szegö, "Problems and Theorems in Analysis," Vol. 1, Springer-Verlag, New York, 1972.
2. G. Dattoli and A. Torre, "Theory and Applications of Generalized Bessel Functions," Aracne, Rome, 1996; H. W. Gould and A. T. Hopper, Operatorial formulas connected with two generalizations of Hermite polynomials, *Duke Math J.* **29** (1962), 51–63

3. G. Dattoli, P. L. Ottaviani, A. Torre, and L. Vazquez, Evolution operator equations: Integration with algebraic and finite difference methods. Applications to physical problems in classical and quantum mechanics and quantum field theory, *Riv. Nuovo Cimento* **20** (1997), 1–133.
4. G. Dattoli, S. Lorenzutta, H. M. Srivastava, and A. Torre, Some families of mixed generating functions and generalized polynomials, *Matematiche (Catania)* **54** (1999), 147–157.
5. L. Carlitz, A class of generating functions, *SIAM J. Math. Anal*. **8** (1977), 518–532.
6. G. Dattoli and A. Torre, Operational methods and two-variable Laguerre polynomials, *Atti Accad. Torino Rend. Cl. Sci. Mat. Fis*. **132** (1998), 1–7.
7. J. Dieudonne, "Foundations of Modern Analysis," Academic Press, New York, 1969.
8. L. C. Andrews, "Special Functions for Engineers and Applied Mathematicians," Macmillan, New York, 1985.
9. G. Dattoli, S. Lorenzutta, G. Maino, and A. Torre, Theory of multiindex multivariable Bessel functions and Hermite polynomials, *Matematiche (Catania)* **52** (1997), 179–197.
10. H. M. Srivastava, Some generalizations of Carlitz's theorem, *Pacific J. Math*. **85** (1979), 471–477.
11. H. M. Srivastava and H. L. Manocha, "A Treatise on Generating Functions," Ellis Horwood/Wiley, New York, 1984.
12. G. M. Constantine, "Combinatorial Theory and Statistical Design," Wiley, New York, 1967.
13. G. Dattoli, S. Lorenzutta, and A. Torre, Operational identities and properties of ordinary and generalized special functions, *J. Math. Anal. Appl*. **236** (1999), 399–414.