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A survey of the Fine numbers

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Abstract

The Fine numbers and the Catalan numbers are intimately related. Two manifestations are the identity $C_n = 2F_n + F_{n-1}$, $n \geqslant 1$, and the generating function identities $F = C/(1 + zC)$, $C = F/(1 - zF)$. In this paper we collect and organize the previous literature, present many new settings, and develop the theory and generating functions as well as asymptotics. Among the topics developed are the hill-killer involution, the set of all path pairs, and some new results about noncrossing partitions. © 2001 Elsevier Science B.V. All rights reserved.

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0. Introduction

We started this project with the idea of giving a unified presentation and history of the eight or so previous results in the literature that mentioned the Fine numbers. Along the way we discovered many more results and many new settings for the Fine numbers.

In this paper we have included the unified presentation and a selection of new results. Those results, such as the Fine–Narayana numbers, that require Lagrange inversion will be written up later since our discarded title, “More, much more, than you wanted to know about the Fine numbers”, may already be totally applicable.

1. Examples

In this section we want to set out many examples of occurrences of the Fine numbers (M1624 [1]). The first few Fine numbers are $1, 0, 1, 2, 6, 18, 57, \ldots$ and in typical Catalan

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[1] The number Mxxxx are identifiers of the sequences in the *Encyclopedia of Integer Sequences* [43].

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fashion we will list the $F_4=6$ examples. For background on the Catalan numbers (M1459) there are many fine sources and we mention, for example, [8,21,23,39,45,46] as readable introductions.

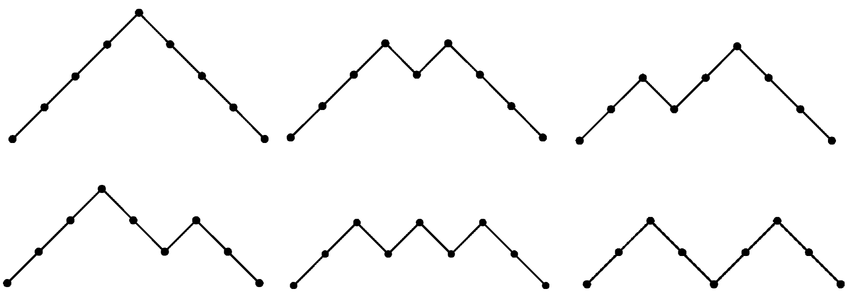
An extensive bibliography up to 1980 has been compiled by Gould [22].

We will discuss generating functions in Section 2 but one way to define the Fine numbers is by their generating function

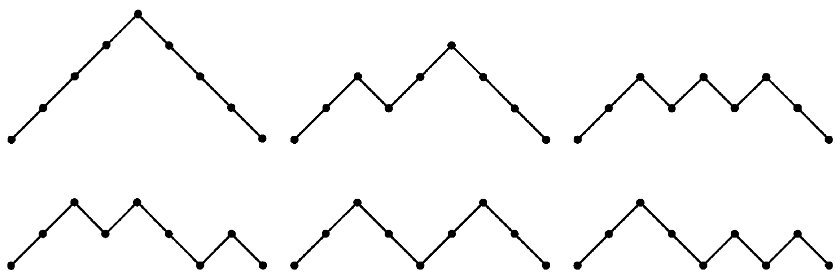
$$F(z)=\sum_{n=0}^{\infty} F_n z^n = \frac{1}{z}\frac{1-\sqrt{1-4z}}{3-\sqrt{1-4z}}.$$

Here are some combinatorial interpretations of the Fine numbers. Some of them will be discussed in the paper.

(A) Dyck paths (mountain ranges) with no hills. Dyck paths are paths starting and ending on the horizontal axis using steps $(1,1)$ and $(1,-1)$, and never going below the horizontal axis. A *hill* is a pair of consecutive steps giving a peak of height 1.



(B) Dyck paths where the first peak, reading from the left, has even height.

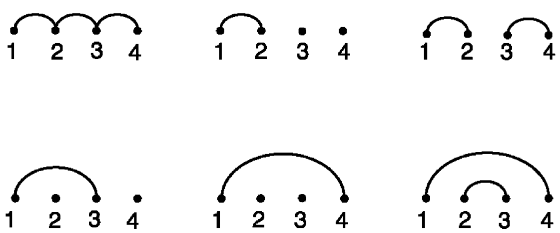


(C) Standard Young tableaux of the shape (n,n) [45], where there is no column of the form $\begin{array}{|c|}\hline k\\\hline k+1\\\hline\end{array}$. Namely,

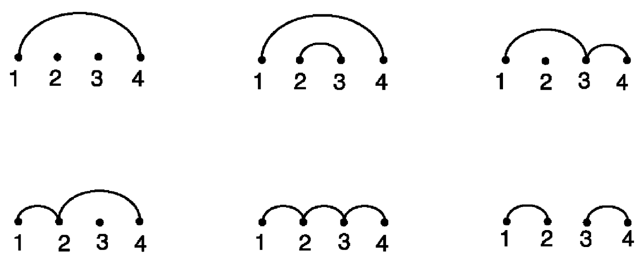
$$\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 5&6&7&8\\\hline\end{array},\quad \begin{array}{|c|c|c|c|}\hline 1&2&3&5\\\hline 4&6&7&8\\\hline\end{array},\quad \begin{array}{|c|c|c|c|}\hline 1&2&3&6\\\hline 4&5&7&8\\\hline\end{array},$$

$$\begin{array}{|c|c|c|c|}\hline 1&2&4&5\\\hline 3&6&7&8\\\hline\end{array},\quad \begin{array}{|c|c|c|c|}\hline 1&2&4&6\\\hline 3&5&7&8\\\hline\end{array},\quad \begin{array}{|c|c|c|c|}\hline 1&2&5&6\\\hline 3&4&7&8\\\hline\end{array}.$$

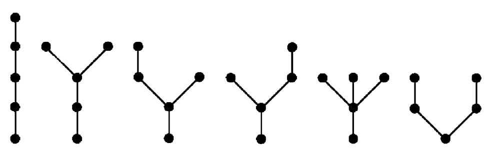
(D) Noncrossing partitions of $[n]$, where the first block has even size. A partition is a collection of nonempty subsets B_i, called blocks, such that $\bigcup B_i = [n]$ and $B_i \cap B_j = \emptyset$ for $i \neq j$. If x and y are elements in the same block, then we write $x \sim y$. A partition is noncrossing if $a < b < c < d$ and $a \sim c$, $b \sim d$ imply $a \sim b \sim c \sim d$. If we connect elements in the same block by arches, the noncrossing condition guarantees that the arches never cross. It is well known that the noncrossing partitions are counted by the Catalan numbers (see, for example, [45]). We use the abbreviation NCP(n) for "noncrossing partition of $[n]$".



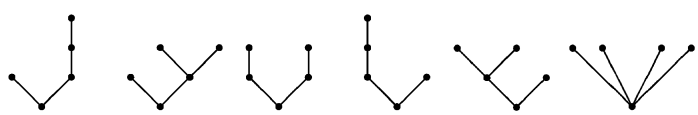
(E) NCP(n) with no visible singletons. A singleton is a block consisting of a single element. It is visible if there are no arches above it.



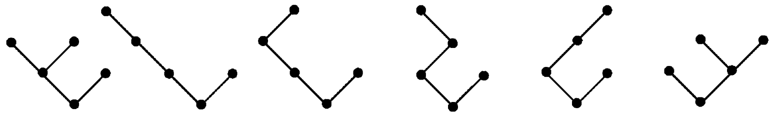
(F) Plane (ordered) trees with no leaves at level 1.



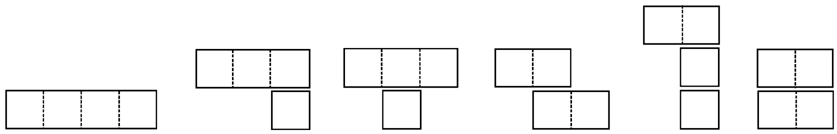
(G) Plane trees with root of even degree.



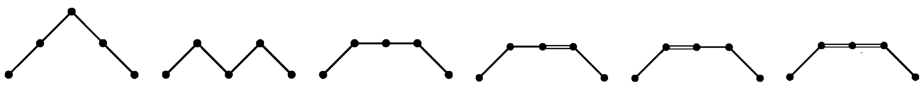
(H) Binary trees (i.e., a rooted tree where each node has either a right child, a left child, both, or neither) where the root and each direct right descendant of the root has outdegree 0 or 2.



(I) Heaps of segments (or stackable $1 \times m$ horizontal rectangles) with a unique left justified maximal (top) piece (these are counted by the Catalan numbers; see, for example, [4,5,48]). The Fine condition is that there are no left justified 1×1 pieces.



(J) Two-Motzkin paths are paths starting and ending on the horizontal axis but never going below it with possible steps $(1,1)$, $(1,0)$, and $(1,-1)$, where the level steps $(1,0)$ can be either of two colors. (Regular Motzkin paths, counted by the Motzkin sequence $(1,1,2,4,9,21,51,\ldots;\mathrm{M}1184)$ [45, p. 238], arise when the level steps have but one color [17]). Use of three colors gives the tree-like polyhexes of Harary and Read [24]. The Fine numbers occur when there are no level steps on the horizontal axis.

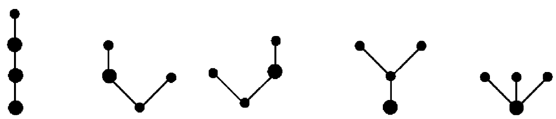


We mention briefly one other occurrence (for details see Section 5).

(K) The total number of nodes of odd outdegree over all plane trees with n edges is

$$\frac{2}{3}\binom{2n-1}{n} + \frac{1}{3}F_{n-1}.$$

For $n=3$ the $\frac{2}{3}10 + \frac{1}{3}1 = 7$ cases are illustrated by



2. A brief history of the Fine numbers

The Fine numbers seem to have first appeared in a paper of Terrence Fine [19] where he studied an abstract theory of interpolation. He considered *similarity relations* (see also [32,35,47]), i.e., relations $\sim$ on the set $[n] = \{1, 2, \dots, n\}$, which are reflexive, symmetric, and such that if $a \sim b$ and $a < x < b$, then $a \sim x$ and $x \sim b$. For instance, assume $n = 5$ and $1 \sim 2, 2 \sim 3, 4 \sim 5$. The diagram is

5				×	×
4				×	×
3		×	×		
2	×	×	×		
1	×	×			
	1	2	3	4	5

The lower boundary is a subdiagonal (Catalan) path, showing that the number of similarity relations is a Catalan number. Without going into details (see [19] for these), it makes intuitive sense that interpolation from a single point is meaningless, so Fine excluded blocks consisting of a single element. The number of similarity relations without singleton blocks is counted by the Fine sequence. We would like to point out, as shown by the above example, that a similarity relation need not be transitive.

The next appearance of the Fine numbers (with the exception of the first two) was as diagonal sums of the Catalan triangle given in Table 3 [40]. The connection between these two appearances would probably not have occurred without Sloane's Handbook of Integer Sequences [42] which had just been published. Properties of the Fine numbers arising from the study of similarity relations are considered also by Strehl [47], Rogers [35], Kim et al. [28], and Moon [32]. The next time the Fine numbers appear in a new context is in a paper by Meir and Moon [30], which we discuss in Section 4. The context is degrees of vertices in plane trees. Much more recent is the paper of Dobrow and Fill [16] which discusses the move-to-root algorithm for binary search trees. The paper [14] considers the enumeration of Dyck paths according to various statistics and the Fine numbers make several appearances. We would appreciate hearing about other appearances of the Fine numbers in the literature. Since the Fine numbers and Catalan numbers are so tightly linked, other occurrences seem likely.

Table 1

$n\backslash s$	0	1	2	3	4	5
0	1	1	1	1	1	1
1	0	1	2	3	4	5
2	0	2	5	9	14	20
3	0	5	14	28	48	75
4	0	14	42	90	165	275
5	0	42	132	297	572	1001

3. Generating functions and Dyck paths

The number of Dyck paths of length $2n$ is the Catalan number $C_n = [1/(n+1)]\binom{2n}{n}$. Recall also that the Catalan generating function $C = C(z) = \sum_{n=0}^{\infty} C_n z^n$ satisfies

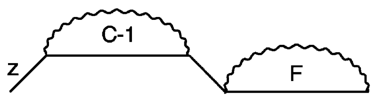
$$C = 1 + zC^2 = \frac{1}{1-zC} = \frac{1-\sqrt{1-4z}}{2z}. \tag{1}$$

If $f(z) = f = \sum_{n=0}^{\infty} f_n z^n$, then let $[z^n]f = f_n$. Another useful fact about the Catalan numbers is that $[z^n]C^s = [s/(2n+s)]\binom{2n+s}{n}$. This can be shown via Lagrange inversion or more simply by induction using the array in Table 1 and the fact that for the ballot numbers $C_{n,s} = [s/(2n+s)]\binom{2n+s}{n}$ the equality $C^s = C^{s-1} + zC^{s+1}$ yields at once the recursion relation $C_{n,s} = C_{n,s-1} + C_{n-1,s+1}$. Some references for the ballot numbers are [7, p. 21; 18, p. 69; 25,39, p. 128].

Let us now define the Fine number F_n as the number of Dyck paths of length $2n$ with no hills. We then have the generating function relation

$$F = 1 + z(C-1)F, \tag{2}$$

where $F = \sum_{n=0}^{\infty} F_n z^n$. To see this, note that 1 counts the empty path that starts and ends at the origin. The picture then is



We have n up steps and n down steps so we can mark each up step with a z. If the path is nonempty, it must start with an up step. The section of the path between the first up step and the first down step returning to the horizontal axis must be a nontrivial Dyck path, so that we do not have a hill. Once the path returns to the horizontal axis, it must again avoid hills, so the generating function for this part of the path is F. More formally, the path has a unique factorization of the form (up step, nontrivial Dyck path, down step, hill-free path), which, following the usual combinatorial interpretation for multiplying generating functions (see, for example, [20,39]) gives $1+z(C-1)F$. Thus,

$$F = 1 + z(C-1)F = 1 + z(zC^2)F,$$

from where

$$F = \frac{1}{1-z^2C^2} = \frac{C}{1+zC}, \tag{3}$$

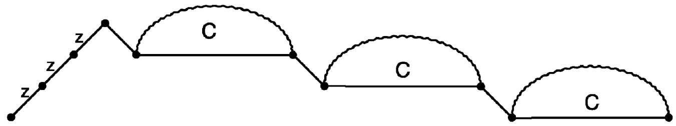
where we have made use of (1). Making use of the last expression in (1), we obtain

$$F = \frac{1}{z}\frac{1-\sqrt{1-4z}}{3-\sqrt{1-4z}}.$$

Proposition 1. *The generating function for Dyck paths whose initial peak is at height k is $z^k C^k$.*

Proof (Idea).

We draw the picture for $k=3$. $\square$



Proposition 2. *The Fine numbers count the number of Dyck paths whose first peak has even height.*

Proof. By Proposition 1 the generating function is

$$1+z^2C^2+z^4C^4+z^6C^6+\cdots=\frac{1}{1-z^2C^2}=F. \qquad \square$$

Proposition 3. *The generating function for Dyck paths whose first peak has odd height is zCF.*

Proof.

$$zC+z^3C^3+z^5C^5+\cdots=zCF. \qquad \square$$

Proposition 4. *The bivariate generating function $\Psi(t,z)$ for Dyck paths by number of hills (marked by t) and number of up steps*[2] *(marked by z) is given by*

$$\Psi(t,z)=\frac{F}{1-tzF}.$$

Proof. The relation

$$\Psi=F+Ftz\Psi$$

can be easily justified. Indeed, the first term in the right-hand side counts the Dyck paths with no hills, while the Dyck paths with at least one hill can be obtained by concatenating a no-hill Dyck path, a hill (the first one; marked by tz), and an arbitrary Dyck path. $\square$

For the number $f_{n,k}$ of Dyck paths of length $2n$ with exactly k hills we obtain the values from Table 2.

[2] The number of up steps of a Dyck path is sometimes called its *semilength*.

Table 2

$n \backslash k$	0	1	2	3	4	5	6
0	1	0	0	0	0	0	0
1	0	1	0	0	0	0	0
2	1	0	1	0	0	0	0
3	2	2	0	1	0	0	0
4	6	4	3	0	1	0	0
5	18	13	6	4	0	1	0
6	57	40	21	8	5	0	1

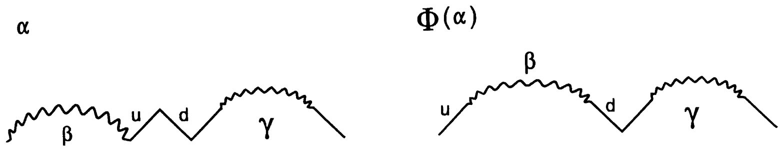


Fig. 1.

The infinite lower triangular matrix $(f_{n,k})_{n,k\geqslant 0}$ is a Riordan array.[3] Namely, $(f_{n,k})_{n,k\geqslant 0}=(F,zF)$.

There is an involution on the set of Dyck paths which will be called the *hill-killer involution*. First we partition the set $\mathscr{D}_n$ of Dyck paths of length $2n$ into

$\mathscr{A}_n=$ set of Dyck paths in $\mathscr{D}_n$ with no hills;

$\mathscr{B}_n=$ set of Dyck paths in $\mathscr{D}_n$ that start with a hill and have no later hills;

$\mathscr{C}_n=$ set of Dyck paths in $\mathscr{D}_n$ that have at least one nonstarting hill.

Note that $|\mathscr{A}_n|=F_n$ and $|\mathscr{B}_n|=F_{n-1}$. We define the mapping $\Phi:\mathscr{B}_n\cup\mathscr{C}_n\to\mathscr{A}_n\cup\mathscr{B}_n$ in the following manner: for a Dyck path $\alpha=\beta ud\gamma\in\mathscr{B}_n\cup\mathscr{C}_n$, where β is a Dyck path, γ is a hill-free Dyck path, while u and d are the steps $(1,1)$ and $(1,-1)$, respectively, we set $\Phi(\alpha)=u\beta d\gamma$. For a pictorial definition see Fig. 1.

Clearly, the restriction of Φ to $\mathscr{B}_n$ is the identity mapping and the restriction of Φ to $\mathscr{C}_n$ is a bijection between $\mathscr{C}_n$ and $\mathscr{A}_n$. Consequently, $|\mathscr{C}_n|=|\mathscr{A}_n|=F_n$ and, therefore,

$$C_n=2F_n+F_{n-1} \quad \text{for } n\geqslant 1. \tag{4}$$

Let $\Phi^{-1}:\mathscr{A}_n\cup\mathscr{B}_n\to\mathscr{B}_n\cup\mathscr{C}_n$ be the inverse mapping of Φ. Now the mapping

$$\psi:\mathscr{D}_n\to\mathscr{D}_n$$

[3] An infinite lower triangular matrix A is called a Riordan array if its bivariate generating function $G(t,z)$ is of the form $G(t,z)=g(z)/(1-th(z))$. We denote $A=(g(z),h(z))$. For more details see [41,44], as well as [26,27,29,37,38]. With an emphasis more on rows and orthogonal polynomials rather than columns, the Riordan group was known as the Scheffer group or umbral group. Several examples go back much further.

defined by

$$\psi(\alpha)=\begin{cases}\Phi^{-1}(\alpha) & \text{if } \alpha\in\mathscr{A}_n,\\ \Phi(\alpha)=\alpha & \text{if } \alpha\in\mathscr{B}_n,\\ \Phi(\alpha) & \text{if } \alpha\in\mathscr{C}_n\end{cases}$$

is an involution on $\mathscr{D}_n$, called the *hill-killer involution*. Its fixed points are the paths in $\mathscr{B}_n$.

Remark. Relation (4) follows at once also from the identity

$$1+C=(2+z)F, \tag{5}$$

which, in turn, is an easy consequence of (1) and (3). Incidentally, relation (5) has an elegant combinatorial proof, suggested by Rogers. Namely, Dyck paths have first peak either of odd or of even height. Consequently, by Propositions 2 and 3, $C=F+zCF$. Now, a little algebra gives $C=F+zF(1+zC^2)=F+zF+z^2FC^2=F+zF+F-1$.

Eq. (4) allows us to prove a useful asymptotic result.

Proposition 5.

$$\frac{F_n}{C_n}\sim\frac{4}{9}. \tag{6}$$

Proof. From $C_n=[1/(n+1)]\binom{2n}{n}$ we obtain at once that $\lim_{n\to\infty}C_{n+1}/C_n=4$. Now, from (4) we have

$$1=2\frac{F_n}{C_n}+\frac{F_{n-1}}{C_n}=2\frac{F_n}{C_n}+\frac{F_{n-1}}{C_{n-1}}\frac{C_{n-1}}{C_n}$$

and letting $\lim_{n\to\infty}F_n/C_n=L$, we have

$$1=2L+\tfrac{1}{4}L=\tfrac{9}{4}L.$$

We have blandly assumed that L exists but the interested reader can fill in this gap by applying, for example, the following theorem found in [2, p. 496].

Suppose that $A(z)=\Sigma a_nz^n$ and $B(z)=\Sigma b_nz^n$ are power series with radii of convergence $\alpha>\beta\geqslant 0$, respectively. Suppose b_{n-1}/b_n approaches the limit b as $n\to\infty$. If $A(b)\neq 0$, then $c_n\sim A(b)b_n$, where $\Sigma c_nz^n=A(z)B(z)$.

We know that $1+C=(z+2)F$ so we can take $A(z)=1/(z+2)$ and $B(z)=1+C$. $\square$

There are about 60 or 70 known appearances of the Catalan numbers, the best reference being Stanley's [45]. Assuming that the hill-killer involution can be translated successfully to them, each should yield a partition $\mathscr{C}_n=\mathscr{F}_n\cup\mathscr{F}_n^*\cup\mathscr{F}_{n-1}$, where there is an involution ψ on $\mathscr{C}_n$ interchanging $\mathscr{F}_n$ and $\mathscr{F}_n^*$ and fixing $\mathscr{F}_{n-1}$, with $|\mathscr{C}_n|=C_n$, $|\mathscr{F}_n|=|\mathscr{F}_n^*|=F_n$, and $|\mathscr{F}_{n-1}|=F_{n-1}$.

For example, returning to the (n,n) standard Young tableau, example (C) in Section 1, assume first that the tableau has at least one column of consecutive integers and let

$$\begin{array}{|c|}\hline 2k-1 \\ \hline 2k \\ \hline\end{array}$$

be the last such column. Then we define

$$\psi\left(\begin{array}{|c|c|c|c|c|c|c|c|}\hline a_1 & a_2 & \cdots & a_{k-1} & 2k-1 & c_1 & \cdots & c_{n-k} \\ \hline b_1 & b_2 & \cdots & b_{k-1} & 2k & d_1 & \cdots & d_{n-k} \\ \hline\end{array}\right)$$

$$=\begin{array}{|c|c|c|c|c|c|c|c|c|}\hline 1 & a_1+1 & a_2+1 & \cdots & a_{k-2}+1 & a_{k-1}+1 & c_1 & \cdots & c_{n-k} \\ \hline b_1+1 & b_2+1 & b_3+1 & \cdots & b_{k-1}+1 & 2k & d_1 & \cdots & d_{n-k} \\ \hline\end{array}.$$

If the tableau has no column of consecutive integers, then let k be the smallest positive integer such that $2k$ is in the kth column. Then we define

$$\psi\left(\begin{array}{|c|c|c|c|c|c|c|c|}\hline a_1 & a_2 & \cdots & a_{k-1} & a_k & c_1 & \cdots & c_{n-k} \\ \hline b_1 & b_2 & \cdots & b_{k-1} & 2k & d_1 & \cdots & d_{n-k} \\ \hline\end{array}\right)$$

$$=\begin{array}{|c|c|c|c|c|c|c|c|c|}\hline a_2-1 & a_3-1 & a_4-1 & \cdots & a_k-1 & 2k-1 & c_1 & \cdots & c_{n-k} \\ \hline b_1-1 & b_2-1 & b_3-1 & \cdots & b_{k-1}-1 & 2k & d_1 & \cdots & d_{n-k} \\ \hline\end{array}.$$

The fixed points have first column $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}$ and no column of consecutive numbers thereafter. For $n=3$ we have

$$\begin{array}{|c|c|c|}\hline 1&3&5\\ \hline 2&4&6\\ \hline\end{array} \leftrightarrow \begin{array}{|c|c|c|}\hline 1&2&4\\ \hline 3&5&6\\ \hline\end{array}$$

$$\begin{array}{|c|c|c|}\hline 1&2&5\\ \hline 3&4&6\\ \hline\end{array} \leftrightarrow \begin{array}{|c|c|c|}\hline 1&2&3\\ \hline 4&5&6\\ \hline\end{array}$$

$$\begin{array}{|c|c|c|}\hline 1&3&4\\ \hline 2&5&6\\ \hline\end{array} \hookleftarrow$$

4. A compendium of formulas

In this section we present a collection of formulas involving either the Fine function $F(z)$ or the Fine numbers F_n. To most of those that were not derived in the previous section, we indicate references or we sketch the proof:

$$F(z)=\frac{1}{z}\frac{1-\sqrt{1-4z}}{3-\sqrt{1-4z}},$$

$$F(z)=\frac{1+2z-\sqrt{1-4z}}{2z(2+z)},$$

$$F=\frac{1}{1-z^2C^2}=\frac{C}{1+zC},$$

$$C = \frac{F}{1 - zF},$$

$$(2 + z)F = C + 1,$$

$$3BF = 2BC + F,$$

where $B = 1/\sqrt{1 - 4z}$,

$$2F_n + F_{n-1} = C_n$$

(see [14,32,35,40,47]).

$$2(n+1)F_n = (7n - 5)F_{n-1} + 2(2n - 1)F_{n-2}, \quad n \geqslant 2$$

(use the previous relation and the recurrence relation $(n+1)C_n = 2(2n-1)C_{n-1}$);

$$\frac{F_n}{C_n} \sim \frac{4}{9},$$

$$F_n \sim \frac{4^{n+1}}{9n\sqrt{n\pi}}$$

(in the previous relation use Stirling's formula for the factorials hidden in C_n);

$$F_n = \frac{1}{n+1}\left[\binom{2n}{n} - 2\binom{2n-1}{n} + 3\binom{2n-2}{n} - \cdots + (-1)^n(n+1)\binom{n}{n}\right]$$

(see [14,32]);

$$F_n = \frac{1}{n-1}\binom{2n-2}{n} + \frac{2}{n-2}\binom{2n-4}{n} + \frac{3}{n-3}\binom{2n-6}{n} + \cdots \quad \text{if } n \geqslant 2$$

(see [14,32,40,47]);

$$F_n = \frac{1}{2}\left[C_n - \frac{1}{2}C_{n-1} + \frac{1}{2^2}C_{n-2} - \cdots + (-1)^{n-2}\frac{1}{2^{n-2}}C_2\right]$$

(see [14,32,35,40]);

$$F_n = \sum_{\substack{i_1+i_2+\cdots+i_q+q=n,\\ i_1>0,\ldots,i_q>0,q>0}} C_{i_1}C_{i_2}\ldots C_{i_q}$$

(see [40]);

$$F_n = (-1)^n \begin{vmatrix} C_0 & C_1 & C_2 & \cdots & C_{n-1} & C_n \\ 1 & C_0 & C_1 & \cdots & C_{n-2} & C_{n-1} \\ 0 & 1 & C_0 & \cdots & C_{n-3} & C_{n-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & C_0 \end{vmatrix}$$

(see [14]);

$$\begin{vmatrix} F_0 & F_1 & F_2 & \cdots & F_{n-1} & F_n \\ F_1 & F_2 & F_3 & \cdots & F_n & F_{n+1} \\ F_2 & F_3 & F_4 & \cdots & F_{n+1} & F_{n+2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ F_n & F_{n+1} & F_{n+2} & \cdots & F_{2n-1} & F_{2n} \end{vmatrix} = 1,$$

$$\begin{vmatrix} F_1 & F_2 & F_3 & \cdots & F_{n-1} & F_n \\ F_2 & F_3 & F_4 & \cdots & F_n & F_{n+1} \\ F_3 & F_4 & F_5 & \cdots & F_{n+1} & F_{n+2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ F_n & F_{n+1} & F_{n+2} & \cdots & F_{2n-2} & F_{2n-1} \end{vmatrix} = 1 - n.$$

Let $\sigma = (s_0, s_1, s_2, \ldots)$ be a sequence of nonnegative integers. Consider all paths starting from the origin $(0,0)$, with diagonal steps $(1,1)$, $(1,-1)$ and s_i types of horizontal steps $(1,0)$ at height i, and never going below the horizontal axis. Then the numbers a_n of these paths ending at $(n,0)$ are called *Catalan-like numbers of type* σ (see [1]).[4] Clearly, for $\sigma = (1,1,1,\ldots)$ we obtain the Motzkin numbers and for $\sigma = (0,0,0,\ldots)$ we obtain the aerated Catalan numbers $1,0,1,0,2,0,5,\ldots$.

From our example (J) of Section 1 it follows that the Fine numbers are Catalan-like numbers of type $0,2,2,2,\ldots$. Now, Propositions 6 and 7 of [1] yield the above two results.

5. Path pairs

In this section we want to discuss a lesser known setting for the Catalan numbers which is in some sense a home for most of the Catalan relatives such as Motzkin numbers (M1184) [45, p. 238], γ numbers (M2587), [3,34] (called Riordan numbers in [3]), Schröder numbers (large (M1659) and small (M2898)) [45, p. 178], and so on. It also leads to a very natural way to see the Fine numbers and another instance of the $C_n = 2F_n + F_{n-1}$ decomposition.

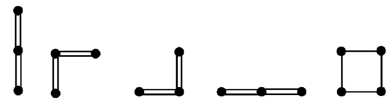
Definition 1. The set of all path pairs (APP) is the set of all pairs of paths such that

(A) both paths are composed of unit east and north steps;

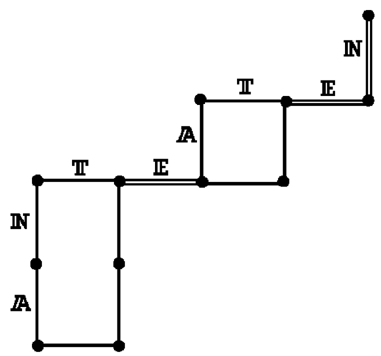
[4] In [1] there is a different definition; however, this one, mentioned in [1], is more suitable for our purposes.

(B) both paths start at $(0,0)$ and have a common endpoint;
(C) the upper path never goes strictly below the lower path.

Here are the 5 path pairs of length 2.



We can encode a path pair as follows:
$\mathbb{A} =$ step apart, upper path goes north, lower path east;
$\mathbb{T} =$ step together, upper path goes east, lower path north;
$\mathbb{E} =$ both steps east;
$\mathbb{N} =$ both steps north.
For example, $\mathbb{ANTEATEN}$ is the code for the path pair below:



Reading from the left we must never have more $\mathbb{T}$'s than $\mathbb{A}$'s and at the end the number of $\mathbb{T}$'s is the same as the number of $\mathbb{A}$'s.

By a *joint* step of a path pair we mean a pair of superposed steps (one from each path of the pair). For example, in the path pair $\mathbb{ANTEATEN}$ both $\mathbb{E}$'s and the last $\mathbb{N}$ represent joint steps.

Proposition 6. *If* APP(n) *is the set of path pairs of length* n, *then* $|\mathrm{APP}(n)| = C_{n+1}$.

Proof. There is a simple bijection between APP(n) and the set of Dyck paths of length $2n+2$.[5] As before, let u and d denote the steps $(1,1)$ and $(-1,1)$, respectively. Then we convert a path pair to a Dyck path by converting $\mathbb{A} \mapsto uu$, $\mathbb{T} \mapsto dd$, $\mathbb{E} \mapsto du$, and $\mathbb{N} \mapsto ud$ and then adding a u to the start and a d to the end of the resulting word. $\square$

Remark. Let us remove the condition on the APPs that the two paths have a common endpoint. Let $a_{n,k}$ be the number of such path pairs of length n having endpoints $k\sqrt{2}$

[5] This bijection was suggested by one of the referees.

Table 3

$n\backslash k$	0	1	2	3	4	5
0	1					
1	2	1				
2	5	4	1			
3	14	14	6	1		
4	42	48	27	8	1	
5	132	165	110	44	10	1

apart. It is easy to see that

$$a_{n+1,k} = a_{n,k-1} + 2a_{n,k} + a_{n,k+1} \quad \text{for } k \geqslant 1, \tag{7}$$

since the two paths can get one step further apart ($\mathbb{A}$), both go $\mathbb{E}$ or $\mathbb{N}$, or become one step closer ($\mathbb{T}$). Similarly, $a_{n+1,0} = 2a_{n,0} + a_{n,1}$. Using these recursions we can verify by induction that

$$a_{n,k} = \frac{k+1}{n+1}\binom{2n+2}{n-k}. \tag{8}$$

When $k=0$, we recover APP and we obtain again $|\mathrm{APP}(n)| = [1/(n+1)]\binom{2n+2}{n} = C_{n+1}$. The first few values of $a_{n,k}$ (these are again ballot numbers) are given in Table 3.

Remark. The matrix $A = (a_{n,k})_{n,k\geqslant 0}$ is a Riordan array and can be written $A = (C^2, zC^2)$. Indeed, this is guaranteed by the relation (7) (see [31,36]) leading to $A = (C^2, zC^2)$. This, in turn, yields the following two properties of the matrix A (see [44, pp. 269–270]) : (i) the generating function of the diagonal sums of A is $C^2/(1 - z^2C^2) = (F-1)/z^2$ and thus, these diagonal sums are Fine numbers (this property of the matrix A can be found in [40]); (ii) the generating function of the alternating row sums of the matrix A is $C^2/(1+zC^2) = C$.

Closely related to the APP is the subset of path pairs meeting only at the origin and the endpoint. We will call these *fat path pairs* (denoted FPP) but they are usually called parallelogram polyominoes and there is an extensive literature on them. See, for example, [4,10] and the many references there.

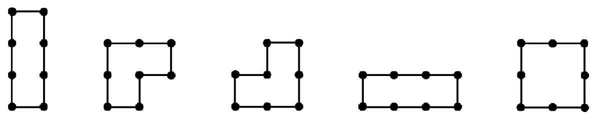
There is a straightforward bijection between APP(n) and FPP($n+2$). Namely, take a path in APP(n) and add an $\mathbb{A}$ at the beginning and a $\mathbb{T}$ at the end. Returning to our example following the definition of APP, we have

$$\mathrm{APP}(2) = \{\mathbb{NN}, \mathbb{NE}, \mathbb{EN}, \mathbb{EE}, \mathbb{AT}\},$$

which becomes

$$\mathrm{FPP}(4) = \{\mathbb{ANNT}, \mathbb{ANET}, \mathbb{AENT}, \mathbb{AEET}, \mathbb{AATT}\}$$

or, equivalently,



With the extra two steps, the generating function for FPP is z^2C^2.

By placing other restrictions or modifications on the APP, we obtain interesting sequences.

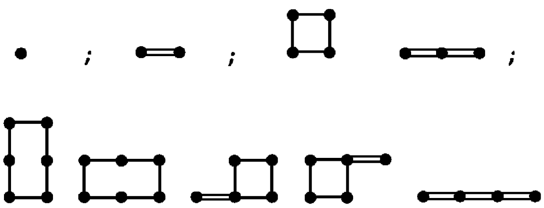
(A) No $\mathbb{E}$ steps. This yields the Motzkin numbers 1, 1, 2, 4, 9, 21, 51, 127,... .

(B) No $\mathbb{E}$ or $\mathbb{N}$ steps. Aerated Catalan numbers 1, 0, 1, 0, 2, 0, 5, 0, 14, 0, 42,... .

(C) Bicolored columns (i.e. each column, including columns of height 0, is either green or red). Small Schröder numbers 1, 3, 11, 45, 197,... (M2898). (If only parallelogram polyominoes are considered, then the sequence 1, 2, 6, 22, 90, 394,... (M1659) results which is the sequence of large Schröder numbers).

(D) Removing the restriction that the upper path never goes strictly below the lower path, we obtain the central binomial coefficients 1, 2, 6, 20, 70, 252,... (M1645).

(E) To realize the Catalan sequence 1, 1, 2, 5, 14, 42,... itself the appropriate condition is: no joint $\mathbb{N}$'s (i.e., no $\mathbb{N}$ steps are possible when the two paths are together.)



(F) Let the paths cross and bicolor the columns again, including columns of height 0. This yields the central Delannoy sequence 1, 3, 13, 63, 321,... (M2942; see [7, p. 81; 45, p. 185] for more details).

The case of greatest interest for this paper is as follows:

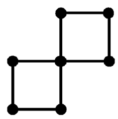
Proposition 7. *The number of path pairs of length n with no joint steps is the Fine number F_n.*

Proof. We consider again the bijection between APP(n) and the set of Dyck paths of length $2n+2$, defined in the proof of Proposition 6. It is easy to see that, in this bijection, to joint $\mathbb{E}$ steps there correspond valleys at level 0 and to joint $\mathbb{N}$ steps there correspond peaks at level 2. Consequently, to a path pair of length n with no joint steps there correspond elevated Dyck paths (i.e., with the exception of the end-points, they stay strictly above the horizontal axis) of length $2n+2$, with no peaks at level 2. Removing the first and the last step of this Dyck path, we obtain a Dyck path of length $2n$ with no hills. $\square$

Remark. As an alternate proof, we can view these path pairs (called Fine pairs) as concatenated fat pairs and so the appropriate generating function is

$$1 + z^2C^2 + (z^2C^2)^2 + (z^2C^2)^3 + \cdots = \frac{1}{1 - z^2C^2} = F.$$

For $n = 4$ we have the 5 fat path pairs above and the sixth is $\mathbb{A}\mathbb{T}\mathbb{A}\mathbb{T}$ i.e.,



We get a pretty picture for a decomposition $C_{n+1} = 2F_{n+1} + F_n$ in this setting. We have $|\mathrm{APP}(n)| = C_{n+1}$ and the Fine pairs are counted by F_n. Consequently, the other path pairs, namely those that have at least one joint step, are counted by $2F_{n+1}$. For example, two disjoint sets counted by F_{n+1} are

$\mathscr{F}^{\mathbb{E}}_{n+1}$ = all path pairs whose first joint step is $\mathbb{E}$,

$\mathscr{F}^{\mathbb{N}}_{n+1}$ = all path pairs whose first joint step is $\mathbb{N}$.

The "hill-killer" involution takes the first joint $\mathbb{E}$ step or joint $\mathbb{N}$ step and toggles $\mathbb{E}$ and $\mathbb{N}$.

It should be noted that there is a very simple bijection between APP and the 2-Motzkin paths of example (I). Namely, *up* $\leftrightarrow$ $\mathbb{A}$, *down* $\leftrightarrow$ $\mathbb{T}$, and the 2 colors for the level steps correspond to $\mathbb{E}$ and $\mathbb{N}$.

6. Odd blocks

In this section we find the total number of blocks of odd size counting over all the C_n possible noncrossing partitions of $[n]$.

Lemma 1.

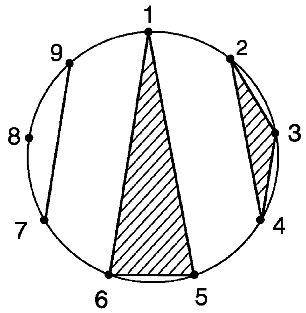
$$[z^n](zBF) = \frac{2}{3}\binom{2n-1}{n} + \frac{1}{3}F_{n-1}.$$

Proof. We need the equality $3BF = 2BC + F$ (see Section 4; it follows at once from the definitions of B, C, and F, namely $B = 1/q$, $C = (1-q)/2z$, $F = (1-q)/z(3-q)$ where $q = \sqrt{1-4z}$). Then, making use of the well-known equality $[z^n]BC^s = \binom{2n+s}{n}$ (see, for example, [33, p. 154]; [49, p. 54]),[6] we have

$$[z^n](zBF) = \frac{2}{3}[z^{n-1}](BC) + \frac{1}{3}[z^{n-1}]F = \frac{2}{3}\binom{2n-1}{n} + \frac{1}{3}F_{n-1}. \qquad \square$$

[6] The following combinatorial proof has been supplied by one of the referees. Every path from (0,0) to $(n, n+s)$ (with (1,0) and (0,1) steps) can be factored by cutting it at the last time it returns to the main diagonal; the first part is counted by B and the second part can be further factored into s parts (each going up by "one diagonal"), each counted by C.

We now return to noncrossing partitions which we now view as points 1 through n arranged in order around a circle. The noncrossing condition can be viewed as follows. If for each block we form the convex hull generated by the points in the block, then these convex hulls must be disjoint. Here is an example.



1 5 6 | 2 3 4 | 7 9 | 8

How many NCP's have 1 as a singleton block? The appropriate generating function is zC with z marking the point 1 and C accounting for all NCP's on the remaining elements.

The next question is: how many singleton blocks are there counting over all of NCP(n)?

Lemma 2. *The generating function for the total number of singleton blocks in all* NCP'*s is zB and thus there are $\binom{2n-2}{n-1}$ singleton blocks arising from* NCP(n).

Proof. If the generating function of a sequence $(a_n)_{n\geqslant 0}$ is $f(z)$, then, obviously, the generating function of the sequence $(na_n)_{n\geqslant 0}$ is $z\mathrm{d}f/\mathrm{d}z$. We have just seen that 1 is a singleton block in $[z^n](zC)$ NCPs. The same holds for each of the points $1, 2, \ldots, n$ and, consequently, the number of singleton blocks in all NCP(n)s is $n[z^n](zC) = [z^n]z(zC)' = [z^n](zB)$. $\square$

Let $\mathcal{O}_n$ denote the total number of blocks of odd size over all the C_n possible noncrossing partitions of $[n]$ and let $\mathcal{O} = \mathcal{O}(z)$ be the corresponding generating function, i.e., $\mathcal{O} := \sum_{n=0}^{\infty} \mathcal{O}_n z^n$.

Theorem 1. *The generating function $\mathcal{O}$ for the total number of odd blocks in all of* NCP *is zBF and the total number $\mathcal{O}_n$ of odd blocks in all of* NCP(n) *is* $[z^n]zBF = \frac{2}{3}\binom{2n-1}{n} + \frac{1}{3}F_{n-1}$.

Proof. In Lemma 2 we have seen that the generating function for the total number of singleton blocks in all of NCP is zB.

Next we ask for the number of NCPs where 1 is in a block of size 3. The picture in Fig. 2 indicates that the appropriate generating function is z^3C^3.

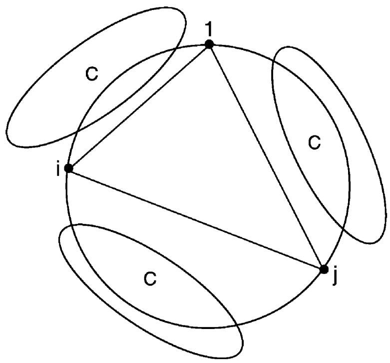


Fig. 2.

If we want to count the total number of blocks of size 3 in all NCP's, then the generating function is

$$z\frac{(z^3C^3)'}{3}=z(zC)^2B.$$

Similar reasoning shows that the generating function counting blocks of size $2m+1$ is $z(zC)^{2m}B$. Now

$$\mathcal{O}=zB+(zC)^2zB+(zC)^4zB+\cdots=zB\frac{1}{1-z^2C^2}=zBF.$$

The second statement of the theorem follows from Lemma 1. □

It turns out that $\frac{2}{3}\binom{2n-1}{n}+\frac{1}{3}F_{n-1}$ is also the answer to the question of how many vertices of odd outdegree are there in all plane trees with n edges. Noting that $zBF=z+2z^2+7z^3+24z^4+\cdots$, the seven cases for $n=3$ are illustrated in Section 1, example (K).

There is a lovely bijection of Dershowitz and Zaks [12] between plane trees and noncrossing partitions which leads to an immediate proof. Number the nodes in preorder (the worm climbs the tree, as Martin Gardner puts it) starting by labeling the root by 0. Then, looking on this as a family tree, put siblings in the same block. An example is given in Fig. 3. Using this correspondence, we see directly that a node of odd outdegree corresponds directly to a block of odd size, the set of its descendants.

This result is closely related to a result of Meir and Moon [30]:

Proposition 8. *The total number of nodes of odd degree* (*i.e., outdegree* $+1$, *except at the root*) *over all plane trees with n edges is*

$$\frac{4}{3}\binom{2n-1}{n}+\frac{2}{3}F_{n-1}=2\mathcal{O}_n.$$

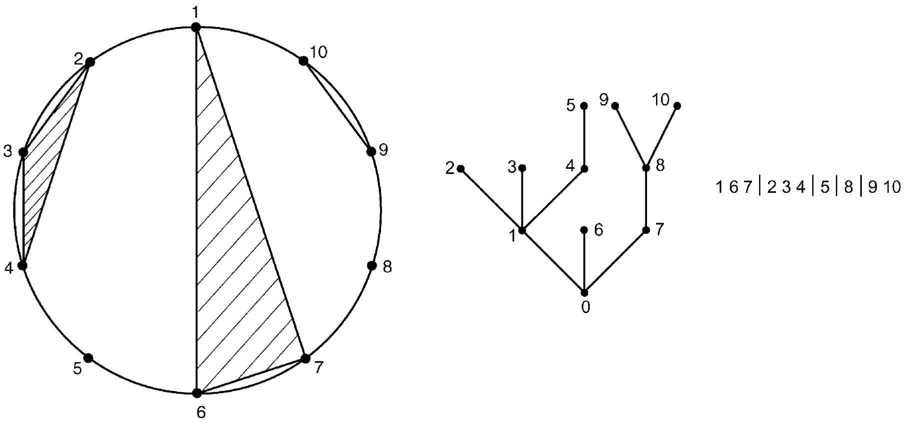
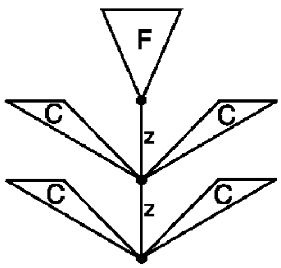


Fig. 3.

Proof (Outline). (i) The generating function for the total number of roots of odd degree in all plane trees is zCF.

(ii) The generating function for the total number of nonroot nodes of even outdegree (and thus of odd degree), at height $k \geqslant 1$ in all plane trees is $z^k C^{2k} F$. The picture for $k=2$ is



(iii) Thus, the generating function is

$$zCF + \sum_{k=1}^{\infty} Fz^k C^{2k} = zCF + F\frac{zC^2}{1 - zC^2} = zCF + zFCB$$

$$= zCF(1 + B) = 2zFB = 2\mathcal{O}.$$

This requires two identities, $C/(1 - zC^2) = B$ and $C(1 + B) = 2B$, along the way. These are easily established and we omit their verification. □

It would be good to have a direct two-to-one correspondence but we have not found one.

7. Fine path statistics

A *Fine path* is a Dyck path without hills. In this section we will prove some results about the average behavior of certain statistics. We also compare the results for Fine paths and Dyck paths. We are assuming that all paths are equally likely to be chosen.

First, we consider the statistic *number of returns*, denoted X_{R}. A return (to the horizontal axis) consists of a nontrivial path, a point on the horizontal axis, and another path (possibly trivial). Then the generating function for the total number of returns of all Dyck paths of a given length is

$$(C-1)C = C^2 - C = \frac{1}{z}C - \frac{1}{z} - C.$$

To find the expected number of returns, we divide the total number of returns by the total number of paths. We obtain

$$E(X_{\mathrm{R}}) = \frac{C_{n+1} - C_n}{C_n} = \frac{3n}{n+2} \to 3.$$

Dyck paths are closely related to plane trees. Through a well-known bijection (the "glove" bijection), the number of returns of a Dyck path corresponds to the degree of the root of the corresponding tree. In this case the last result goes back to Dershowitz and Zaks [11], namely,

$$E(\text{degree of the root}) = \frac{3n}{n+2}.$$

In the same way, for Fine paths we obtain the generating function $(F-1)F$. However, in this case exact results in closed form do not seem to exist and so we settle for asymptotic results as n gets large. Denote $[z^n]F^2 = g_n$. Since, as one can easily check, $z(z+2)F^2 - (1+2z)F + 1 = 0$, we have

$$g_{n-2} + 2g_{n-1} = F_n + 2F_{n-1} \quad \text{for } n \geqslant 1.$$

Dividing by F_{n-1}, yields

$$\frac{g_{n-2}}{F_{n-2}}\frac{F_{n-2}}{F_{n-1}} + 2\frac{g_{n-1}}{F_{n-1}} = \frac{F_n}{F_{n-1}} + 2$$

and now taking limits and denoting $\lim_{n\to\infty} g_n/F_n = L$, yields $\frac{1}{4}L + 2L = 4 + 2$, from where $L = 8/3$. We again have casually assumed that $\lim_{n\to\infty} g_n/F_n$ exists. Tannery's theorem [6] could be applied to fill in the gap. Now

$$E(X_{\mathrm{R}}) = \frac{[z^n](F^2 - F)}{[z^n]F} \to \frac{8}{3} - 1 = \frac{5}{3}.$$

Next, we consider the statistic *height of the first peak*, denoted X_{H}. The generating function for Dyck paths whose initial peak is at height k is $z^k C^k$ (see Proposition 1). Then the generating function for the sum of the heights of the first peaks of all Dyck

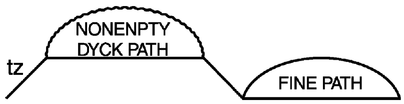


Fig. 4.

paths of length $2n$ is

$$zC + 2z^2C^2 + 3z^3C^3 + \cdots = \frac{zC}{(1-zC)^2} = zC^3 = C^2 - C. \tag{9}$$

This is the same as the result obtained at the statistic "number of returns" and, therefore,

$$E(X_{\mathrm{H}}) = \frac{3n}{n+2} \to 3.$$

Alternatively, there are several bijections on the set of Dyck paths that show that the statistics "height of first peak" and "number of returns" are equidistributed (for recent examples see [13,15]).

For Fine paths we might expect the number of returns to be smaller (as we saw 5/3 compared to 3) while the height of the first peak to be larger than for Dyck paths. We proceed as in the case of Dyck paths. The generating function for Fine paths whose initial peak is at height k is $z^kC^{k-1}F$. Then the generating function for the sum of the heights of the first peaks of all Fine paths of length $2n$ is

$$\begin{aligned}
&2z^2CF + 3z^3C^2F + 4z^4C^3F + \cdots \\
&\quad = \frac{z^2CF(2-zC)}{(1-zC)^2} \\
&\quad = z^2C^3F(2-zC) = z^2C^2F(1+C) = zF(C+1)(C-1) \\
&\quad = zF(C^2-1) = F(C-1) - zF = \frac{1}{z}F - \frac{1}{z} - zF.
\end{aligned} \tag{10}$$

Here we have used repeatedly relations (1) and for the last equality relation (2). Now

$$E(X_{\mathrm{H}}) = \frac{F_{n+1} - F_{n-1}}{F_n} \to 4 - \frac{1}{4} = \frac{15}{4},$$

since from Proposition 5 it follows that $\lim_{n\to\infty} F_{n+1}/F_n = 4$.

Alternatively, we can obtain the generating functions for the sum of the heights of the first peaks for all Dyck/Fine paths of length $2n$ from the corresponding bivariate generating functions. Marking each up step by z and each up step before the first peak by t, for Dyck paths we obtain the bivariate generating function

$$\Delta(t,z) = \Delta = \frac{1}{1-tzC} \tag{11}$$

(see, for example, [14] for details). For Fine paths from Fig. 4 we obtain the generating function

$$\Omega(t,z) = \Omega = 1 + tz(\Delta - 1)F = 1 + \frac{t^2z^2C}{1-tzC}F. \tag{12}$$

Table 4

$n\backslash k$	0	1	2	3	4	5	6	7
0	1							
1	0	0						
2	0	0	1					
3	0	0	1	1				
4	0	0	3	2	1			
5	0	0	8	6	3	1		
6	0	0	24	18	10	4	1	
7	0	0	75	57	33	15	5	1

Differentiating (11) and (12) with respect to t and setting $t=1$, we obtain again the expressions in (9) and (10), respectively. From (12) we obtain $[t^k]\Omega = z^k C^{k-1} F$ (as before) and then

$$[t^k z^n]\Omega = [z^{n-k}]C^{k-1}F = \sum_{\alpha=0}^{\lfloor (n-k)/2 \rfloor} \frac{k-1+2\alpha}{2n-k-1-2\alpha} \binom{2n-k-1-2\alpha}{n-1}$$

(see formula (B7) in [14]). The first few values are given in Table 4.

The row sums are the Fine numbers, while deleting the first two rows and columns yields the Riordan array (CF, zC). This follows at once from $[t^k]\Omega = z^k C^{k-1} F$.

Now we consider the statistic *number of peaks*, denoted X_{P}. We use the method of bivariate generating functions. Let $\Lambda(t,z)$ be the bivariate generating function for Dyck paths according to semilength (marked by z) and number of peaks (marked by t). From a picture similar to that in Fig. 4 we obtain at once

$$\Lambda = 1 + z(\Lambda - 1 + t)\Lambda$$

and, taking into account that $\Lambda(1,z) = C$, by implicit differentiation we obtain

$$\left(\frac{\partial \Lambda}{\partial t}\right)_{t=1} = \frac{zC}{1-2zC} = zBC = \frac{B-1}{2},$$

where again $B = 1/\sqrt{1-4z} = \sum_{n=0}^{\infty} \binom{2n}{n} z^n$. Thus

$$E(X_{\mathrm{P}}) = \frac{1}{C_n}[z^n]\left(\frac{\partial \Lambda}{\partial t}\right)_{t=1} = \frac{\frac{1}{2}\binom{2n}{n}}{C_n} = \frac{n+1}{2}.$$

This is a known result (see, for example, [9,11,14]).[7]

In the case of Fine paths, in a similar manner, for the bivariate generating function $\Gamma(t,z)$ we find

$$\Gamma = 1 + z(\Lambda - 1)\Gamma,$$

[7] A simple combinatorial proof has been supplied by one of the referees: just take into account that the total number of Dyck paths of length $2n$ having a peak with a prescribed abscissa is C_{n-1} and there are $2n-1$ possible abscissae for a peak.

and, taking into account that $\Gamma(1,z)=F$, by implicit differentiation we obtain

$$\left(\frac{\partial \Gamma}{\partial t}\right)_{t=1} = z^2BCF^2 = \frac{1}{2}zF',$$

where the last equality follows, for example, by differentiating the relation $1/F = 1 - z^2C^2$ (see (3)). Now,

$$[z^n]\left(\frac{\partial \Gamma}{\partial t}\right)_{t=1} = \frac{1}{2}nF_n$$

and then

$$E(X_{\mathrm{P}}) = \frac{n}{2}.$$

8. A partial list of Fine number occurrences

We conclude with a partial list of occurrences of the Fine numbers. The criterion for inclusion was a very brief description. The interested reader can consider them as exercises. In some of the examples there are variations with the initial terms. We would like to mention that an interesting occurrence in the setting of staircase polyominoes had also been developed by one of the referees:

Dyck paths with no hills.
Dyck paths with leftmost peak of even height.
Dyck paths with an even number of returns.
Dyck paths with no hills (i.e. Fine paths) with leftmost peak of height 3.
Plane trees with no leaves at level 1.
Plane trees with root of even degree.
Plane trees with no node of outdegree 1 on the leftmost path.
Plane trees with root of degree 3 and no node of outdegree 1 on the leftmost path.
Plane trees with no leaves at level 1 and leftmost leaf is at level 3,
Plane trees with root of degree at least two and leftmost subtree has no leaf at level 1.
Plane trees in which the leftmost subtree has a leaf at level 1.
Plane trees having the leftmost leaf at even level.
Plane trees having at least one leaf at level 1 that is not the rightmost child of the root.
Noncrossing partitions with no visible singletons.
Noncrossing partitions with an even number of visible blocks.
Noncrossing partitions with no visible singletons and first block has size 3.
Noncrossing partitions in which the size of the first block is even.
Noncrossing partitions in which the first block has at least two consecutive points.
Noncrossing partitions in which the first point where a block ends is even.

Noncrossing partitions in which the first block has no cyclically consecutive points (i.e. consecutive in the circular representation).
Two-Motzkin paths with no level steps at level zero.
Two-Motzkin paths having a red level step that precedes all green level steps and all down steps.
Two-Motzkin paths with an odd number of red level steps at level zero.
Two-Motzkin paths with no red level steps at the beginning or at the end and having no consecutive red level steps at level zero.
APPs with no joint steps.
APPs with an odd number of joint $\mathbb{E}$ steps.
APPs with no joint $\mathbb{E}$ step at the beginning or at the end and having no consecutive joint $\mathbb{E}$ steps.
APPs having an $\mathbb{N}$ step that precedes all $\mathbb{E}$ and $\mathbb{T}$ steps.
Parallelogram polyominoes with no columns of height 1.
Parallelogram polyominoes in which the number of edges shared by two consecutive columns or two consecutive rows is at least two.
321-avoiding permutations without fixed points.

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