Trilateral Generating Function for Hermite, Jacobi and Bessel Polynomials

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Abstract

In this paper we have obtained a new class of trilateral generating function involving Hermite, Jacobi and Bessel polynomials from a given class of trilateral generating function. As in particular cases we have obtained bilateral and unilateral generating functions. The bilateral generating function involves the Hermite and Jacobi polynomials. Whereas, an unilateral generating function involves Hermite polynomials.

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1   Introduction

In a theoretical connection with the unification of generating functions has great importance in the study of special functions. With the steps forward in this directions has been made by some researchers [3, 4, 6, 15]. Also, the special functions has great deal with applications in pure and applied mathematics. They appears in different frameworks. They are often used in combinatorial analysis [13], and even in statistics [9]. In his study Mukherjee [10] extend the idea of bilateral generating function involving Jacobi polynomials derived by Chongdar [7] and it has been well presented by group-theoretic method. Also, he have been proved the existence of quasi bilinear generating function implies the existence of a more general generating function. In their paper [1], Alam and Chongdar obtained some results on bilateral and trilateral generating functions of modified Laguerre polynomials. Furthermore, they made some

comments on the results of Laguerre polynomials obtained by Das and Chatterjea [8]. Banerji and Mohsen [2] established a result on generating relation involving modified Bessel polynomials. On other hand Mukherjee obtained trilateral generating function from given class of bilateral generating function for some special kinds of functions in his paper [11].

This papers aims at obtaining a new class of trilateral generating function involving Hermite polynomials $H_{n+m}(x)$, Jacobi polynomials $P_k^{(n,\beta)}(u)$ and Bessel Polynomials $Y_s(v, n, \gamma)$ from given class of trilateral generating function. As in particular cases of this generating function we have obtained bilateral and unilateral generating functions. An unilateral generating function as a special case of our trilateral generating function is the result of Rainville (cf. [12], pp. 197).

2   Main Results

In this section we have obtained a new class trilateral generating function from given class of trilateral generating function. We have summarized this result in the form of theorem 2.1. In section 3, we gave particular cases of theorem 2.1 and that are summarized in the form of corollaries 3.1 and 3.2.

Now extending the general theorem on trilateral generating function for the modified Hermite polynomials $H_{n+m}(x)$, Jacobi polynomials $P_k^{(n,\beta)}(u)$ and Bessel polynomials $Y_s(v, n, \gamma)$ by means of the relation

$$G(x, u, v, w) = \sum_{n=0}^{\infty} a_n w^n H_{m+n}(x) P_k^{(n,\beta)}(u) Y_s(v, n, \gamma).$$

Here the Bessel polynomials $Y_n^{(\alpha)}$ are defined as in [16]

$$Y_n(x, \alpha, \beta) = {}_2F_1\left(-n, \alpha + n - 1; -; -\frac{x}{\beta}\right).$$

Also, the Hermite polynomials are defined as in [12] as follows

$$H_n(x) = \sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k n! (2x)^{n-2k}}{k!(n-2k)!}.$$

The Jacobi polynomials as defined in [12] are given by

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!} {}_2F_1\left(-n, 1 + \alpha + \beta + n; 1 + \alpha; \frac{1-x}{2}\right).$$

The object of this paper is to establish some general class of trilateral generating functions involving Bessel, Hermite and Jacobi polynomials. The following is the theorem that gives us a new class of trilateral generating function.

Theorem 2.1 *If there exists a trilateral generating function for the Hermite, Jacobi and Bessel polynomials in the form of*

$$G(x, u, v, w) = \sum_{n=0}^{\infty} a_n w^n H_{m+n}(x) P_k^{(n,\beta)}(u) Y_s(v, n, \gamma). \tag{1}$$

Then the following more general class of generating function holds:

$$(1-w)^{-(1+\beta+k)}(1-wv)^s \exp(2wx - w^2 + w\gamma) G\left(x - w, \tfrac{u+w}{1-w}, \tfrac{v}{1-wv}, \tfrac{w}{1-w}\right)$$

$$= \sum_{n,p,q,r=0}^{\infty} a_n \frac{w^{n+p+q+r}}{p!q!r!}(1+n+\beta+k)_q \gamma^r H_{m+n+p}(x) P_k^{(n+q,\beta)}(u) Y_s(v, n-r, \gamma). \tag{2}$$

Proof: We consider the trilateral generating function for Hermite, Jacobi and Bessel polynomials

$$G(x, u, v, w) = \sum_{n=0}^{\infty} a_n w^n H_{m+n}(x) P_k^{(n,\beta)}(u) Y_s(v, n, \gamma). \tag{3}$$

we replace w by $wygt$ in above equation and then multiplying both sides of resultant equation with $y^m h^s$, we get

$$y^m h^s G(x, u, v, wygt) = \sum_{n=0}^{\infty} a_n w^n [H_{m+n}(x) y^{m+n}][P_k^{(n,\beta)}(u) t^n][Y_s(v, n, \gamma) g^n h^s]. \tag{4}$$

Now, we choose the following three partial differential operators (cf. [5, 10, 14]):

$$A_1 = 2xy - y\frac{\partial}{\partial x}, \tag{5}$$

$$A_2 = (1-u)t\frac{\partial}{\partial u} + t^2\frac{\partial}{\partial t} + (1+\beta+m)t, \tag{6}$$

and

$$A_3 = v^2 g^{-1}\frac{\partial}{\partial v} - vg^{-1}h\frac{\partial}{\partial h} + g^{-1}\gamma. \tag{7}$$

Hence, we have

$$A_1\big[H_{m+n}(x) y^{m+n}\big] = H_{m+n+1}(x) y^{m+n+1}, \tag{8}$$

$$A_2\left[P_k^{(n,\beta)}(u)t^n\right] = (1 + n + \beta + k)P_k^{(n+1,\beta)}(u)t^{n+1}, \tag{9}$$

and

$$A_3[Y_s(v, n, \gamma)g^n h^s] = \gamma Y_s(v, n - 1, \gamma)g^{n-1}h^s. \tag{10}$$

Consequently, we have following exponential forms of above operators.

$$\exp(wA_1)f(x, y) = \exp(2wxy - w^2y^2)f(x - wy, y), \tag{11}$$

$$\exp(wA_2)f(u, t) = (1 - wt)^{-(1+\beta+k)}f\left(\frac{u + wt}{1 - wt}, \frac{t}{1 - wt}\right) \tag{12}$$

and

$$\exp(wA_3)f(v, g, h) = \exp\left(\frac{w\gamma}{g}\right)f\left(\frac{vg}{g - wv}, g, \frac{h(g - wv)}{g}\right). \tag{13}$$

Now, we operate both side of (4) with $\exp(wA_1)\exp(wA_2)\exp(wA_3)$, we obtain

$$\begin{aligned}
\exp(wA_1)&\exp(wA_2)\exp(wA_3)\left[y^m h^s G(x, u, v, wygt)\right] \\
&= \exp(wA_1)\exp(wA_2)\exp(wA_3) \\
\times \sum_{n=0}^{\infty} a_n w^n &[H_{m+n}(x)y^{m+n}][P_k^{(n,\beta)}(u)t^n][Y_s(v, n, \gamma)g^n h^s].
\end{aligned} \tag{14}$$

After the simplification left hand side of (14) becomes

$$\begin{aligned}
y^m h^s (1 - wt)^{-(1+\beta+k)} &\left(\frac{g - wv}{g}\right)^s \exp(2wxy - w^2y^2)\exp\left(\frac{w\gamma}{g}\right) \\
&\times G\left(x - wy, \frac{u + wt}{1 - wt}, \frac{vg}{g - wv}, \frac{wygt}{1 - wt}\right).
\end{aligned} \tag{15}$$

On the other hand the right side of (14) is simplified as

$$\begin{aligned}
\sum_{n,p,q,r=0}^{\infty} a_n &\frac{w^{n+p+q+r}}{p!q!r!}(1 + n + \beta + k)_q \gamma^r H_{m+n+p}(x)P_k^{(n+q,\beta)}(u)Y_s(v, n - r, \gamma) \\
&\times y^{m+n+p}t^{n+q}g^{n+r}h^s.
\end{aligned} \tag{16}$$

Now, equating (15) and (16) we get

$$\begin{aligned}
y^m h^s (1 - wt)^{-(1+\beta+k)}&\left(\frac{g - wv}{g}\right)^s \exp(2wxy - w^2y^2)\exp\left(\frac{w\gamma}{g}\right) \\
\times G\left(x - wy, \frac{u + wt}{1 - wt}, \frac{vg}{g - wv}, \frac{wygt}{1 - wt}\right) &= \sum_{n,p,q,r=0}^{\infty} a_n \frac{w^{n+p+q+r}}{p!q!r!}(1 + n + \beta + k)_q \gamma^r \\
\times H_{m+n+p}(x)P_k^{(n+q,\beta)}(u)Y_s(v, n - r, \gamma)&y^{m+n+p}t^{n+q}g^{n+r}h^s.
\end{aligned} \tag{17}$$

If we put $y = t = g = h = 1$ in equation (17), we get required theorem.

3   Application

In our theorem 2.1, if we put $s = 0$ we notice that $G(x, u, v, w)$ becomes $G(x, u, w)$ for $Y_0(v, n - r, \gamma) = 1$. Hence we obtain the following result which we hope a new bilateral generating function for Hermite polynomials.

Corollary 3.1 *If there exists a bilateral generating function*

$$G(x, u, w) = \sum_{n=0}^{\infty} a_n w^n H_{m+n}(x) P_k^{(n,\beta)}(u), \tag{18}$$

then the following more general bilateral generating relation is hold

$$\exp(2wx - w^2) G\left(x - w, \frac{u + w}{1 - w}, w\right) = \sum_{n,p=0}^{\infty} a_n \frac{w^{n+p}}{p!} H_{m+n+p}(x) P_k^{(n+q,\beta)}(u). \tag{19}$$

In corollary 3.1, if we put $k = m = 0$ then we notice that $G(x, u, w)$ becomes $G(x, w)$, and for $P_0^{(n+q,\beta)} = 1$. Hence, we obtain the following result.

Corollary 3.2 *If there exists a unilateral generating function*

$$G(x, w) = \sum_{n=0}^{\infty} a_n w^n H_n(x), \tag{20}$$

then the following generating relation holds.

$$\exp(2wx - w^2) H_n(x - w) = \sum_{n,p=0}^{\infty} a_n \frac{w^p}{p!} H_{n+p}(x). \tag{21}$$

If we put $a_n = 1$ in (21), we get

$$\exp(2wx - w^2) H_n(x - w) = \sum_{p=0}^{\infty} \frac{w^p}{p!} H_{n+p}(x). \tag{22}$$

Which is obtained by Rainville (cf. [12], pp. 197).

4   Conclusion

In section 2, we have obtained a new class of trilateral generating function in the form of (2) from given class of trilateral generating function (1). In section 3, we have the particular case of generating function (2) for $s = 0$, which is a new class of bilateral generating function (19) that involves Hermite and Jacobi polynomials. Also, the particular case of corollary 3.1 for $k = m = 0$ is an unilateral generating function (21) and thereby substituting $a_n = 1$, we recover the result of Rainville (cf. [12], pp. 197).

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