ON GENERALIZATION OF CERTAIN CONTINUED FRACTIONS

REMY Y. DENIS

Department of Mathematics and Statistics, University of Gorakhpur
Gorakhpur 273 009

The main object of this paper is to generalize Nörlund's continued fraction by establishing its basic analogue.

1. INTRODUCTION

In this paper we establish the q-analogue of a known result due to Nörlund[7] (see also Perron[8]) which provides the q-analogue of the third part of entry 21 of chapter XII of Ramanujan's[9] second Notebook.

The notations and definitions appearing in this paper carry thier usual meaning. The interested reader is referred to the existing .relevant literature.

2. GENERALIZATION OF NORLUND'S CONTINUED FRACTION

It is easy to verify that (cf. Hahn[3])

$$(1 - c) \; {}_2\varphi_1 \left[\begin{array}{c} a, \, b; \, x \\ c \end{array} \right] \Big/ {}_2\varphi_1 \left[\begin{array}{c} aq, \, bq; \, x \\ cq \end{array} \right] = 1 - c + \{ (1 + q) \, ab - a - b \} \, x$$

$$+ \; \frac{x \, (c - abqx) \, (1 - aq) \, (1 - bq)}{(1 - cq) \; {}_2\varphi_1 \left[\begin{array}{c} aq, \, bq; \, x \\ cq \end{array} \right] \Big/ {}_2\varphi_1 \left[\begin{array}{c} aq^2, \, bq^2; \, x \\ cq^2 \end{array} \right]} \qquad \ldots(2.1)$$

with the help of which we get the following result, after some simplification

$$ {}_2\varphi_1 \left[\begin{array}{c} aq, \, bq; \, x \\ cq \end{array} \right] \Big/ (1 - c) \; {}_2\varphi_1 \left[\begin{array}{c} a, \, b; \, x \\ c \end{array} \right] = \frac{1}{\alpha_0} + \frac{\beta_1}{\alpha_1} + \frac{\beta_2}{\alpha_2} + \frac{\beta_3}{\alpha_3} + \ldots, $$

$$\ldots(2.2)$$

where for $i = 1, 2, 3, \ldots$,

$$\alpha_{i-1} = 1 - cq^{i-1} + \{ (1 + q) \, abq^{2i-2} - aq^{i-1} - bq^{i-1} \} \, x$$

and

$$\beta_i = x \, (cq^{i-1} - abxq^{2i-1}) \, (1 - aq^i) \, (1 - bq^i).$$

The result (2.2) holds in the region common to $|x| < 1$ and $4|x(c - abxq)| < |1 - c - \{(1 + q)ab - a - b\}x|^2$, for proper choice of a, b and c (cf. Wall[11], §, 10).

If $q \to 1 - 0$ in (2.2), we get the following known result due to Nörlund[7] (see also Perron[8]),

$$_2F_1 \begin{bmatrix} 1+a, \ 1+b; \ x \\ 1+c \end{bmatrix} \Big/ c_2F_1 \begin{bmatrix} a, \ b; \ x \\ c \end{bmatrix}$$

$$= \frac{1}{c - (1+a+b)x} + \frac{(a+1)(b+1)(x-x^2)}{c+1 - (3+a+b)x} + \frac{(a+2)(b+2)(x-x^2)}{c+2 - (5+a+b)x} + \ldots$$
$$\ldots(2.3)$$

Again, if we replace x by $-x$, put $a = 1$ in (2.2) and let $q \to 1 - 0$ after multiplying both sides by $x(1 - b)$, we get,

$$\frac{xb}{c} \ _2F_1 \begin{bmatrix} 1, \ b+1; \ -x \\ c+1 \end{bmatrix}$$

$$= \frac{bx}{c + (b+1)x} - \frac{1(b+1)x(1+x)}{c+1 + (b+3)x} - \frac{2.(b+2)x(1+x)}{c+2 + (b+5)x} - \qquad \ldots(2.4)$$

which is the third part of the entry 21 referred to earlier.

Further, if in (2.2), we make use of a known transformation due to Jackson[5] to replace the $_2\varphi_1$'s on the left by their equivalent abnormal $_2\varphi_2$'s and then let $q \to 1 - 0$ in the result thus obtained, we get the following known result (cf. Berndt *et al.*) last equation on page 264 with α, β, γ replaced by a, b, c, respectively)

$$\frac{x}{c(1-x)} \cdot \frac{_2F_1 \begin{bmatrix} c-a, \ b+1; \ -x/(1-x) \\ c+1 \end{bmatrix}}{_2F_1 \begin{bmatrix} c-a, \ b; \ -x/(1-x) \\ c \end{bmatrix}}$$

$$= \frac{x}{c - (1+a+b)x} + \frac{(a+1)(b+1)(x-x^2)}{c+1 - (3+a+b)x} + \frac{(a+2)(b+2)(x-x^2)}{c+2 - (5+a+b)x} + \ldots$$
$$\ldots(2.5)$$

[There are certain misprints in the result given by Berndt *et al.* (p. 264). The arguments of the $_2F_1$'s should be $x/(x - 1)$ instead of $1/(x - 1)$ and the denominator of the third term on the right should contain $\gamma + 2$ instead of $\gamma + 1$].

If we start with yet another known result (cf. Hahn[4]), we can easily get the following

$$_2\varphi_1 \begin{bmatrix} \alpha, \ \beta; \ xq \\ \gamma \end{bmatrix} \Big/ _2\varphi_1 \begin{bmatrix} \alpha, \ \beta; \ x \\ \gamma \end{bmatrix}$$

$$= \frac{q(1-x)}{C_0} + \frac{D_1}{C_1} + \frac{D_2}{C_2} + \frac{D_3}{C_3} + \ldots \qquad \ldots(2.6)$$

where, for $i = 1, 2, 3,...,$

$$C_{i-1} = \gamma + q - (\alpha + \beta) xq^i, \quad D_i = (\gamma - \alpha\beta xq^i) \, q \, (1 - xq^i)$$

and the result being valid, because of uniform convergence in the region common to $|x| < 1$ and $|x| < |\sqrt{\gamma} - \sqrt{q}|/|\alpha + \beta|$ for a proper choice of α, β and γ (cf. Wall[11], § 10).

We have the following result due to the author[2]

$$_2\phi_1 \left[\begin{array}{c} \alpha, \, \beta; \, xq \\ \gamma \end{array} \right] \Big/ {}_2\phi_1 \left[\begin{array}{c} \alpha, \, \beta; \, x \\ \gamma \end{array} \right]$$

$$= \frac{1}{1} \; + \; \frac{A_0}{1 - x} \; + \; \frac{B_0}{1} \; + \; \frac{A_1}{1 - x} \; + \; \frac{B_1}{1} \; + \; ... \qquad ...(2.7)$$

where, for $i = 0, 1, 2,...,$

$$A_i = (1 - \alpha q^i) \, (1 - \beta q^i) \, x, \quad \beta_i = \alpha\beta xq^{2i+1} - \gamma q^i$$

and the result being valid, because of uniform convergence in the region common to $|x| < 1$ and max. $\{4|x|, 4|\alpha\beta xq - \gamma|\} < |1 - x|$ for a proper choice of α, β and γ (cf. Wall[11], §10).

Now (2.6) and (2.7) provide the continued fraction representation of the same function. As such the right sides of the above two results may be equivalent to each other in a common region of validity.

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