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A GENERAL CLASS OF GENERATING FUNCTIONS OF LAGUERRE POLYNOMIALS

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Abstract. In this paper we have obtained a new general class of generating functions for the generalized modified Laguerre polynomials $L_n^{(\alpha)}(x)$ by group theoretic method. Also, we introduce the bilateral generating function for the generalized modified Laguerre and Jacobi polynomials with the help of two linear partial differential operators. Consequently we recover the result of Majumdar [1] and notice that the result of Das and Chatterjea [2] is the particular case of our result.

1. Introduction

In a theoretical connection with the unification of generating functions has great importance in the study of special functions. With the steps forward in this directions has been made by some researchers [3, 4, 5, 6]. Also, the special functions has great deal with applications in pure and applied mathematics. They are appears in different frameworks. They are often used in combinatorial analysis [7], and even in statistics [8]. Moreover, the Laguerre polynomials have been applied in many other contexts, such as the Blissard problem (see [3]), the representation of Lucas polynomials of the first and second kinds [9, 10], the representation formulas of Newton sum rules for polynomials zeros [11, 12], the recurrence relations for a class of Freud-type polynomials [13], the representation of symmetric functions of a countable set of numbers, generalizing the classical algebraic Newton-Girard formulas [14]. Consequently they were also used [15] in order to find reduction formulas for the orthogonal invariants of a strictly positive compact operator, deriving in a simple way so called Robert formulas [16]. In their study Darus and Ibrahim [17] used deformed calculus to define generalized Laguerre polynomials and other special functions. Moreover, they gave the explicit representation formulas for the deformed Laguerre-type derivative of a composite function and illustration with applications. While Mukherjee [18] extend the bilateral generating function involving Jacobi polynomials derived by Chongdar [19] is well presented by group-theoretic method. Also, he had been proved the existence of quasi bilinear generating function implies the existence of a more general generating function. In their paper

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[20], Alam and Chongdar obtained some results on bilateral and trilateral generating functions of modified Laguerre polynomials. Furthermore, they made some comments on the results of Laguerre polynomials obtained by Das and Chatterjea [2]. Further, Banerji and Mohsen [21] established a result on generating relation involving modified Bessel polynomials.

In this paper we have obtained new general class of generating functions for the generalized modified Laguerre polynomials $L_n^{(\alpha)}(x)$. Also, we have introduced the bilateral generating function for the generalized modified Laguerre and Jacobi polynomials, which has been established by two linear partial differential operators. Consequently we recovered the results of Majumdar [1]. Furthermore, we notice that result of Das and Chatterjea [2] is the particular case of our result.

This paper is organized in four sections. In the first section, we gave the introduction to the problem. While in section two, we develop the new general class of modified Laguerre and Jacobi polynomials. Also, there we have introduce bilateral generating function. In the third section of this article, we gave the applications to our results, and we conclude the results in section four.

2. Generating Functions of Laguerre Polynomials

In this section we develop the new general class of generating functions for modified Laguerre polynomials. Also, we introduce the bilateral generating function for modified Laguerre and Jacobi polynomials.

In [2], Das and Chatterjea have claimed that the operator R_1, obtained by double interpretations to both the index (n) and the parameter (α) of the Laguerre polynomial in Weisner's group-theoretic method, also in [1], Majumdar has studied the quasi bilinear generating function for the Laguerre polynomials.

While extending the generalized modified Laguerre polynomials $L_n^{(\alpha)}(x)$, we introduce the bilateral generating function for the generalized modified Laguerre and Jacobi polynomials by means of Theorem 2.1 and Theorem 2.2. The Laguerre polynomials, as introduce by the later author [22], are defined as,

$$L_n^{(\alpha)}(x) = \frac{(1+\alpha)_n}{n!}\,{}_1F_1(-n; 1+\alpha; x), \quad Re(\alpha) > -1.$$

Theorem 2.1. *If there exists a generating function of the form*

$$G(x,u,w) = \sum_{n=0}^{\infty} a_n w^n L_n^{(\alpha)}(x) P_m^{(n,\beta)}(u), \tag{2.1}$$

then

$$\begin{aligned} &\exp(-wx)(1-wt)^{-(1+\beta+m)}(1+w)^{\alpha} G\left(x(1+w), \tfrac{u+wt}{1-wt}, \tfrac{w}{1-wt}\right) \\ &= \sum_{n,p,q=0}^{\infty} a_n w^{n+p+q} \frac{(1+n)_p(1+n+\alpha+m)_q}{p!q!} L_{(n+p)}^{(\alpha-p)}(x) P_m^{(n+q,\beta)}(u) t^q. \end{aligned} \tag{2.2}$$

Proof. Let us carry forward with the following linear partial differential operators, which has been referred from [18, 20].

$$R_1 = xy^{-1}z\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} - xy^{-1}z, \tag{2.3}$$

and

$$R_2 = (1+u)t\frac{\partial}{\partial u} + t^2\frac{\partial}{\partial t} + (1+\beta+m)t. \tag{2.4}$$

So that

$$R_1[y^\alpha z^n L_n^{(\alpha)}(x)] = (1+n)L_{(n+1)}^{(\alpha-1)}(x)y^{(\alpha-1)}z^{(n+1)}, \tag{2.5}$$

and

$$R_2[t^n P_m^{(n,\,\beta)}(u)] = (1+n+\beta+m)P_m^{(n+1,\beta)}(u)t^{(n+1)}. \tag{2.6}$$

Also, we have

$$\exp(wR_1)f(x,y,z) = \exp\left(\frac{-wxz}{y}\right)f(x+wxy^{-1}z, y+wz, z), \tag{2.7}$$

and

$$\exp(wR_2)f(u,t) = (1-wt)^{-(1+\beta+m)}f\left(\frac{u+wt}{1-wt}, \frac{t}{1-wt}\right). \tag{2.8}$$

Now, we consider the following generating relation

$$G(x,u,w) = \sum_{n=0}^{\infty} a_n w^n L_n^{(\alpha)}(x)P_m^{(n,\,\beta)}(u), \tag{2.9}$$

replacing w by wtz and then multiplying both sides by y^α, we get

$$y^\alpha G(x,u,wtz) = y^\alpha \sum_{n=0}^{\infty} a_n (wtz)^n L_n^{(\alpha)}(x)P_m^{(n,\,\beta)}(u). \tag{2.10}$$

Operating $\exp(wR_1)$, $\exp(wR_2)$ on both sides of (2.10), we have

$$\begin{aligned}&\exp(wR_1)\exp(wR_2)\,[y^\alpha G(x,u,wtz)]\\ &= \exp(wR_1)\exp(wR_2)\sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(x)y^\alpha P_m^{(n,\beta)}(u)(wtz)^n.\end{aligned} \tag{2.11}$$

With the help of (2.7) and (2.8) the left hand side of (2.11) can be simplified as

$$\exp\left(\tfrac{-wxz}{y}\right)(1-wt)^{-(1+\beta+m)}(y+wz)^\alpha G\left(x+wxy^{-1}z, \tfrac{u+wt}{1-wt}, \tfrac{wtz}{1-wt}\right). \tag{2.12}$$

Also, the right hand side of (2.11) with the help of (2.5) and (2.6) is simplified as

$$\begin{aligned}&\sum_{n,p,q=0}^{\infty} a_n w^{n+p+q}\frac{(1+n)_p}{p!}L_{n+p}^{(\alpha-p)}(x)y^{\alpha-p}\frac{(1+n+\beta+m)_q}{q!}\\ &\times P_m^{(n+q,\,\beta)}(u)z^{n+p}t^{n+q}.\end{aligned} \tag{2.13}$$

Therefore, the simplified form of (2.11) is

$$\begin{aligned}&\exp\left(\tfrac{-wxz}{y}\right)(1-wt)^{-(1+\beta+m)}(y+wz)^\alpha G\left(x+wxy^{-1}z, \tfrac{u+wt}{1-wt}, \tfrac{wtz}{1-wt}\right) =\\ &\sum_{n,p,q=0}^{\infty} a_n w^{n+p+q}\frac{(1+n)_p(1+n+\beta+m)_q}{p!q!}L_{n+p}^{(\alpha-p)}(x)P_m^{(n+q,\,\beta)}(u)y^{\alpha-p}z^{n+p}t^{n+q}.\end{aligned} \tag{2.14}$$

Finally substituting $z/y = 1$ in (2.14), we obtain bilateral generating function (2.15) for generalized modified Laguerre and Jacobi polynomials.

$$\begin{aligned}&\exp(-wx)(1-wt)^{-(1+\beta+m)}(1+w)^\alpha G\Big(x+wx, \tfrac{u+wt}{1-wt}, \tfrac{w}{1-wt}\Big) =\\ &\sum_{n,p,q=0}^{\infty} a_n w^{n+p+q}\frac{(1+n)_p(1+n+\beta+m)_q}{p!q!}L_{n+p}^{(\alpha-p)}(x)P_m^{(n+q,\,\beta)}(u)t^q.\end{aligned} \tag{2.15}$$

This completes the proof of the theorem. □

Theorem 2.2. *If there exists bilateral generating relation of the form*

$$G(x,v,w)=\sum_{n=0}^{\infty}a_n w^n P_n^{(\alpha,\,\beta)}(x)L_n^{(\alpha)}(v), \tag{2.16}$$

then

$$\begin{aligned}&\left(\frac{1+w}{1+2w}\right)^{\alpha}\exp\left(-wv\right)\,G\!\left(\frac{x+2w}{1+2w},v+wv,w\right)\\ =&\sum_{n,p,q=0}^{\infty}a_n w^{n+q}\frac{(1+n)_q}{q!}\,P_{n+p}^{(\alpha,\,\beta-p)}(x)L_{(n+q)}^{(\alpha-q)}(v).\end{aligned} \tag{2.17}$$

Proof. Now we replace the variables x, y and z in the operator R_1 by v, s and t respectively. With this replacement we can rewrite the operator R_1 as;

$$R_1=vs^{-1}t\frac{\partial}{\partial v}+t\frac{\partial}{\partial s}-vs^{-1}t.$$

So that

$$R_1\left(s^{\alpha}t^{n}L_n^{(\alpha)}(v)\right)=(1+n)L_{(n+1)}^{(\alpha-1)}(v)s^{(\alpha-1)}t^{(n+1)}. \tag{2.18}$$

Let us define the operator R_3

$$\begin{aligned}R_3 &= (1-x^2)y^{-1}z\frac{\partial}{\partial x}-z(x-1)\frac{\partial}{\partial y}-(1+x)y^{-1}z^2\frac{\partial}{\partial z}\\ &- (1+\alpha)(1+x)y^{-1}z.\end{aligned} \tag{2.19}$$

(One may concern [18] for more details about the operator R_3.) Operating R_3 on $y^{\beta}z^{n}P_n^{(\alpha,\beta)}(x)$, we get

$$R_3\left(y^{\beta}z^{n}P_n^{(\alpha,\beta)}(x)\right)=-2(1+n)P_{n+1}^{(\alpha,\beta-1)}(x)y^{\beta-1}z^{n+1}. \tag{2.20}$$

Also, we have

$$\begin{aligned}&\exp(wR_3)f(x,y,z)=\\ &\left(\frac{y}{y+2wz}\right)^{\alpha+1}f\left(\frac{xy+2wz}{y+2wz},\frac{y(y+2wz)}{y+2wz},\frac{yz}{y+2wz}\right),\end{aligned} \tag{2.21}$$

and

$$\exp(wR_1)f(v,s,t)=\exp\left(\frac{-wvt}{s}\right)f(v+wvs^{-1}t,s+wt,t). \tag{2.22}$$

Now we consider the generating relation

$$G(x,v,w)=\sum_{n=0}^{\infty}a_n w^n P_n^{(\alpha,\beta)}(x)L_n^{(\alpha)}(v). \tag{2.23}$$

In above relation replacing w by wtz and then multiplying both sides by $y^{\beta}s^{\alpha}$, we get

$$y^{\beta}s^{\alpha}G(x,v,wtz)=y^{\beta}s^{\alpha}\sum_{n=0}^{\infty}a_n(wtz)^n P_n^{(\alpha,\beta)}(x)L_n^{(\alpha)}(v). \tag{2.24}$$

Operating $\exp(wR_1)$, $\exp(wR_3)$ on both sides of (2.24), we have

$$\begin{aligned}&\exp(wR_1)\exp(wR_3)\left[y^{\beta}s^{\alpha}G(x,v,wtz)\right]\\ &=\exp(wR_1)\exp(wR_3)\sum_{n=0}^{\infty}a_n(wtz)^n P_n^{(\alpha,\beta)}(x)L_n^{(\alpha)}(v)y^{\beta}s^{\alpha}.\end{aligned} \tag{2.25}$$

With the help of (2.18) and (2.20) the right hand side of (2.25) can be simplified as

$$\sum_{n,p,q=0}^{\infty} a_n w^{n+p+q} \frac{(1+n)_p}{p!} \frac{(1+n)_q}{q!} (-2)^p P_{n+p}^{(\alpha,\beta-p)}(x) L_{n+q}^{(\alpha-q)}(v) \times y^{(\beta-p)} s^{(\alpha-q)} z^{(n+p)} t^{(n+q)}. \tag{2.26}$$

Also, the left hand side of (2.25) with the help of (2.21) and (2.22) is simplified as

$$y^{\beta}(s+wt)^{\alpha} \exp\left(\frac{-wvt}{s}\right) \left(\frac{y}{y+2wz}\right)^{\alpha+1} G\left(\frac{xy+2wz}{y+2wz}, v+wvs^{-1}t, \frac{wtyz}{y+2wz}\right). \tag{2.27}$$

Therefore, the simplified form of (2.25) is

$$\begin{aligned} & y^{(\alpha+\beta+1)} \left(\frac{s+wt}{y+2wz}\right)^{\alpha} \exp\left(\frac{-wvt}{s}\right) (y+2wz)^{-1} G\left(\frac{xy+2wz}{y+2wz}, v+wvs^{-1}t, \frac{wtyz}{y+2wz}\right) \\ & = \sum_{n,p,q=0}^{\infty} a_n w^{n+p+q} \frac{(1+n)_p}{p!} \frac{(1+n)_q}{q!} (-2)^p P_{n+p}^{(\alpha,\,\beta-p)}(x) L_{n+q}^{(\alpha-q)}(v) \\ & \qquad \times y^{(\beta-p)} s^{(\alpha-q)} z^{(n+p)} t^{(n+q)}. \end{aligned} \tag{2.28}$$

Finally substituting $s = y = z = t = 1$ in (2.28), we arrive at the proof of theorem. □

3. Application

If we use $m = 0$, we notice from our theorem 2.1 that $G(x, u, w)$ for $P_o^{(n,\beta)}(u) = 1$. Hence, from theorem 2.1, we deduce that

$$(1+w)^{\alpha} \exp(-wx) G(x+wx, w) = \sum_{n,p=0}^{\infty} a_n w^{n+p} \frac{(1+n)_p}{p!} L_{n+p}^{(\alpha-p)}(x). \tag{3.1}$$

(1) If we put $a_n = 1$, in (3.1), we obtain

$$(1+w)^{\alpha} \exp(-wx) L_n^{(\alpha)}(x(1+w)) = \sum_{p=0}^{\infty} w^p \binom{n+p}{p} L_{n+p}^{(\alpha-p)}(x). \tag{3.2}$$

This result is as same as obtained by Das and Chatterjea in their paper [2].

(2) If we multiply both sides of (3.1) by r^n, we get

$$\begin{aligned} (1+w)^{\alpha} \exp(-wx) G(x+wx, wr) &= \sum_{n,p=0}^{\infty} a_n w^{n+p} \frac{(1+n)_p}{p!} r^n L_{n+p}^{(\alpha-p)}(x), \\ &= \sum_{n=0}^{\infty} w^n \sum_{p=0}^{n} a_{n-p} \binom{n}{p} r^{n-p} L_n^{(\alpha-p)}(x), \\ &= \sum_{n=0}^{\infty} w^n \sigma_n(x, r), \end{aligned} \tag{3.3}$$

where

$$\sigma_n(x, r) = \sum_{s=0}^{\infty} \binom{n}{s} a_s r^s L_n^{(\alpha-n+s)}(x).$$

Incidentally this happens to be the theorem 1 in the paper of Majumdar (p:195 cf.[1]).

4. Conclusion

In this article, we have introduced a new general class of generating functions in the form of (2.1), for modified Laguerre and Jacobi's polynomials. Whereas, (2.16) is bilateral generating function for Laguerre and Jacobi's polynomials. Also, we have shown that Das and Chaterjea's (one may refer [2]) result is a particular case of theorem 2.1 for $m = 0$ and $a_n = 1$. Also by multiplying both sides of (3.1) by r^n, we have obtained the result of Majumdar [1] given by (3.3).

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