



Genocchi polynomials associated with the Umbral algebra

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ABSTRACT

The aim of this paper is to study on the Genocchi polynomials of higher order on P , the algebra of polynomials in the single variable x over the field \mathbb{C} of characteristic zero and P^* , the vector spaces of all linear functional on P . By using the action of a linear functional L on a polynomial $p(x)$ Sheffer sequences and Appell sequences, we obtain some fundamental properties of the Genocchi polynomials. Furthermore, we give relations between, the first and second kind Stirling numbers, Euler polynomials of higher order and Genocchi polynomials of higher order.

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1. Introduction

Throughout of this paper, we can use the following notations and definitions, which are given by Roman [3, pp. 1–125].

Let P be the algebra of polynomials in the single variable x over the field of complex numbers. Let P^* be the vector space of all linear functionals on P . Let $\langle L|p(x) \rangle$ be the action of a linear functional L on a polynomial $p(x)$. Let \mathcal{F} denote the algebra of formal power series

$$f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k. \quad (1.1)$$

Such algebra is called Umbral algebra. Each $f \in \mathcal{F}$ defines a linear functional on P and

$$a_k = \langle f(t)|x^k \rangle \quad (1.2)$$

for all $k \geq 0$.

The order $o(f(t))$ of a power series $f(t)$ is the smallest integer k for which the coefficient of t^k does not vanish. A series $f(t)$ for which $o(f(t)) = 1$ will be called a delta series. When we are considering a delta series $f(t)$ in \mathcal{F} as a linear functional we will refer to it as a delta functional.

It is well-known that $\langle t^k|x^n \rangle = n! \delta_{n,k}$ where δ denotes Kronecker symbol. For all $f(t)$ in \mathcal{F}

$$f(t) = \sum_{k=0}^{\infty} \frac{\langle f(t)|x^k \rangle}{k!} t^k.$$

Let $f(t), g(t)$ be in \mathcal{F} , then we have

$$\langle f(t)g(t)|p(x) \rangle = \langle f(t)|g(t)p(x) \rangle. \quad (1.3)$$

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For $y \in \mathbb{C}$ the evaluation functional is defined to be the power series e^{yt} . By (1.2), we have

$$\langle e^{yt} | p(x) \rangle = p(y), \tag{1.4}$$

for all $p(x)$ in P . The forward difference functional is the delta functional $e^{yt} - 1$ and

$$\langle e^{yt} - 1 | p(x) \rangle = p(y) - p(0). \tag{1.5}$$

The Abel functional is the delta functional te^{yt} . We have

$$\langle te^{yt} | p(x) \rangle = p'(y).$$

The Sheffer polynomials are defined by means of the following generating function

$$\sum_{k=0}^{\infty} \frac{s_k(x)}{k!} t^k = \frac{1}{g(t)} e^{xt}.$$

cf. [3] see also [1,2]).

Roman [3] proved the following theorem which is represented by the Sheffer polynomials (or Sheffer sequences) explicitly:

Theorem 1. *Let $f(t)$ be a delta series and let $g(t)$ be an invertible series. Then there exist a unique sequence $s_n(x)$ of polynomials satisfying the orthogonality conditions*

$$\langle g(t)f(t)^k | s_n(x) \rangle = n! \delta_{n,k} \tag{1.6}$$

for all $n, k \geq 0$.

The sequence $s_n(x)$ in (1.6) is the Sheffer polynomials for pair $(g(t), f(t))$, where $g(t)$ must be invertible and $f(t)$ must be delta series. The Sheffer polynomials for pair $(g(t), t)$ is the Appell polynomials or the Appell sequences for $g(t)$.

The Appell polynomials, the Bernoulli polynomials, the Euler polynomials and the Genocchi polynomials belong to the family of the Sheffer polynomials cf. [1–4].

The Sheffer polynomials satisfy the following relations:

$$s_n(x) = g(t)^{-1} x^n, \tag{1.7}$$

derivative formula

$$t s_n(x) = s'_n(x) = n s_{n-1}(x), \tag{1.8}$$

recurrence formula

$$s_{n+1}(x) = \left(x - \frac{g'(t)}{g(t)} \right) s_n(x), \tag{1.9}$$

expansion theorem

$$h(t) = \sum_{k=0}^{\infty} \frac{\langle h(t) | s_k(x) \rangle}{k!} g(t) t^k, \tag{1.10}$$

multiplication theorem, for $\alpha \neq 0$

$$s_n(\alpha x) = \alpha^n \frac{g(t)}{g(\frac{t}{\alpha})} s_n(x), \tag{1.11}$$

and

$$\langle h(t) | p(ax) \rangle = \langle h(at) | p(x) \rangle. \tag{1.12}$$

2. Genocchi Polynomials of higher order on \mathcal{F}

In this section, by using properties of the Sheffer sequences and also the Appell sequences, we prove many fundamental properties of the Genocchi polynomials of higher order $G_n^{(b)}(x)$, which are defined by means of the following generating function:

$$\left(\frac{2t}{e^t + 1} \right)^a e^{xt} = \sum_{n=0}^{\infty} G_n^{(a)}(x) \frac{t^n}{n!}, \tag{2.1}$$

where $|t| < \pi$. $G_n^{(1)}(x) = G_n(x)$ denotes the Genocchi polynomials.

By using (1.7) and (2.1), we arrive at the following Lemma:

Lemma 1

$$G_n^{(b)}(x) = \left(\frac{2t}{e^t + 1} \right)^b x^n.$$

Theorem 2

$$\langle (e^t + 1)^k | G_n(x) \rangle = 2n(k-1)! \sum_{j=0}^{k-1} 2^{k-j-1} \frac{S(n-1, j)}{(k-j-1)!},$$

where $G_n(x)$ and $S(u, v)$ denote the Genocchi polynomials and the Stirling numbers of the second kind, respectively.

By Lemma 1, we obtain

$$\langle (e^t + 1)^k | G_n(x) \rangle = \left\langle (e^t + 1)^k \left| \frac{2t}{e^t + 1} x^n \right. \right\rangle.$$

By using (1.3) and (1.8), we get

$$\langle (e^t + 1)^k | G_n(x) \rangle = 2n \sum_{j=0}^{k-1} \frac{(k-1)!}{(k-j-1)!} 2^{k-j-1} \left\langle \frac{(e^t - 1)^j}{j!} | x^{n-1} \right\rangle. \quad (2.2)$$

Setting

$$S(n-1, j) = \frac{1}{j!} \left\langle (e^t - 1)^j | x^{n-1} \right\rangle,$$

where $S(n-1, j)$ denotes the Stirling numbers of second kind cf [3, pp. 59], [5], in (2.2), we arrive at the desired result.

By using (1.8), we arrive at the following lemma:

Lemma 2

$$tG_n^{(a)}(x) = nG_{n-1}^{(a)}(x)$$

Remark 1. A second proof of Lemma 2 is also obtained from (2.1) by using derivative with respect to x . By Lemma 2, one can see that

$$\frac{1}{t} G_n^{(a)}(x) = \frac{1}{n+1} G_{n+1}^{(a)}(x).$$

Lemma 3

$$\left(\frac{1}{e^t + 1} \right) G_n^{(a)}(x) = \frac{1}{2(n+1)} G_{n+1}^{(a+1)}(x)$$

Proof. By (2.1) and Lemma 1, we obtain

$$\left(\frac{1}{e^t + 1} \right) G_n^{(a)}(x) = \frac{1}{e^t + 1} \left(\frac{2t}{e^t + 1} \right)^a x^n.$$

After some calculations in the above equation, we get

$$\left(\frac{1}{e^t + 1} \right) G_n^{(a)}(x) = \frac{1}{2t} \left(\frac{2t}{e^t + 1} \right)^{a+1} x^n.$$

Using Lemma 2, we obtain the desired result. \square

An integral representation of $\left\langle \frac{e^{ta}-1}{2t} | G_n^{(b)}(x) \right\rangle$ is given by the following theorem.

Theorem 3

$$\left\langle \frac{e^{ta}-1}{2t} | G_n^{(b)}(x) \right\rangle = \frac{1}{2} \int_0^a G_n^{(b)}(x) dx.$$

Proof. By using Lemma 2, we have

$$\left\langle \frac{e^{ta} - 1}{2t} | G_n^{(b)}(x) \right\rangle = \left\langle \frac{e^{ta} - 1}{2t} \middle| \frac{1}{n+1} t G_{n+1}^{(b)}(x) \right\rangle.$$

By (1.3), we obtain

$$\left\langle \frac{e^{ta} - 1}{2t} | G_n^{(b)}(x) \right\rangle = \frac{1}{n+1} \left\langle e^{at} - 1 | G_{n+1}^{(b)}(x) \right\rangle.$$

The desired result follows now from (1.5). \square

A recurrence formula for $G_n^{(a)}(x)$ is given by the next theorem.

Theorem 4 (Recurrence formula).

$$G_{n+1}^{(a+1)}(x) = \frac{2}{a} \left((n-a+1) G_{n+1}^{(a)}(x) + (a-x)(n+1) G_n^{(a)}(x) \right).$$

Proof. Setting

$$g(t) = \left(\frac{e^t + 1}{2t} \right)^a \tag{2.3}$$

in (1.9), one can obtain

$$\begin{aligned} G_{n+1}^{(a)}(x) &= \left(x - a \frac{e^t t - (e^t + 1)}{t(e^t + 1)} \right) G_n^{(a)}(x) \\ &= x G_n^{(a)}(x) - a \left(\frac{e^t}{(e^t + 1)} - \frac{1}{t} \right) G_n^{(a)}(x) \\ &= x G_n^{(a)}(x) + \frac{a}{2(n+1)} G_{n+1}^{(a+1)}(x). \end{aligned}$$

Consequently, in the above equations using Lemma 3, Remark 1 and $e^t G_n^{(a)}(x) = 2n G_{n-1}^{(a-1)}(x) - G_n^{(a)}(x)$, we arrive at the desired result. \square

We now ready to prove multiplication formula for the Genocchi polynomials as follows:

Theorem 5 (Multiplication formula). For every positive odd integer α ,

$$G_n(\alpha x) = \alpha^{n-1} \sum_{j=0}^{\alpha-1} (-1)^j G_n \left(x + \frac{j}{\alpha} \right).$$

Proof. If we substitute (2.3) into (1.11) and let $a = 1$, then we obtain

$$G_n(\alpha x) = \alpha^n \left(\frac{e^t + 1}{2t} \right) \frac{\left(\frac{2t}{\alpha} \right)}{e^{\frac{t}{\alpha}} + 1} G_n(x) = \alpha^{n-1} \frac{e^t + 1}{e^{\frac{t}{\alpha}} + 1} G_n(x).$$

From the above equation, we get

$$G_n(\alpha x) = \alpha^{n-1} \sum_{j=0}^{\alpha-1} (-1)^j e^{\frac{jt}{\alpha}} G_n(x).$$

By (1.4), we arrive at the desired result. \square

Recall that the Euler polynomials $E_n^{(a)}(x)$ of higher order a are defined by the generating series

$$\left(\frac{2}{e^t + 1} \right)^a e^{xt} = \sum_{n=0}^{\infty} E_n^{(a)}(x) \frac{t^n}{n!}.$$

Lemma 4

$$E_n^{(a)}(x) = \left(\frac{2}{e^t + 1} \right)^a x^n.$$

Proof of Lemma 4 was given by Roman [3, pp. 101].

The next theorem expresses the Genocchi polynomials in terms of the Euler polynomials and the Stirling numbers of the first kind.

Theorem 6

$$G_n^{(a)}(x) = \sum_{j=0}^a \sum_{k=0}^j \sum_{m=0}^k \binom{a}{j} \binom{j}{k} 2^{k-a} (-1)^{j-k} s(k, m) n^m E_{n-k}^{(a)}(x).$$

Proof. From Lemma 1 we find

$$\begin{aligned} G_n^{(a)}(x) &= \left(\frac{1}{e^t + 1} + \frac{2t - 1}{e^t + 1} \right)^a x^n \\ &= \sum_{j=0}^a \binom{a}{j} \frac{1}{(e^t + 1)^a} \sum_{k=0}^j \binom{j}{k} 2^k (-1)^{j-k} t^k x^n. \end{aligned}$$

Using

$$t^k x^n = (n)_k x^{n-k},$$

where $(n)_k = n(n-1)\cdots(n-k+1)$ in the above, we have

$$G_n^{(a)}(x) = \sum_{j=0}^a \binom{a}{j} \frac{1}{2^a} \sum_{k=0}^j \binom{j}{k} 2^k (-1)^{j-k} (n)_k \left(\frac{2}{e^t + 1} \right)^a x^{n-k}.$$

By using Lemma 4 and

$$(y)_k = \sum_{m=0}^k s(k, m) y^m,$$

where $s(k, m)$ denotes the Stirling numbers of the first kind, in the above, we obtain the desired result. \square

Theorem 7

$$(e^t + 1)G_n^{(a)}(x) = 2nG_{n-1}^{(a-1)}(x).$$

Proof. By using (1.7), we obtain

$$(e^t + 1)G_n^{(a)}(x) = (2t) \left(\frac{2t}{e^t + 1} \right)^{a-1} x^n.$$

By using Lemma 1 and Lemma 2, we obtain the desired result. \square

By substituting $a = 1$ into Theorem 7, we arrive at the following corollary:

Corollary 1

$$G_n(x+1) + G_n(x) = 2nx^{n-1}. \quad (2.4)$$

Remark 2. If we set $a = 1$ in (2.1), then one can arrive at the proof of the above Corollary 1 as follows:

$$2te^{xt} = \sum_{n=0}^{\infty} (G_n(x+1) + G_n(x)) \frac{t^n}{n!}$$

From the above we arrive at (2.4) cf. ([1,2]).

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References

- [1] P. Blasiak, G. Dattoli, A. Horzela, K.A. Penson, Representations of monomiality principle with Sheffer-type polynomials and boson normal ordering, Phys. Lett. A 352 (2006) 7–12.

- [2] G. Dattoli, M. Migliorati, H.M. Srivastava, Sheffer polynomials, monomiality principle, algebraic methods and the theory of classical polynomials, *Math. Comput. Model.* 45 (2007) 1033–1041.
- [3] S. Roman, *The Umbral Calculus*, Dover Publ. Inc., New York, 2005.
- [4] Y. Simsek, q -Hardy–Berndt type sums associated with q -Genocchi type zeta and q - l -functions, *Nonlinear Anal-Theor* 71 (2009) e377–e395.
- [5] W. Wang, Generalized higher order Bernoulli number pairs and generalized Stirling number pairs, *J. Math. Anal. Appl.* 364 (2010) 255–274.