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ON GENERALIZATION OF EULER'S CONTINUED FRACTIONS

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In this paper we generalize Euler's continued fraction by establishing its basic bilateral analogue.

1. INTRODUCTION

In a recent communication Berndt *et al.*[2] proved the results of chapter 12 of Ramanujan's second Notebook. This entire chapter contains only the results on continued fractions. The most beautiful results in this chapter are the ones involving hypergeometric functions. Here we shall provide the basic bilateral analogue of the entry 22 of this chapter which besides involving hypergeometric functions plays an important role in proving a number of other results of this chapter.

2. NOTATIONS AND DEFINITIONS

We shall use the following notations and definitions throughout this paper.

For α real or complex and $n \in N$, the notation $[\alpha; q]_n$ shall stand for

$$(1-\alpha)(1-\alpha q)(1-\alpha q^2)\ldots(1-\alpha q^{n-1}),\ [\alpha; q]_0 = 1,\ |q| < 1$$

and $[\alpha; q]_{-n}$ shall mean

$$\frac{(-)^n q^{n(n-1)/2}}{\alpha^n [q/\alpha; q]_n}.$$

We shall study a basic bilateral hypergeometric series, defined by

$${}_2\Psi_2\left[\begin{matrix} a, b; \\ c, d; \end{matrix}\ q, x\right] = \sum_{n=-\infty}^{\infty} \frac{[a; q]_n [b; q]_n x^n}{[c; q]_n [d; q]_n}\ (|cd/ab| < |x| < 1).$$

It is obvious that ${}_2\Psi_2$ reduces to ${}_2\Phi_1$ when any of the denominator parameters reduces to q.

In what follows the other notations carry their usual meaning.

3. MAIN RESULT

Here we shall establish our main result. It is easy to verify the following result, for non-negative integral i,

$$(1 - cq^i)(1 - cq^{i+1}) \left\{ \frac{d - bq^{i+1}}{q(1 - bq^i)} \, {}_2\Psi_2 \left[\begin{matrix} a, bq^i; \\ d, cq^i; \end{matrix} \, q, x \right] - \frac{d - cq^{i+1}}{q(1 - cq^i)} \, {}_2\Psi_2 \left[\begin{matrix} a, bq^{i+1}; \\ d, cq^{i+1}; \end{matrix} \, q, x \right] \right\}$$

$$= \frac{ax}{q} \left\{ \left(1 - \frac{bq^{i+1}}{a}\right)(1 - cq^{i+1}) \, {}_2\Psi_2 \left[\begin{matrix} a, bq^{i+1}; \\ d, cq^{i+1}; \end{matrix} \, q, x \right] - \left(1 - \frac{cq^{i+1}}{a}\right)(1 - bq^{i+1}) \, {}_2\Psi_2 \left[\begin{matrix} a, bq^{i+2}; \\ d, cq^{i+2}; \end{matrix} \, q, x \right] \right\}$$

$$\times \left(|cd/ab| < |x| < 1 \right). \qquad \ldots(3.1)$$

From the above relation we easily have

$$(1 - cq^i) \, {}_2\Psi_2 \left[\begin{matrix} a, bq^i; \\ d, cq^i; \end{matrix} \, q, x \right] \Big/ \, {}_2\Psi_2 \left[\begin{matrix} a, bq^{i+1}; \\ d, cq^{i+1}; \end{matrix} \, q, x \right]$$

$$= \alpha_i + \frac{\beta_i}{(1 - cq^{i+1}) \, {}_2\Psi_2 \left[\begin{matrix} a, bq^{i+1}; \\ d, cq^{i+1}; \end{matrix} \, q, x \right] \Big/ \, {}_2\Psi_2 \left[\begin{matrix} a, bq^{i+2}; \\ d, cq^{i+2}; \end{matrix} \, q, x \right]} \qquad \ldots(3.2)$$

where

$$\alpha_i = \frac{(1 - bq^i)(d - cq^{i+1}) + ax(1 - bq^i)\left(1 - \frac{bq^{i+1}}{a}\right)}{d - bq^{i+1}}$$

and

$$\beta_i = \frac{-ax\left(1 - \frac{cq^{i+1}}{a}\right)(1 - bq^i)(1 - bq^{i+1})}{d - bq^{i+1}}.$$

Now, repeated application of (3.2) yields the following general result

$$(1 - cq^i) \, {}_2\Psi_2 \left[\begin{matrix} a, bq^i; \\ d, cq^i; \end{matrix} \, q, x \right] \Big/ \, {}_2\Psi_2 \left[\begin{matrix} a, bq^{i+1}; \\ d, cq^{i+1}; \end{matrix} \, q, x \right]$$

$$= \alpha_i + \frac{\beta_i}{\alpha_{i+1}} + \frac{\beta_{i+1}}{\alpha_{i+1}} + \frac{\beta_{i+2}}{\alpha_{i+3}} + \ldots \qquad \ldots(3.3)$$

where

$$|cd/ab| < |x| < 1$$

and α_i and β_i are the same as given in (3.2).

Our main result (3.3) provides a basic bilateral analogue of Euler's continued fraction.

4. Special Cases

In this section we shall deduce a few interesting special cases of our main result (3.3).

If in (3.3) we take $i = 0$, $d = q$, we get the following known result (cf. Adiga *et al.*[1], Lemma 1, p. 17)

$$(1 - c)\,{}_2\Phi_1\left[\begin{matrix} a, b; \\ c \end{matrix}\; q, x\right] \Big/ {}_2\Phi_1\left[\begin{matrix} a, bq; \\ cq \end{matrix}\; q, x\right] \qquad \ldots(4.1)$$

$$= \alpha_0 + \frac{\beta_0}{\alpha_1} + \frac{\beta_1}{\alpha_2} + \frac{\beta_2}{\alpha_3} + \ldots$$

where, for $i = 0, 1, 2$

$$\alpha_i = 1 - cq^i + \frac{ax}{q}\left(1 - \frac{bq^{i+1}}{a}\right)$$

and

$$\beta_i = -\frac{ax}{q}\left(1 - \frac{cq^{i+1}}{q}\right)(1 - bq^{i+1}).$$

Again, if in (3.3), we take $i = 0$, $d = q$, replace a, b, c and x by $-\alpha, \beta, \gamma$ and $-x$, respectively, then take the reciprocal of both sides, and now multiply both sides by $(1 - \beta)\,x$ and let $q \to 1$, we get the following known result, the Euler's continued fraction (cf. Ramanujan[3] entry 22, p. 147)

$$\beta x\; {}_2F_1[-\alpha, \beta + 1; \gamma + 1; -x]/\gamma\; {}_2F_1[-\alpha, \beta; \gamma; -x]$$

$$= \frac{\beta x}{\gamma - (\alpha + \beta + 1)\,x} + \frac{(\beta + 1)(\alpha + \gamma + 1)\,x}{\gamma + 1 - (\alpha + \beta + 2)\,x}$$

$$+ \frac{(\beta + 2)(\alpha + \gamma + 2)\,x}{\gamma + 2 - (\alpha + \beta + 3)\,x} + \ldots \qquad \ldots(4.2)$$

A number of other interesting special cases could also be deduced by specializing the parameters.

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