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ON THE q-BESSEL FOURIER TRANSFORM

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Abstract. In this work, we are interested by the q-Bessel Fourier transform with a new approach. Many important results of this q-integral transform are proved with a new constructive demonstrations and we establish in particular the associated q-Fourier-Neumen expansion which involves the q-little Jacobi polynomials.

1. Introduction

In the recent mathematical literature one finds many articles which deal with the theory of q-Fourier analysis associated with the q-Hankel transform. This theory was elaborated first by Koornwinder and R.F. Swarttouw [12] and then by Fitouhi and Al [5, 8].
It should be noticed that in [5] we provided the mains results of q-Fourier analysis in particular that the q-Hankel transform is extended to the $\mathcal{L}_{q,2,\nu}$ space like an isometric operator. Often we use the crucial properties namely the positivity of the q-Bessel translation operator to prove some results but these last property is not ensured for any q in the interval $]0,1[$. Thus, we will prove some main results of q-Fourier analysis without the positivity argument especially the following statments:
- Inversion Formula in the $\mathcal{L}_{q,p,\nu}$ spaces with $p \geq 1$.
- Plancherel Formula in the $\mathcal{L}_{q,p,\nu} \cap \mathcal{L}_{q,1,\nu}$ spaces with $p > 2$.
- Plancherel Formula in the $\mathcal{L}_{q,2,\nu}$ spaces.

Note that in the paper [7] we have proved that the positivity of the q-Bessel translation operator is ensured in all points of the interval $]0,1[$ when $\nu \geq 0$. In this article we will try to show in a clear way the part in which the positivity of the q-Bessel translation operator plays a role in q-Bessel Fourier analysis. In particular, when we try to prove a q-version of the Young's inequality for the associated convolution.

Many interesting result about the uncertainty principle for the q-Bessel transform was proved in the last years. We cite for examples [2, 3, 4, 9]. There are some differences of the results cited above and our result:

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In this paper the Heisenberg uncertainty inequality is established for functions in $\mathcal{L}_{q,2,\nu}$ space.
The Hardy's inequality discuss here is a quantitative uncertainty principles which give an information about how a function and its q-Bessel Fourier transform are linked.

In the end of this paper we use the remarkable work in [1] to establish a new result about the q-Fourier-Neumen expansion involving the q-little Jacobi polynomials.

2. The q-Bessel transform

The reader can see the references [10, 11, 16] about q-series theory. The references [5, 8, 12] are devoted to the q-Bessel Fourier analysis. Throughout this paper, we consider $0 < q < 1$ and $\nu > -1$. We denote by

$$\mathbb{R}_q^+ = \{q^n, \quad n \in \mathbb{Z}\}.$$

The q-Bessel operator is defined as follows [5]

$$\Delta_{q,\nu} f(x) = \frac{1}{x^2}\left[f(q^{-1}x) - (1+q^{2\nu})f(x) + q^{2\nu} f(qx)\right].$$

The eigenfunction of $\Delta_{q,\nu}$ associated with the eigenvalue $-\lambda^2$ is the function $x \mapsto j_\nu(\lambda x, q^2)$, where $j_\nu(., q^2)$ is the normalized q-Bessel function defined by [5, 8, 10, 14, 16]

$$j_\nu(x, q^2) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)}}{(q^{2\nu+2}, q^2)_n (q^2, q^2)_n} x^{2n}.$$

The q-Jackson integral of a function f defined on $\mathbb{R}^+$ is

$$\int_0^\infty f(t) d_q t = (1-q) \sum_{n\in\mathbb{Z}} q^n f(q^n).$$

We denote by $\mathcal{L}_{q,p,\nu}$ the space of functions f defined on $\mathbb{R}_q^+$ such that

$$\|f\|_{q,p,\nu} = \left(\int_0^\infty |f(x)|^p x^{2\nu+1} d_q x\right)^{1/p} \text{exist.}$$

We denote by $\mathcal{C}_{q,0}$ the space of functions defined on $\mathbb{R}_q^+$ tending to 0 as $x \to \infty$ and continuous at 0 equipped with the topology of uniform convergence. The space $\mathcal{C}_{q,0}$ is complete with respect to the norm

$$\|f\|_{q,\infty} = \sup_{x\in\mathbb{R}_q^+} |f(x)|.$$

The normalized q-Bessel function $j_\nu(., q^2)$ satisfies the orthogonality relation

$$c_{q,\nu}^2 \int_0^\infty j_\nu(xt, q^2) j_\nu(yt, q^2) t^{2\nu+1} d_q t = \delta_q(x, y), \quad \forall x, y \in \mathbb{R}_q^+ \tag{1}$$

where

$$\delta_q(x,y) = \begin{cases} 0 \text{ if } x \neq y \\ \frac{1}{(1-q)x^{2(\nu+1)}} \text{ if } x = y \end{cases}$$

and

$$c_{q,\nu} = \frac{1}{1-q} \frac{(q^{2\nu+2}, q^2)_\infty}{(q^2, q^2)_\infty}.$$

Let f be a function defined on $\mathbb{R}_q^+$ then

$$\int_0^\infty f(y)\delta_q(x,y)y^{2\nu+1}d_qy = f(x).$$

The normalized q-Bessel function $j_\nu(.,q^2)$ satisfies

$$|j_\nu(q^n,q^2)| \leq \frac{(-q^2;q^2)_\infty(-q^{2\nu+2};q^2)_\infty}{(q^{2\nu+2};q^2)_\infty}\begin{cases} 1 & \text{if} \quad n \geq 0 \\ q^{n^2-(2\nu+1)n} & \text{if} \quad n < 0 \end{cases}.$$

The q-Bessel Fourier transform $\mathcal{F}_{q,\nu}$ is defined by [5, 8, 12]

$$\mathcal{F}_{q,\nu}f(x) = c_{q,\nu}\int_0^\infty f(t)j_\nu(xt,q^2)t^{2\nu+1}d_qt, \quad \forall x \in \mathbb{R}_q^+.$$

Proposition 1. *Let $f \in \mathcal{L}_{q,1,\nu}$ then $\mathcal{F}_{q,\nu}f \in \mathcal{C}_{q,0}$ and we have*

$$\|\mathcal{F}_{q,\nu}(f)\|_{q,\infty} \leq B_{q,\nu}\|f\|_{q,1,v}$$

where

$$B_{q,\nu} = \frac{1}{1-q}\frac{(-q^2;q^2)_\infty(-q^{2v+2};q^2)_\infty}{(q^2;q^2)_\infty}.$$

Theorem 1. *Let f be a function in the $\mathcal{L}_{q,p,\nu}$ space where $p \geq 1$ then*

$$\mathcal{F}_{q,\nu}^2 f = f. \tag{2}$$

Proof. If $f \in \mathcal{L}_{q,p,\nu}$ then $\mathcal{F}_{q,\nu}f$ exist, and we have

$$\begin{aligned}
\mathcal{F}_{q,\nu}^2 f(x) &= c_{q,\nu}\int_0^\infty \mathcal{F}_{q,\nu}f(t)j_\nu(xt,q^2)t^{2\nu+1}d_qt \\
&= \int_0^\infty f(y)\left[c_{q,\nu}^2\int_0^\infty j_\nu(xt,q^2)j_\nu(yt,q^2)t^{2\nu+1}d_qt\right]y^{2\nu+1}d_qy \\
&= \int_0^\infty f(y)\delta_q(x,y)y^{2\nu+1}d_qy \\
&= f(x).
\end{aligned}$$

The computations are justified by the Fubuni's theorem: If $p > 1$ then we use the Hölder's inequality

$$\begin{aligned}
&\int_0^\infty |f(y)|\left[\int_0^\infty |j_\nu(xt,q^2)j_\nu(yt,q^2)|t^{2\nu+1}d_qt\right]y^{2\nu+1}d_qy \\
&\leq \left[\int_0^\infty |f(y)|^p y^{2\nu+1}d_qy\right]^{1/p} \times \left[\int_0^\infty \sigma(y)^{\overline{p}}y^{2\nu+1}d_qy\right]^{1/\overline{p}}.
\end{aligned}$$

The numbers p and $\overline{p}$ above are conjugates and

$$\sigma(y) = \int_0^\infty |j_\nu(xt,q^2)j_\nu(yt,q^2)|t^{2\nu+1}d_qt,$$

then

$$\begin{aligned}
&\int_0^\infty \sigma(y)^{\overline{p}}y^{2\nu+1}d_qy \\
&= \int_0^1 \sigma(y)^{\overline{p}}y^{2\nu+1}d_qy + \int_1^\infty \sigma y^{\overline{p}}y^{2\nu+1}d_qy.
\end{aligned}$$

Note that

$$\begin{aligned}
&\int_0^1 \sigma(y)^{\overline{p}} y^{2\nu+1} d_q y \\
&\leq \|j_\nu(.,q^2)\|_{q,\infty}^{\overline{p}} \int_0^1 \left[\int_0^\infty |j_\nu(xt,q^2)| t^{2\nu+1} d_q t\right]^{\overline{p}} y^{2\nu+1} d_q y \\
&\leq \|j_\nu(.,q^2)\|_{q,\infty}^{\overline{p}} \|j_\nu(.,q^2)\|_{q,1,\nu}^{\overline{p}} x^{-2(\nu+1)\overline{p}} \left[\int_0^1 y^{2\nu+1} d_q y\right] < \infty,
\end{aligned}$$

and

$$\begin{aligned}
&\int_1^\infty \sigma(y)^{\overline{p}} y^{2\nu+1} d_q y \\
&\leq \|j_\nu(.,q^2)\|_{q,\infty}^{\overline{p}} \|j_\nu(.,q^2)\|_{q,1,\nu}^{\overline{p}} \int_1^\infty \frac{y^{2\nu+1}}{y^{2(\nu+1)\overline{p}}} d_q y \\
&\leq \|j_\nu(.,q^2)\|_{q,\infty}^{\overline{p}} \|j_\nu(.,q^2)\|_{q,1,\nu}^{\overline{p}} \int_1^\infty \frac{1}{y^{2(\nu+1)(\overline{p}-1)+1}} d_q y < \infty.
\end{aligned}$$

If $p = 1$ then

$$\begin{aligned}
&\int_0^\infty \|f(y)\| \left[\int_0^\infty |j_\nu(xt,q^2) j_\nu(yt,q^2)| t^{2\nu+1} d_q t\right] y^{2\nu+1} d_q y \\
&\leq \|f\|_{q,1,\nu} \|j_\nu(.,q^2)\|_{q,\infty} \|j_\nu(.,q^2)\|_{q,1,\nu} \times \frac{1}{x^{2(\nu+1)}}.
\end{aligned}$$

□

Theorem 2. *Let f be a function in the $\mathcal{L}_{q,1,\nu} \cap \mathcal{L}_{q,p,\nu}$ space, where $p > 2$ then*

$$\|\mathcal{F}_{q,\nu} f\|_{q,2,\nu} = \|f\|_{q,2,\nu}.$$

Proof. Let $f \in \mathcal{L}_{q,1,\nu} \cap \mathcal{L}_{q,p,\nu}$ then by Theorem 1 we see that

$$\mathcal{F}_{q,\nu}^2 f = f.$$

This implies

$$\begin{aligned}
\int_0^\infty \mathcal{F}_{q,\nu} f(x)^2 x^{2\nu+1} d_q x &= \int_0^\infty \mathcal{F}_{q,\nu} f(x) \left[c_{q,\nu} \int_0^\infty f(t) j_\nu(xt,q^2) t^{2\nu+1} d_q t\right] x^{2\nu+1} d_q x \\
&= \int_0^\infty f(t) \left[c_{q,\nu} \int_0^\infty \mathcal{F}_{q,\nu} f(x) j_\nu(xt,q^2) x^{2\nu+1} d_q x\right] t^{2\nu+1} d_q t \\
&= \int_0^\infty f(t)^2 t^{2\nu+1} d_q t.
\end{aligned}$$

The computations are justified by the Fubuni's theorem

$$\begin{aligned}
&\int_0^\infty |f(t)| \left[c_{q,\nu} \int_0^\infty |\mathcal{F}_{q,\nu} f(x)| |j_\nu(xt,q^2)| x^{2\nu+1} d_q x\right] t^{2\nu+1} d_q t \\
&\leq \left[\int_0^\infty |f(t)|^p t^{2\nu+1} d_q t\right]^{1/p} \times \left[\int_0^\infty |\phi(t)|^{\overline{p}} t^{2\nu+1} d_q t\right]^{1/\overline{p}},
\end{aligned}$$

where

$$\phi(t) = c_{q,\nu} \int_0^\infty |\mathcal{F}_{q,\nu} f(x)| |j_\nu(xt,q^2)| x^{2\nu+1} d_q x,$$

then

$$\begin{aligned}\|\mathcal{F}_{q,\nu}f(x)\| &\leq c_{q,\nu}\int_0^\infty |f(y)||j_\nu(xy,q^2)|y^{2\nu+1}d_qy\\ &\leq c_{q,\nu}\left[\int_0^\infty |f(y)|^p y^{2\nu+1}d_qy\right]^{1/p}\times\left[\int_0^\infty |j_\nu(xy,q^2)|^{\overline{p}}y^{2\nu+1}d_qy\right]^{1/\overline{p}}\\ &\leq c_{q,\nu}\left[\int_0^\infty |f(y)|^p y^{2\nu+1}d_qy\right]^{1/p}\times\left[\int_0^\infty |j_\nu(y,q^2)|^{\overline{p}}y^{2\nu+1}d_qy\right]^{1/\overline{p}}x^{-2(\nu+1)/\overline{p}}\\ &\leq c_{q,\nu}\|f\|_{q,p,\nu}\|j_\nu(.,q^2)\|_{q,\overline{p},\nu}x^{-2(\nu+1)/\overline{p}}.\end{aligned}$$

This gives

$$\begin{aligned}\phi(t) &\leq c_{q,\nu}^2\|f\|_{q,p,\nu}\|j_\nu(.,q^2)\|_{q,\overline{p},\nu}\int_0^\infty |j_\nu(xt,q^2)|x^{(2\nu+1)-2(\nu+1)/\overline{p}}d_qx\\ &\leq c_{q,\nu}^2\|f\|_{q,p,\nu}\|j_\nu(.,q^2)\|_{q,\overline{p},\nu}\left[\int_0^\infty |j_\nu(x,q^2)|x^{2(\nu+1)/p-1}d_qx\right]t^{-2(\nu+1)/p}\\ &\leq C_1t^{-2(\nu+1)/p},\end{aligned}$$

and

$$\begin{aligned}\phi(t) &= c_{q,\nu}\int_0^\infty |\mathcal{F}_{q,\nu}f(x)||j_\nu(xt,q^2)|x^{2\nu+1}d_qx\\ &= \left[c_{q,\nu}\int_0^\infty |\mathcal{F}_{q,\nu}f(x/t)||j_\nu(x,q^2)|x^{2\nu+1}d_qx\right]t^{-2(\nu+1)}\\ &\leq c_{q,\nu}\|\mathcal{F}_{q,\nu}f\|_{q,\infty}\times\|j_\nu(.,q^2)\|_{q,1,\nu}\times t^{-2(\nu+1)}\\ &\leq C_2t^{-2(\nu+1)}.\end{aligned}$$

Note that

$$\begin{cases}-1<-2(\nu+1)\frac{\overline{p}}{p}+2v+1\\ -2(\nu+1)\overline{p}+2v+1<-1\end{cases}\Leftrightarrow\begin{cases}0<-2(\nu+1)(\overline{p}-2)\\ -2(\nu+1)(\overline{p}-1)<0\end{cases}\Leftrightarrow 1<\overline{p}<2\Leftrightarrow p>2.$$

Hence

$$\begin{aligned}\int_0^\infty |\phi(t)|^{\overline{p}}t^{2\nu+1}d_qt &= \int_0^1 |\phi(t)|^{\overline{p}}t^{2\nu+1}d_qt+\int_1^\infty |\phi(t)|^{\overline{p}}t^{2\nu+1}d_qt\\ &\leq C_1\int_0^1 t^{-2(v+1)\overline{p}/p}t^{2\nu+1}d_qt+C_2\int_1^\infty t^{-2(v+1)\overline{p}}t^{2\nu+1}d_qt<\infty,\end{aligned}$$

which prove the result. □

Theorem 3. *Let f be a function in the $\mathcal{L}_{q,2,\nu}$ space then*

$$\|\mathcal{F}_{q,\nu}f\|_{q,2,\nu}=\|f\|_{q,2,\nu}.$$

Proof. We introduce the function ψ_x as follows

$$\psi_x(t)=c_{q,\nu}j_\nu(tx,q^2).$$

The inner product $\langle,\rangle$ in the Hilbert space $\mathcal{L}_{q,2,\nu}$ is defined by

$$f,g\in\mathcal{L}_{q,2,\nu}\Rightarrow\langle f,g\rangle=\int_0^\infty f(t)g(t)t^{2\nu+1}d_qt. \tag{3}$$

Using (1) we write

$$x\neq y\Rightarrow\langle\psi_x,\psi_y\rangle=0$$

$$\|\psi_x\|^2_{q,2,\nu} = \frac{1}{1-q} x^{-2(\nu+1)}.$$

We have

$$\mathcal{F}_{q,\nu} f(x) = \langle f, \psi_x \rangle,$$

and by Theorem 1

$$f \in \mathcal{L}_{q,2,\nu} \Rightarrow \mathcal{F}^2_{q,\nu} f = f,$$

then

$$\langle f, \psi_x \rangle = 0, \forall x \in \mathbb{R}^+_q \Rightarrow \mathcal{F}_{q,\nu} f(x) = 0, \forall x \in \mathbb{R}^+_q \Rightarrow f = 0.$$

Hence, $\{\psi_x, x \in \mathbb{R}^+_q\}$ form an orthogonal basis of the Hilbert space $\mathcal{L}_{q,2,\nu}$ and we have

$$\overline{\{\psi_x, \quad \forall x \in \mathbb{R}^+_q\}} = \mathcal{L}_{q,2,\nu}.$$

Now

$$f \in \mathcal{L}_{q,2,\nu} \Rightarrow f = \sum_{x \in \mathbb{R}^+_q} \frac{1}{\|\psi_x\|^2_{q,2,\nu}} \langle f, \psi_x \rangle \psi_x,$$

and then

$$\|f\|^2_{q,2,\nu} = \sum_{x \in \mathbb{R}^+_q} \frac{1}{\|\psi_x\|^2_{q,2,\nu}} \langle f, \psi_x \rangle^2 = (1-q) \sum_{x \in \mathbb{R}^+_q} x^{2(\nu+1)} \mathcal{F}_{q,\nu} f(x)^2 = \|\mathcal{F}_{q,\nu} f\|^2_{q,2,\nu},$$

which achieve the proof. □

Proposition 2. *Let $f \in \mathcal{L}_{q,p,\nu}$ where $p \geq 1$ then $\mathcal{F}_{q,\nu} f \in \mathcal{L}_{q,\overline{p},\nu}$. If $1 \leq p \leq 2$ then*

$$\|\mathcal{F}_{q,\nu} f\|_{q,\overline{p},\nu} \leq B^{\frac{2}{p}-1}_{q,\nu} \|f\|_{q,p,\nu}. \tag{4}$$

Proof. This is an immediate consequence of Proposition 1, Theorem 3, the Riesz-Thorin theorem and the inversion formula (2). □

The q-translation operator is given as follow

$$T^\nu_{q,x} f(y) = c_{q,\nu} \int_0^\infty \mathcal{F}_{q,\nu} f(t) j_\nu(yt, q^2) j_\nu(xt, q^2) t^{2\nu+1} d_q t.$$

Let us now introduce

$$Q_\nu = \left\{ q \in]0,1[, \quad T^\nu_{q,x} \quad \text{is positive for all} \quad x \in \mathbb{R}^+_q \right\}$$

the set of the positivity of $T^\nu_{q,x}$. We recall that $T^\nu_{q,x}$ is called positive if $T^v_{q,x} f \geq 0$ for $f \geq 0$. In a recent paper [6] it was proved that if $-1 < \nu < \nu'$ then $Q_\nu \subset Q_{\nu'}$. As a consequence :

- **-:** If $0 \leq \nu$ then $Q_\nu =]0,1[$.
- **-:** If $-\frac{1}{2} \leq \nu < 0$ then $]0, q_0] \subset Q_{-\frac{1}{2}} \subset Q_\nu \subsetneq]0,1[, \quad q_0 \simeq 0.43$.
- **-:** If $-1 < \nu \leq -\frac{1}{2}$ then $Q_\nu \subset Q_{-\frac{1}{2}}$.

Theorem 4. *Let $f \in \mathcal{L}_{q,p,\nu}$ then $T^\nu_{q,x} f$ exists and we have*

$$\int_0^\infty T^\nu_{q,x} f(y) y^{2\nu+1} d_q y = \int_0^\infty f(y) y^{2\nu+1} d_q y.$$

and

$$T^\nu_{q,x} f(y) = \int_0^\infty f(z) D_\nu(x,y,z) z^{2\nu+1} d_q z,$$

where

$$D_\nu(x,y,z)=c_{q,\nu}^2\int_0^\infty j_\nu(xs,q^2j_\nu(ys,q^2j_\nu(zs,q^2)s^{2\nu+1}d_qs.$$

If we suppose that $T_{q,x}^\nu$ is a positive operator then for all $p\geq 1$ we have

$$\|T_{q,x}^\nu f\|_{q,p,\nu}\leq\|f\|_{q,p,\nu}. \tag{5}$$

Proof. We write the operator $T_{q,x}^\nu$ in the following form

$$\begin{aligned}T_{q,x}^\nu f(y) &= c_{q,\nu}\int_0^\infty \mathcal{F}_{q,v}f(z)j_v(xz,q^2)j_v(yz,q^2)z^{2v+1}d_qz\\ &= \mathcal{F}_{q,\nu}\left[\mathcal{F}_{q,\nu}f(z)j_\nu(xz,q^2)\right](y).\end{aligned}$$

So we have

$$\begin{aligned}\int_0^\infty T_{q,x}^\nu f(y)y^{2\nu+1}d_qy &= \int_0^\infty \mathcal{F}_{q,\nu}\left[\mathcal{F}_{q,\nu}f(z)j_\nu(xz,q^2)\right](y)y^{2\nu+1}d_qy\\ &= \frac{1}{c_{q,\nu}}c_{q,\nu}\int_0^\infty \mathcal{F}_{q,\nu}\left[\mathcal{F}_{q,\nu}f(z)j_\nu(xz,q^2)\right](y)j_\nu(0,q^2)y^{2\nu+1}d_qy\\ &= \frac{1}{c_{q,\nu}}\mathcal{F}_{q,\nu}^2\left[\mathcal{F}_{q,\nu}f(z)j_\nu(xz,q^2)\right](0)\\ &= \frac{1}{c_{q,\nu}}\mathcal{F}_{q,\nu}f(0)\\ &= \int_0^\infty f(y)y^{2v+1}d_qy.\end{aligned}$$

On the other hand

$$\begin{aligned}T_{q,x}^\nu f(y) &= c_{q,\nu}\int_0^\infty \mathcal{F}_{q,\nu}f(z)j_\nu(xz,q^2)j_\nu(yz,q^2)z^{2\nu+1}d_qz\\ &= c_{q,\nu}\int_0^\infty\left[c_{q,\nu}\int_0^\infty f(t)j_\nu(tz,q^2)t^{2\nu+1}d_qt\right]j_\nu(xz,q^2)j_\nu(yz,q^2)z^{2v+1}d_qz\\ &= \int_0^\infty\left[c_{q,\nu}^2\int_0^\infty j_\nu(xz,q^2)j_\nu(yz,q^2)j_\nu(tz,q^2)z^{2\nu+1}d_qz\right]f(t)t^{2\nu+1}d_qt\\ &= \int_0^\infty D_{q,\nu}(x,y,t)f(t)t^{2\nu+1}d_qt.\end{aligned}$$

The computations are justified by the Fubuni's theorem

$$\begin{aligned}&\int_0^\infty\left[\int_0^\infty |f(t)|\left|j_\nu(tz,q^2)\right|t^{2\nu+1}d_qt\right]\left|j_v(xz,q^2)\right|\left|j_\nu(yz,q^2)\right|z^{2v+1}d_qz\\ \leq\;& \|f\|_{q,p,\nu}\int_0^\infty\left[\int_0^\infty\left|j_\nu(tz,q^2)\right|^{\overline{p}}t^{2\nu+1}d_qt\right]^{\frac{1}{\overline{p}}}\left|j_\nu(xz,q^2)\right|\left|j_v(yz,q^2)\right|z^{2\nu+1}d_qz\\ \leq\;& \|f\|_{q,p,\nu}\left\|j_\nu(.,q^2)\right\|_{q,\overline{p},\nu}\int_0^\infty\left|j_\nu(xz,q^2)\right|\left|j_\nu(yz,q^2)\right|z^{2(\nu+1)\left(1-\frac{1}{p}\right)-1}d_qz.\end{aligned}$$

Now suppose that $T^{\nu}_{q,x}$ is positive. Given a function $f \in \mathcal{C}_{q,0}$ we obtains

$$\begin{aligned}
\left|T^{\nu}_{q,x}f(y)\right| &= \left|\int_0^\infty D_{q,\nu}(x,y,t)f(t)t^{2\nu+1}d_qt\right| \\
&\leq \int_0^\infty |D_{q,\nu}(x,y,t)|\,|f(t)|\,t^{2v+1}d_qt \\
&\leq \left[\int_0^\infty D_{q,\nu}(x,y,t)t^{2v+1}d_qt\right]\|f\|_{q,\infty} = \|f\|_{q,\infty}
\end{aligned}$$

which implies

$$\left\|T^{\nu}_{q,x}f\right\|_{q,\infty} \leq \|f\|_{q,\infty}.$$

If the function $f \in \mathcal{L}_{q,1,\nu}$ then we obtains

$$\begin{aligned}
\left\|T^{\nu}_{q,x}f\right\|_{q,1,\nu} &= \int_0^\infty \left|T^{\nu}_{q,x}f(y)\right| y^{2\nu+1}d_qy \\
&\leq \int_0^\infty \left[\int_0^\infty |D_{q,\nu}(x,y,t)|\,|f(t)|\,t^{2\nu+1}d_qt\right] y^{2\nu+1}d_qy \\
&\leq \int_0^\infty \left[\int_0^\infty D_{q,\nu}(x,y,t)y^{2\nu+1}d_qy\right] |f(t)|\,t^{2\nu+1}d_qt \\
&\leq \int_0^\infty |f(t)|\,t^{2\nu+1}d_qt = \|f\|_{q,1,\nu}.
\end{aligned}$$

The result is a consequence of the Riesz-Thorin theorem.

Notice that the kernel $D_{q,\nu}(x,y,t)$ can be written as follows

$$\begin{aligned}
D_{q,\nu}(x,y,t) &= c^2_{q,\nu}\int_0^\infty j_\nu(xz,q^2)j_\nu(yz,q^2)j_\nu(tz,q^2)z^{2\nu+1}d_qz \\
&= c_{q,\nu}\mathcal{F}_{q,\nu}\left[j_\nu(xz,q^2)j_\nu(yz,q^2)\right](t),
\end{aligned}$$

which implies

$$\begin{aligned}
\int_0^\infty D_{q,\nu}(x,y,t)t^{2\nu+1}d_qt &= c_{q,\nu}\int_0^\infty \mathcal{F}_{q,\nu}\left[j_\nu(xz,q^2)j_\nu(yz,q^2)\right](t)t^{2\nu+1}d_qt \\
&= \mathcal{F}^2_{q,\nu}\left[j_\nu(xz,q^2)j_\nu(yz,q^2)\right](0) = 1.
\end{aligned}$$

□

The q-convolution product is defined by

$$f *_q g = \mathcal{F}_{q,\nu}\left[\mathcal{F}_{q,\nu}f \times \mathcal{F}_{q,\nu}g\right].$$

Theorem 5. *Let* $1 \leq p, r, s$ *such that*

$$\frac{1}{p} + \frac{1}{r} - 1 = \frac{1}{s}$$

Given two functions $f \in \mathcal{L}_{q,p,\nu}$ *and* $g \in \mathcal{L}_{q,r,\nu}$ *then* $f *_q g$ *exists and we have*

$$f *_q g(x) = c_{q,\nu}\int_0^\infty T^{\nu}_{q,x}f(y)g(y)y^{2\nu+1}d_qy.$$

and

$$f *_q g \in \mathcal{L}_{q,s,\nu}.$$
$$\mathcal{F}_{q,\nu}(f *_q g) = \mathcal{F}_{q,\nu}(f) \times \mathcal{F}_{q,\nu}(g).$$

If $s \geq 2$ then

$$\left\|f *_q g\right\|_{q,s,\nu} \leq B_{q,\nu} \left\|f\right\|_{q,p,\nu} \left\|g\right\|_{q,r,\nu}. \tag{6}$$

If we suppose that $T_{q,x}^{\nu}$ is a positive operator then

$$\left\|f *_q g\right\|_{q,s,\nu} \leq c_{q,\nu} \left\|f\right\|_{q,p,\nu} \left\|g\right\|_{q,r,\nu}. \tag{7}$$

Proof. We have

$$\begin{aligned}
f *_q g(x) &= \mathcal{F}_{q,\nu}\left[\mathcal{F}_{q,\nu} f \times \mathcal{F}_{q,\nu} g\right](x) \\
&= c_{q,\nu} \int_0^\infty \mathcal{F}_{q,\nu} f(y) \times \mathcal{F}_{q,\nu} g(y) j_\nu(xy, q^2) y^{2\nu+1} d_q y \\
&= c_{q,\nu} \int_0^\infty \mathcal{F}_{q,\nu} f(y) \times \left[c_{q,\nu} \int_0^\infty g(z) j_\nu(zy, q^2) z^{2\nu+1} d_q z\right] j_v(xy, q^2) y^{2\nu+1} d_q y \\
&= c_{q,\nu} \int_0^\infty \left[c_{q,\nu} \int_0^\infty \mathcal{F}_{q,\nu} f(y) j_v(zy, q^2) j_\nu(xy, q^2) y^{2\nu+1} d_q y\right] g(z) z^{2\nu+1} d_q z \\
&= c_{q,\nu} \int_0^\infty T_{q,x}^{\nu} f(z) g(z) z^{2\nu+1} d_q z.
\end{aligned}$$

The computations are justified by the Fubuni's theorem

$$\begin{aligned}
&\int_0^\infty \left|F_{q,\nu} f(y)\right| \times \left[\int_0^\infty |g(z)| \times \left|j_\nu(zy, q^2)\right| z^{2\nu+1} d_q z\right] \left|j_\nu(xy, q^2)\right| y^{2\nu+1} d_q y \\
\leq\ & \|g\|_{q,r,\nu} \int_0^\infty \left|F_{q,\nu} f(y)\right| \times \left[\int_0^\infty \left|j_\nu(zy, q^2)\right|^{\overline{r}} z^{2\nu+1} d_q z\right]^{\frac{1}{\overline{r}}} \left|j_\nu(xy, q^2)\right| y^{2\nu+1} d_q y \\
\leq\ & \|g\|_{q,r,\nu} \left\|j_\nu(., q^2)\right\|_{q,\overline{r},\nu} \int_0^\infty \left|F_{q,\nu} f(y)\right| \times \left[\left|j_v(xy, q^2)\right| y^{-\frac{2\nu+2}{\overline{r}}}\right] y^{2\nu+1} d_q y \\
\leq\ & \|g\|_{q,r,\nu} \left\|j_\nu(., q^2)\right\|_{q,\overline{r},\nu} \left\|F_{q,\nu} f\right\|_{q,\overline{p},v} \left(\int_0^\infty \left[\left|j_\nu(xy, q^2)\right| y^{-\frac{2\nu+2}{\overline{r}}}\right]^p y^{2\nu+1} d_q y\right)^{\frac{1}{p}} \\
\leq\ & \|g\|_{q,r,\nu} \left\|j_\nu(., q^2)\right\|_{q,\overline{r},\nu} \left\|F_{q,\nu} f\right\|_{q,\overline{p},\nu} \left(\int_0^\infty \left|j_\nu(xy, q^2)\right|^p y^{2(\nu+1)\left(1-\frac{p}{\overline{r}}\right)-1} d_q y\right)^{\frac{1}{p}}.
\end{aligned}$$

From Proposition 2 we deduce that

$$\mathcal{F}_{q,\nu} f \in \mathcal{L}_{q,\overline{p},\nu} \text{ and } \mathcal{F}_{q,\nu} g \in \mathcal{L}_{q,\overline{r},\nu}.$$

Then, using the Hölder inequality and the fact that

$$\frac{1}{\overline{p}} + \frac{1}{\overline{r}} = \frac{1}{\overline{s}}$$

to conclude that

$$\mathcal{F}_{q,\nu} f \times \mathcal{F}_{q,\nu} g \in \mathcal{L}_{q,\overline{s},\nu}.$$

Which implies that

$$f *_q g = \mathcal{F}_{q,\nu}\left[\mathcal{F}_{q,\nu} f \times \mathcal{F}_{q,\nu} g\right] \in \mathcal{L}_{q,s,\nu}$$

and by the inversion formula (2) we obtain

$$\mathcal{F}_{q,\nu}\left(f *_q g\right) = \mathcal{F}_{q,\nu} f \times \mathcal{F}_{q,\nu} g.$$

Suppose that $s \geq 2$, so $1 \leq \overline{s} \leq 2$ and we can write

$$\begin{aligned}\|f *_q g\|_{q,s,\nu} &= \|\mathcal{F}_{q,\nu}[\mathcal{F}_{q,\nu}f \times \mathcal{F}_{q,\nu}g]\|_{q,s,\nu} \\ &\leq B_{q,\nu}^{\frac{2}{\overline{s}}-1} \|\mathcal{F}_{q,\nu}f\|_{q,\overline{p},\nu} \|\mathcal{F}_{q,\nu}g\|_{q,\overline{r},\nu} \\ &\leq B_{q,\nu}^{\frac{2}{\overline{s}}-1} B_{q,\nu}^{\frac{2}{p}-1} B_{q,\nu}^{\frac{2}{r}-1} \|f\|_{q,p,\nu} \|g\|_{q,r,\nu} \\ &\leq B_{q,\nu} \|f\|_{q,p,\nu} \|g\|_{q,r,\nu}.\end{aligned}$$

Now suppose that $T^{\nu}_{q,x}$ is a positive operator.
We introduce the operator K_f as follows

$$K_f g(x) = c_{q,\nu} \int_0^{\infty} T^{\nu}_{q,x} f(z) g(z) z^{2\nu+1} d_q z.$$

By the Hölder inequality and (5) we get

$$\|K_f g\|_{q,\infty} \leq c_{q,\nu} \|f\|_{q,p,\nu} \|g\|_{q,\overline{p},\nu}.$$

The Minkowski inequality leads to

$$\|K_f g\|_{q,p,\nu} \leq c_{q,\nu} \|f\|_{q,p,\nu} \|g\|_{q,1,\nu}.$$

Hence we have

$$K_f : \mathcal{L}_{q,\overline{p},\nu} \to \mathcal{C}_{q,0}, \quad K_f : \mathcal{L}_{q,1,\nu} \to \mathcal{L}_{q,p,\nu}.$$

Then the operator K_f satisfies

$$K_f : \mathcal{L}_{q,r,\nu} \to \mathcal{L}_{q,s,\nu}$$

and

$$\|f *_q g\|_{q,s,\nu} = \|K_f g\|_{q,s,\nu} \leq c_{q,\nu} \|f\|_{q,p,\nu} \|g\|_{q,r,\nu}.$$

□

Remark 1. *We discuss here the sharp results for the Hausdorf-Young inequality provided above. An inequality already sharper than (6) is given in formula (7). In fact we have $c_{q,\nu} < B_{q,\nu}$.*
To obtained (7) without the positivity argument, we can do by using which is a q-Riemann-Liouville fractional integral generalizing the q-Mehler integral representation for the q-Bessel function $j_\nu(.,q^2)$ which can be proved in a straightforward way [8]

$$j_\nu(\lambda, q^2) = [2\nu]_q \int_0^1 \frac{(q^2t^2, q^2)_\infty}{(q^{2\nu}t^2, q^2)_\infty} j_0(\lambda t, q^2) t d_q t$$

together with the inequalities for the q-Bessel function which is given as formula (24) in the paper [4]

$$|j_0(x; q^2)| \leq 1, \quad \forall x \in \mathbb{R}_q^+.$$

Combine this formulas we arrive at

$$|j_\nu(x; q^2)| \leq 1, \quad \forall x \in \mathbb{R}_q^+, \quad \nu \geq 0.$$

Then the inequalities (4) can be written as follows

$$\|\mathcal{F}_{q,\nu}f\|_{q,\overline{p},\nu} \leq c_{q,\nu}^{\frac{2}{p}-1} \|f\|_{q,p,\nu}.$$

This should give the sharpest version of (6) in the cases $\nu \geq 0$. Unfortunately the positivity of the operator $T^{\nu}_{q,x}$ is satisfied in this case.

In fact we can prove that if we are in the positivity cases then

$$\left\|j_\nu(.,q^2)\right\|_{q,\infty} \le 1.$$

To prove this recalling that

$$T^\nu_{q,x} j_\nu(y,q^2) = j_\nu(x,q^2) j_\nu(y,q^2).$$

So we have

$$\int_0^\infty D_{q,\nu}(x,y,t) j_v(t,q^2) t^{2v+1} d_q t = j_\nu(x,q^2) j_\nu(y,q^2).$$

We obtains for all $x, y \in \mathbb{R}_q^+$

$$\begin{aligned} \left|j_\nu(x,q^2)\right| \times \left|j_\nu(y,q^2)\right| &\le \int_0^\infty D_{q,\nu}(x,y,t) \left|j_\nu(t,q^2)\right| t^{2\nu+1} d_q t \\ &\le \left[\int_0^\infty D_{q,\nu}(x,y,t) t^{2\nu+1} d_q t\right] \left\|j_\nu(.,q^2)\right\|_{q,\infty}. \end{aligned}$$

The fact that

$$\int_0^\infty D_{q,\nu}(x,y,t) t^{2\nu+1} d_q t = 1$$

implies

$$\left\|j_\nu(.,q^2)\right\|^2_{q,\infty} \le \left\|j_\nu(.,q^2)\right\|_{q,\infty}$$

which gives the result.

3. Uncertainty principle

We introduce two q-difference operators

$$\partial_q f(x) = \frac{f(q^{-1}x) - f(x)}{x}$$

and

$$\partial_q^* f(x) = \frac{f(x) - q^{2\nu+1} f(qx)}{x}.$$

Then we have

$$\partial_q \partial_q^* f(x) = \partial_q^* \partial_q f(x) = \Delta_{q,\nu} f(x).$$

Proposition 3. *If* $\langle \partial_q f, g\rangle$ *exist and* $\lim_{a\to\infty} \left|a^{2\nu+1} f(q^{-1}a) g(a)\right| = 0$ *then*

$$\langle \partial_q f, g\rangle = -\left\langle f, \partial_q^* g\right\rangle.$$

Proof. The following computation

$$\begin{aligned}
& \int_0^a \partial_q f(x) g(x) x^{2\nu+1} d_q x \\
= & \int_0^a \frac{f(q^{-1}x) - f(x)}{x} g(x) x^{2\nu+1} d_q x \\
= & \int_0^a \frac{f(q^{-1}x)}{x} g(x) x^{2\nu+1} d_q x - \int_0^a \frac{f(x)}{x} g(x) x^{2\nu+1} d_q x \\
= & q^{2\nu+1} \int_0^{q^{-1}a} \frac{f(x)}{x} g(qx) x^{2\nu+1} d_q x - \int_0^a \frac{f(x)}{x} \partial_q g(x) x^{2\nu+1} d_q x \\
= & q^{2\nu+1} \int_0^a \frac{f(x)}{x} \partial_q g(qx) x^{2\nu+1} d_q x - \int_0^a \frac{f(x)}{x} g(x) x^{2v+1} d_q x + a^{2\nu+1} f(q^{-1}a) g(a) \\
= & -\int_0^a f(x) \frac{g(x) - q^{2\nu+1} g(qx)}{x} x^{2\nu+1} d_q x + a^{2\nu+1} f(q^{-1}a) g(a) \\
= & = -\int_0^a f(x) \partial_q^* g(x) x^{2\nu+1} d_q x + a^{2\nu+1} f(q^{-1}a) g(a)
\end{aligned}$$

leads to the result. □

Corollary 1. *If $f \in \mathcal{L}_{q,2,\nu}$ such that $x\mathcal{F}_{q,\nu} f \in \mathcal{L}_{q,2,\nu}$ then*

$$\|\partial_q f\|_2 = \|x\mathcal{F}_{q,\nu} f\|_2 .$$

Proof. In fact we have

$$\begin{aligned}
\|\partial_q f\|_2^2 &= \langle \partial_q f, \partial_q f \rangle = -\left\langle f, \partial_q^* \partial_q f \right\rangle \\
&= -\langle f, \Delta_{q,\nu} f \rangle \\
&= -\langle \mathcal{F}_{q,\nu} f, \mathcal{F}_{q,\nu} \Delta_{q,\nu} f \rangle \\
&= \left\langle \mathcal{F}_{q,\nu} f, x^2 \mathcal{F}_{q,\nu} f \right\rangle \\
&= \|x\mathcal{F}_{q,\nu} f\|_2^2 ,
\end{aligned}$$

which prove the result. □

Theorem 6. *Assume that f belongs to the space $\mathcal{L}_{q,2,\nu}$. Then the q-Bessel transform satisfies the following uncertainty principal*

$$\|f\|_2^2 \leq k_{q,v} \|xf\|_2 \|x\mathcal{F}_{q,\nu} f\|_2$$

where

$$k_{q,\nu} = \frac{\left[1 + \sqrt{q} \times q^{\nu+1}\right]}{1 - q^{2(\nu+1)}}.$$

Proof. In fact

$$\partial_q^* x f = f(x) - q^{2\nu+2} f(qx)$$

$$x \partial_q f = f(q^{-1}x) - f(x).$$

We introduce the following operator

$$\Lambda_q f(x) = f(qx),$$

then

$$\langle \Lambda_q f, g \rangle = q^{-2(\nu+1)} \left\langle f, \Lambda_q^{-1} g \right\rangle .$$

So

$$\frac{1}{1-q^{2(\nu+1)}}\left[\partial_q^* xf(x) - q^{2\nu+2}\Lambda_q x\partial_q f(x)\right] = f(x)$$

Assume that xf and $x\mathcal{F}_{q,\nu}f$ belongs to the space $\mathcal{L}_{q,2,\nu}$. Then we have

$$\langle f, f\rangle = -\frac{1}{1-q^{2(\nu+1)}}\langle xf, \partial_q f\rangle - \frac{1}{1-q^{2(\nu+1)}}\left\langle \partial_q f, x\Lambda_q^{-1}f\right\rangle.$$

By Cauchy-Schwartz inequality we get

$$\langle f, f\rangle \leq \frac{1}{1-q^{2(\nu+1)}}\|xf\|_2\|\partial_q f\|_2 + \frac{1}{1-q^{2(\nu+1)}}\|\partial_q f\|_2\left\|x\Lambda_q^{-1}f\right\|_2.$$

On the other hand

$$\left\|x\Lambda_q^{-1}f\right\|_2 = \sqrt{q}\times q^{\nu+1}\|xf\|_2,$$

Corollary 1 leads to the result. □

4. Hardy's theorem

The following Lemma from complex analysis is crucial for the proof of our main theorem.

Lemma 1. *For every $p \in \mathbb{N}$, there exist $\sigma_p > 0$ for which*

$$|z|^{2p}|j_\nu(z,q^2)| < \sigma_p e^{|z|}, \quad \forall z \in \mathbb{C}.$$

Proof. In fact

$$\begin{aligned}|z|^{2p}|j_\nu(z,q^2)| &\leq \frac{1}{(q^2,q^2)_\infty(q^{2\nu+2},q^2)_\infty}\sum_{n=0}^{\infty} q^{n(n-1)}|z|^{2n+2p}\\ &\leq \frac{q^{p(p+1)}}{(q^2,q^2)_\infty(q^{2\nu+2},q^2)_\infty}\sum_{n=p}^{\infty} q^{n(n-2p-1)}|z|^{2n}.\end{aligned}$$

Now using the Stirling's formula

$$n! \sim \sqrt{2\pi n}\frac{n^n}{e^n},$$

we see that there exist an entire $n_0 \geq p$ such that

$$q^{n(n-2p-1)} < \frac{1}{(2n)!}, \quad \forall n \geq n_0,$$

which implies

$$\sum_{n=n_0}^{\infty} q^{n(n-2p-1)}|z|^{2n} < \sum_{n=n_0}^{\infty}\frac{1}{(2n)!}|z|^{2n} < e^{|z|}.$$

Finally there exist $\sigma_p > 0$ such that

$$\frac{|z|^{2p}|j_\nu(z,q^2)|}{e^{|z|}} < \sigma_p, \quad \forall z \in \mathbb{C}$$

This complete the proof. □

Lemma 2. *Let h be an entire function on $\mathbb{C}$ such that*

$$|h(z)| \leq Ce^{a|z|^2}, \quad z \in \mathbb{C},$$

$$|h(x)| \leq Ce^{-ax^2}, \quad x \in \mathbb{R},$$

for some positive constants a and C. Then there exist $C^* \in \mathbb{R}$ such

$$h(x) = C^* e^{-ax^2}.$$

The reader can see the reference [17] for the proof.

Now we are in a position to state and prove the q-analogue of the Hardy's theorem

Theorem 7. *Suppose $f \in \mathcal{L}_{q,1,\nu}$ satisfying the following estimates*

$$|f(x)| \leq Ce^{-\frac{1}{2}x^2}, \quad \forall x \in \mathbb{R}_q^+, \tag{8}$$

$$|\mathcal{F}_{q,\nu} f(x)| \leq Ce^{-\frac{1}{2}x^2}, \quad \forall x \in \mathbb{R},$$

where C is a positive constant. Then there exist $A \in \mathbb{R}$ such that

$$f(z) = Ac_{q,\nu}\mathcal{F}_{q,\nu}\left(e^{-\frac{1}{2}x^2}\right)(z), \quad \forall z \in \mathbb{C}.$$

Proof. We claim that $\mathcal{F}_{q,\nu} f$ is an analytic function and there exist $C' > 0$ such that

$$|\mathcal{F}_{q,\nu} f(z)| \leq C' e^{\frac{1}{2}|z|^2}, \quad \forall z \in \mathbb{C}.$$

We have

$$|\mathcal{F}_{q,\nu} f(z)| \leq c_{q,\nu} \int_0^\infty |f(x)||j_\nu(zx, q^2)| x^{2\nu+1} d_q x.$$

From the Lemma 1, if $|z| > 1$ then there exist $\sigma_1 > 0$ such that

$$x^{2\nu+1}|j_\nu(zx, q^2)| = \frac{1}{|z|^{2\nu+1}}(|z|\,x)^{2\nu+1}|j_\nu(zx, q^2)| < \frac{\sigma_1}{1 + |z|^2 x^2} e^{x|z|}, \quad \forall x \in \mathbb{R}_q^+.$$

Then we obtain

$$|\mathcal{F}_{q,\nu} f(z)| \leq C\sigma_1 c_{q,\nu}\left[\int_0^\infty \frac{e^{-\frac{1}{2}(x-|z|)^2}}{1+|z|^2 x^2} d_q x\right] e^{\frac{1}{2}|z|^2} < C\sigma_1 c_{q,\nu}\left[\int_0^\infty \frac{1}{1+x^2} d_q x\right] e^{\frac{1}{2}|z|^2}.$$

Now, if $|z| \leq 1$ then there exist $\sigma_2 > 0$ such that

$$x^{2\nu+1}|j_\nu(zx, q^2)| \leq \sigma_2 e^x, \quad \forall x \in \mathbb{R}_q^+.$$

Therefore

$$|\mathcal{F}_{q,\nu} f(z)| \leq C\sigma_2 c_{q,\nu}\left[\int_0^\infty e^{-\frac{1}{2}x^2+x} d_q x\right] \leq C\sigma_2 c_{q,\nu}\left[\int_0^\infty e^{-\frac{1}{2}x^2+x} d_q x\right] e^{\frac{1}{2}|z|^2},$$

which leads to the estimate (8). Using Lemma 2, we obtain

$$\mathcal{F}_{q,\nu} f(z) = \text{const.} e^{-\frac{1}{2}z^2}, \quad \forall z \in \mathbb{C},$$

and by Theorem 1, we conclude that

$$f(z) = \text{const.}\mathcal{F}_{q,\nu}\left(e^{-\frac{1}{2}t^2}\right)(z), \quad \forall z \in \mathbb{C}.$$

□

Corollary 2. *Suppose $f \in \mathcal{L}_{q,1,\nu}$ satisfying the following estimates*

$$|f(x)| \le Ce^{-px^2}, \quad \forall x \in \mathbb{R}_q^+,$$

$$|\mathcal{F}_{q,\nu} f(x)| \le Ce^{-\sigma x^2}, \quad \forall x \in \mathbb{R},$$

where C, p, σ are a positive constant and $p\sigma = \frac{1}{4}$. We suppose that there exist $a \in \mathbb{R}_q^+$ such that $a^2 p = \frac{1}{2}$. Then there exist $A \in \mathbb{R}$ such that

$$f(z) = Ac_{q,\nu}\mathcal{F}_{q,\nu}\left(e^{-\sigma t^2}\right)(z), \quad \forall z \in \mathbb{C}.$$

Proof. Let $a \in \mathbb{R}_q^+$, and put

$$f_a(x) = f(ax),$$

then

$$\mathcal{F}_{q,\nu} f_a(x) = \frac{1}{a^{2\nu+2}}\mathcal{F}_{q,\nu} f(x/a).$$

In the end, applying Theorem 7 to the function f_a. □

Corollary 3. *Suppose $f \in \mathcal{L}_{q,1,\nu}$ satisfying the following estimates*

$$|f(x)| \le Ce^{-px^2}, \quad \forall x \in \mathbb{R}_q^+,$$

$$|\mathcal{F}_{q,\nu} f(x)| \le Ce^{-\sigma x^2}, \quad \forall x \in \mathbb{R}, \tag{9}$$

where C, p, σ are a positive constant and $p\sigma > \frac{1}{4}$. We suppose that there exist $a \in \mathbb{R}_q^+$ such that $a^2 p = \frac{1}{2}$. Then $f \equiv 0$.

Proof. In fact there exists $\sigma' < \sigma$ such that $p\sigma' = \frac{1}{4}$. Then the function f satisfying the estimates of Corollary 2, if we replacing σ by σ'. Which implies

$$\mathcal{F}_{q,\nu} f(x) = \text{const.}e^{-\sigma' x^2}, \quad \forall x \in \mathbb{R}.$$

On the other hand, f satisfying the estimates (9), then

$$\left|\text{const.}e^{-\sigma' x^2}\right| \le Ce^{-\sigma x^2}, \quad \forall x \in \mathbb{R}.$$

This implies $\mathcal{F}_{q,\nu} f \equiv 0$, and by Theorem 1 we conclude that $f \equiv 0$. □

5. The q-Fourier-Neumann Expansions

The little q-Jacobi polynomials are defined for $\nu, \beta > -1$ by [15]

$$p_n(x; q^\nu, q^\beta; q) = {}_2\phi_1\left(\left.\begin{matrix} q^{n+\nu+\beta+1}, q^{-n} \\ q^{\nu+1} \end{matrix}\right| q; qx\right).$$

We define the functions

$$P_{\nu,n}(x; q^2) = \sigma_{q,\nu}(n) q^{-n(\nu+1)} \frac{(q^{2+2n}, q^{2\nu+2}; q^2)_\infty}{(q^{2+2n+2\nu}, q^2; q^2)_\infty} p_n(x^2; q^{2\nu}, 1; q^2)$$

and

$$\mathcal{J}_{\nu,n}(x; q^2) = \sigma_{q,\nu}(n) \frac{J_{\nu+2n+1}(q^n x; q^2)}{x^{\nu+1}},$$

where

$$\sigma_{q,\nu}(n) = \sqrt{\frac{1 - q^{2\nu+4n+2}}{1-q}}.$$

Consider $\mathcal{L}_{q,2}^{\nu}$ as an Hilbert space with the inner product

$$\langle f|g\rangle = \int_0^1 f(x)g(x)x^{2\nu+1}d_qx.$$

The q-Paley-Wiener space is defined by

$$PW_q^{\nu} = \left\{ f \in \mathcal{L}_{q,2,\nu} : f(x) = c_{q,\nu}\int_0^1 u(t)j_\nu(xt,q^2)t^{2\nu+1}d_qt, \quad u \in \mathcal{L}_{q,2}^{\nu} \right\}.$$

Proposition 4. *PW_q^{ν} is a closed subspace of $\mathcal{L}_{q,2,\nu}$ and with the inner product given in (3) is an Hilbert space.*

Proof. In fact, given $f \in \mathcal{L}_{q,2,\nu}$ and let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of element of PW_q^{ν} which converge to f in L^2-norm. For $n \in \mathbb{N}$, there exist $u_n \in \mathcal{L}_{q,2}^{\nu}$ such that

$$f_n(x) = c_{q,\nu}\int_0^1 u_n(t)j_\nu(xt,q^2)t^{2\nu+1}d_qt.$$

Moreover

$$\lim_{n\to\infty} \|f_n - f\|_{q,2,\nu} = 0.$$

This give

$$\lim_{n\to\infty} \|\mathcal{F}_{q,\nu}f_n - \mathcal{F}_{q,\nu}f\|_{q,2,\nu} = 0,$$

and then

$$\lim_{n\to\infty}\left[\int_0^1 |\mathcal{F}_{q,\nu}f_n(x) - \mathcal{F}_{q,\nu}f(x)|^2x^{2\nu+1}d_qx + \int_1^\infty |\mathcal{F}_{q,\nu}f(x)|^2x^{2\nu+1}d_qx\right] = 0,$$

which implies

$$\int_1^\infty |\mathcal{F}_{q,\nu}f(x)|^2x^{2\nu+1}d_qx = 0 \Rightarrow \mathcal{F}_{q,\nu}f(x) = 0, \quad \forall x \in \mathbb{R}_q^+ \cap]1,+\infty[.$$

Then $f \in PW_q^{\nu}$. □

Proposition 5. *We have*

$$\mathcal{F}_{q,\nu}(\mathcal{J}_{\nu,n})(x) = P_{\nu,n}(x;q^2)\chi_{[0,1]}(x), \quad \forall x \in \mathbb{R}_q^+.$$

As a consequence

$$\int_0^1 P_{\nu,n}(x;q^2)P_{\nu,m}(x;q^2)x^{2\nu+1}\, d_qx = \delta_{n,m}.$$

Proof. The following proof is identical to the proof of Lemma 1 in [1]. Using an identity established in [12, 13]

$$\begin{aligned}\int_0^\infty & t^{-\lambda}J_\mu(q^mt;q^2)J_\theta(q^nt;q^2)d_qt \\ &= (1-q)q^{n(\lambda-1)+(m-n)\mu}\frac{(q^{1+\lambda+\theta-\mu},q^{2\mu+2};q^2)_\infty}{(q^{1-\lambda+\theta+\mu},q^2;q^2)_\infty} \\ &\quad \times {}_2\phi_1\left(\begin{matrix} q^{1-\lambda+\mu+\theta}, q^{1-\lambda+\mu-\theta} \\ q^{2\mu+2}\end{matrix}\,\middle|\, q^2; q^{2m-2n+1+\lambda+\theta-\mu}\right), \quad (10)\end{aligned}$$

where $n, m \in \mathbb{Z}$ and $\theta, \mu, \lambda \in \mathbb{C}$ such that $\mathrm{Re}(1-\lambda+\theta+\mu) > 0$, θ, μ are not equal to a negative integer and

$$(\lambda+\theta+1-\mu)/2, \quad m-n+(\lambda+\theta+1-\mu)/2$$

are not a non-positive integer [13].

To evaluate $\mathcal{F}_{q,\nu}(\mathcal{J}_{\nu,n})(x)$ when $x = q^m \leq 1$, we take in (10)

$$q^m = x, \mu = \nu, \theta = \nu + 2n + 1, \lambda = 0$$

then

$$\begin{aligned}
\mathcal{F}_{q,\nu}(\mathcal{J}_{\nu,n})(x) &= \sigma_{q,\nu}(n)\frac{x^{-\nu}}{1-q}\int_0^\infty J_\nu(xt;q^2)J_{\nu+2n+1}(q^n t;q^2)\,d_qt \\
&= \sigma_{q,\nu}(n)q^{-n(\nu+1)}\frac{(q^{2+2n},q^{2\nu+2};q^2)_\infty}{(q^{2+2n+2\nu},q^2;q^2)_\infty}\,{}_2\phi_1\left(\left.\begin{matrix} q^{2+2\nu+2n}, q^{-2n} \\ q^{2\nu+2}\end{matrix}\right| q^2;q^2x^2\right) \\
&= P_{\nu,n}(x;q^2).
\end{aligned}$$

To evaluate $\mathcal{F}_{q,\nu}(\mathcal{J}_{\nu,m})(x)$ when $x = q^n > 1$, we consider in (10)

$$q^n = x, \mu = \nu + 2m + 1, \theta = \nu, \lambda = 0$$

In this way, $1 + \lambda + \theta - \mu = -2m$. This gives for $m \in \mathbb{N}$ a factor

$$(q^{-2m};q^2)_\infty = 0$$

on the numerator and then

$$\mathcal{F}_{q,\nu}(\mathcal{J}_{\nu,m})(x) = 0, \quad x > 1$$

By setting $\lambda = 1$, $\theta = \nu + 2n + 1$, and $\mu = \nu + 2m + 1$ in , it is clear that, for $n, m = 0, 1, 2, \ldots$,

$$\int_0^\infty J_{\nu+2n+1}(q^n x;q^2)J_{\nu+2m+1}(q^m x;q^2)\,\frac{d_qx}{x} = \frac{1}{\sigma_{q,\nu}(n)^2}\delta_{n,m}$$

and then

$$\int_0^\infty \mathcal{J}_{\nu,n}(x;q^2)\mathcal{J}_{\nu,m}(x;q^2)\,x^{2\nu+1}d_qx = \delta_{n,m}.$$

Now we use the arguments of q-Bessel Fourier analysis provided in this paper to show that

$$\left\langle P_{\nu,n}\chi_{[0,1]}, P_{\nu,m}\chi_{[0,1]}\right\rangle = \langle \mathcal{F}_{q,\nu}(\mathcal{J}_{v,n}), \mathcal{F}_{q,\nu}(\mathcal{J}_{v,m})\rangle = \langle \mathcal{J}_{\nu,n}, \mathcal{J}_{\nu,m}\rangle = \delta_{n,m}. \quad (11)$$

Another proof of the orthogonality of the little q-Jacobi polynomials can be found in [15] □

Proposition 6. *The systems*

$$\{\mathcal{J}_{\nu,n}\}_{n=0}^\infty, \quad \{P_{\nu,n}\}_{n=0}^\infty$$

forme two orthonormals basis respectively of the Hilbert spaces PW_q^v *and* $\mathcal{L}_{q,2}^\nu$.

Proof. From (11) we derive the orthonormality. To prove that the system $\{\mathcal{J}_{\nu,n}\}_{n=0}^\infty$ is complet in PW_q^ν, given a function $f \in PW_q^\nu$ such that

$$\langle f, \mathcal{J}_{\nu,n}\rangle = 0, \quad \forall n \in \mathbb{N}.$$

Then

$$\langle \mathcal{F}_{q,\nu}(f), \mathcal{F}_{q,\nu}(\mathcal{J}_{\nu,n})\rangle = 0, \quad \forall n \in \mathbb{N},$$

which implies

$$\left\langle \mathcal{F}_{q,\nu}(f), P_{\nu,n}\chi_{[0,1]}\right\rangle = \left\langle \mathcal{F}_{q,\nu}(f)\chi_{[0,1]}, P_{\nu,n}\right\rangle = \langle \mathcal{F}_{q,\nu}(f), P_{\nu,n}\rangle = 0, \quad \forall n \in \mathbb{N}.$$

From the definition of the polynomial $P_{\nu,n}$ we conclude that

$$\left\langle \mathcal{F}_{q,\nu}(f), t^{2n}\right\rangle = 0, \quad \forall n \in \mathbb{N}.$$

Then

$$c_{q,\nu}\sum_{n=0}^{\infty}(-1)^n\frac{q^{n(n+1)}}{(q^2,q^2)_n(q^{2\nu+2},q^2)_n}\left\langle \mathcal{F}_{q,\nu}(f),t^{2n}\right\rangle x^{2n}=0,\quad \forall x\in\mathbb{R}_q^+,$$

which can be written as

$$\mathcal{F}_{q,\nu}^2(f)(x)=0,\quad \forall x\in\mathbb{R}_q^+.$$

By the inversion formula (2) we conclude that $f=0$. From (11) we derive the orthonormality. To prove that the system $\{P_{\nu,n}\}_{n=0}^{\infty}$ is complet in $\mathcal{L}_{q,2}^{\nu}$, given a function $f\in\mathcal{L}_{q,2}^{\nu}$ such that

$$\langle f|P_{\nu,n}\rangle=0,\quad \forall n\in\mathbb{N}$$

Then

$$\langle f|t^{2n}\rangle=0,\quad \forall n\in\mathbb{N}.$$

Which leads to the result. □

Proposition 7. *Let $\lambda\in\mathbb{R}_q^+$ then*

$$c_{q,\nu}j_\nu(\lambda x;q^2)=\sum_{n=0}^{\infty}\mathcal{J}_{n,\nu}(\lambda;q^2)P_{n,\nu}(x),\quad \forall x\in[0,1]\cap\mathbb{R}_q^+.$$

As a consequence we have

$$\sum_{n=0}^{\infty}\left[P_{n,\nu}(x;q^2)\right]^2=\frac{x^{-2(\nu+1)}}{1-q},\quad \forall x\in[0,1]\cap\mathbb{R}_q^+$$

and for all $\lambda\in\mathbb{R}_q^+$

$$\sum_{n=0}^{\infty}\left[\mathcal{J}_{n,\nu}(\lambda;q^2)\right]^2=-\frac{q^\nu}{2(1-q)\lambda^{1+2\nu}}$$
$$\times\left[\frac{\lambda}{q}J_{\nu+1}(\lambda;q^2)J'_\nu(\lambda/q;q^2)-J_{\nu+1}(\lambda;q^2)J_\nu(\lambda/q;q^2)-J'_{\nu+1}(\lambda;q^2)J_\nu(\lambda/q;q^2)\right].$$

Proof. Let $\lambda\in\mathbb{R}$ and consider the function

$$\psi_\lambda:[0,1]\cap\mathbb{R}_q^+\to\mathbb{R},\quad x\mapsto c_{q,\nu}j_\nu(\lambda x;q^2).$$

Then $\psi_\lambda\in\mathcal{L}_{q,2}^{\nu}$ and we can write

$$\psi_\lambda(x)=\sum_{n=0}^{\infty}\langle\psi_\lambda|P_{n,\nu}\rangle P_{n,\nu}(x),\quad \forall x\in[0,1]\cap\mathbb{R}_q^+. \tag{12}$$

Note that

$$\langle\psi_\lambda|P_{n,\nu}\rangle=\left\langle\psi_\lambda,P_{n,\nu}\chi_{[0,1]}\right\rangle=\langle\psi_\lambda,\mathcal{F}_{q,\nu}(\mathcal{J}_{n,\nu})\rangle=\mathcal{F}_{q,\nu}^2(\mathcal{J}_{n,\nu})(\lambda)=\mathcal{J}_{n,\nu}(\lambda;q^2).$$

Then we deduce the result. Using the Parseval's theorem and (12) we obtain

$$\sum_{n=0}^{\infty}\left[P_{n,\nu}(x;q^2)\right]^2=\|\psi_x\|_{q,2,\nu}^2=\frac{x^{-2(\nu+1)}}{1-q}.$$

The second identity is deduced also from the Parseval's theorem

$$\sum_{n=0}^{\infty}\left[\mathcal{J}_{n,\nu}(\lambda;q^2)\right]^2=N_{q,\nu,2}^2(\psi_\lambda),$$

and the following relation proved in [14]

$$\int_0^1 \left[J_\nu(aqt;q^2)\right]^2 t d_q t = -\frac{(1-q)q^{\nu-1}}{2a}$$
$$\times\Big[aJ_{\nu+1}(aq;q^2)J'_\nu(a;q^2) - J_{\nu+1}(aq;q^2)J_\nu(a;q^2) - J'_{\nu+1}(aq;q^2)J_\nu(a;q^2)\Big].$$

□

References

[1] L. D. Abreu, O. Ciaurri and J. L. Varona, A q-linear analogue of the plane wave expansion, Advances in Applied Mathematics, Volume 50, Article 415-428 (2013)

[2] N. Bettaibi, A. Fitouhi and W. Binous, Uncertainty principles for the q-trigonometric Fourier transforms, Math. Sci. Res. J, Volume 11 (2007).

[3] N. Bettaibi, Uncertainty principles in q^2-analogue Fourier analysis, Math. Sci. Res. J, Volume 11 (2007).

[4] N. Bettaibi, F. Bouzeffour, H. Ben Elmonser and W. Binous, Elements of harmonic analysis related to the third basic zero order Bessel function, J. Math. Anal. Appl. Volume 342, Article 1203-1219 (2008).

[5] L. Dhaouadi, A. Fitouhi and J. El Kamel, Inequalities in q-Fourier Analysis, Journal of Inequalities in Pure and Applied Mathematics, Volume 7, Issue 5, Article 171 (2006).

[6] L. Dhaouadi, W. Binous and A. Fitouhi, Paley-Wiener theorem for the q-Bessel transform and associated q-sampling formula, Expo. Math. Volume 27, Issue 1, Article 55-72 (2009).

[7] A. Fitouhi and L. Dhaouadi, Positivity of the Generalized Translation Associated with the q-Hankel Transform, Constructive Approximation, Volume 34, Aricle 453-472 (2011).

[8] A. Fitouhi, M. Hamza and F. Bouzeffour, The q-j_α Bessel function, J. Appr. Theory, Volume 115, Article 144-166 (2002).

[9] A. Fitouhi, N. Bettaibi and R. Bettaieb, On Hardy's inequality for symmetric integral transforms and analogous, Appl. Math. Comput, Volume 198 (2008).

[10] G. Gasper and M. Rahman, Basic hypergeometric series, Encycopedia of mathematics and its applications, Volume 35, Cambridge university press (1990).

[11] F. H. Jackson, On a q-Definite Integrals, Quarterly Journal of Pure and Application Mathematics, Volume 41, Article 193-203 (1910).

[12] T.H. Koornwinder and R. F. Swarttouw, On q-analogues of the Hankel and Fourier Transforms, Trans. A. M. S, Volume 333, Article 445-461 (1992).

[13] T.H. Koornwinder, R.F. Swarttouw, On q-analogues of the Fourier and Hankel transforms, arXiv:1208.2521v1 [math.CA] (2012) (this paper is [12] with some corrections).

[14] H. T. Koelink and R. F. Swarttouw, On the zeros of the Hahn-Exton q-Bessel Function and associated q-Lommel polynomials, J. Math. Anal. Appl, Volume 186, Article 690-710 (1994).

[15] R. Koekoek and R. F. Swarttouw, The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue, Delft University of Technology, Report no. 98-17 (1998).

[16] R. F. Swarttouw, The Hahn-Exton q-Bessel functions, Ph. D. Thesis, Delft Technical University (1992).

[17] A. Sitaram and M. Sundari, An analogue of Hardy's theorem for very rapidly decreasing functions on semi-simple Lie groups, Pacific J, Math. Volume 177, Article 187-200 (1997).

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