Euler Institute for
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Discrete Mathematics and its Applications

An introduction to Umbral Calculus[*]

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Abstract

The aim of these lectures is to give an introduction to combinatorial aspects of Umbral Calculus. Seen in this light, Umbral Calculus is a theory of polynomials that count combinatorial objects. In the first two lectures we present the basics of Umbral Calculus as presented in the seminal papers Mullin and Rota (1970) and Rota, Kahaner, and Odlyzko (1973). In the third lecture we present an extension of the Umbral Calculus due to Niederhausen for solving recurrences and counting lattice paths. An overview of other extensions is given in the fourth lecture.

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1 Introduction

The roots of Umbral Calculus can be traced back to the previous century (see Bell (1938)). The name *Umbral Calculus* was invented by Sylvester (referred to as 'that great inventor of unsuccesful terminology' in Roman and Rota (1978)). This 'calculus' (also known under the names Blissard Calculus or Symbolic Calculus) is a set of heuristic devices in which subscripts are treated as powers. Let us look at an example to see what is meant by this (a recent exposition with more examples can be found in Guinand (1979)).
The Bernoulli numbers B_n are defined by the generating function

$$\sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = \frac{x}{e^x - 1}. \tag{1}$$

The magic trick used in the 19th century Umbral Calculus is to write

$$\sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \simeq \sum_{n=0}^{\infty} B^n \frac{x^n}{n!} = e^{Bx}, \tag{2}$$

where we use the $\simeq$ symbol to stress the purely formal character of this manipulation. A trivial standard algebraic manipulation then yields

$$e^{(B+1)x} - e^{Bx} \simeq x, \tag{3}$$

from which we deduce by equating coefficients of $\frac{x^n}{n!}$ that

$$(B+1)^n - B^n \simeq \delta_{1n}, \tag{4}$$

where δ_{1n} denotes the Kronecker delta. If we now expand (4) using the Binomial Theorem and change the superscripts back to subscripts, we obtain the following relation for the Bernoulli numbers:

$$\sum_{k=0}^{n-1} \binom{n}{k} B_k = \delta_{1n}, \tag{5}$$

which can be shown to be true (a standard direct proof is possible by considering the reciprocal power series $(e^x - 1)/x$). Many more examples of similar manipulations with valid results are known (see e.g. Guinand (1979)). It is interesting to note that this technique is used extensively in the well-known standard works Riordan (1958) and Riordan (1968). Attempts to put this 'calculus' on a firm foundation were unsuccesful (e.g. Bell (1940)). A major obstacle was to explain why it is not allowed to write:

$$\left(e^{Bx}\right)^2 \equiv e^{2Bx},$$

which yields the obviously invalid identity

$$\frac{2x}{e^x - 1} = \left(\frac{x}{e^x - 1}\right)^2.$$

A major breakthrough was achieved in Rota (1964), where the process of lowering superscripts was replaced by the action of functionals on polynomials in order to deal with problems related to partitions. This idea was extended in Mullin and Rota (1970) to a theory of shift-invariant operators acting on polynomials and an associated class of so-called polynomials of binomial type. This paper can also be seen as a direct predecessor to Joyal (1981); a recent extension of the concept of species that relates all polynomials of binomial type (including polynomials with negative coefficients) to species can be found in Senato, Venezia, and Yang (1997). Subsequently this modern form of Umbral Calculus was extended in Rota, Kahaner, and Odlyzko (1973) to the class of Sheffer polynomials.

In Section 2 we will present the main results of Mullin and Rota (1970). Sheffer polynomials and their use by Niederhausen for solving recurrences are the subject of Section 3. An overview of some extensions of the Umbral Calculus (multivariate polynomials, generalized differential operators, classical Umbral Calculus etc.) is given in Section 4.

These introductory lecture notes can only show a glimpse of the many ramifications of the Umbral Calculus. For a major overview of the Umbral Calculus we refer to Di Bucchianico and Loeb (1995). An unofficial HTML version can be found at

`http://www.win.tue.nl/math/bs/statistics/bucchianico/hypersurvey/`.

For biographical information (including pictures) on mathematicians, consult the excellent MacTutoor History of Mathematics archive at

`http://www-groups.dcs.st-and.ac.uk/ history//`.

Notation and conventions

The degree of a polynomial, is defined as usual. However, the degree of a nonzero constant is defined to be zero and the degree of the zero polynomial is defined to be -1.

$\mathbb{N}$ is defined to be the set $\{0, 1, 2, \ldots\}$.

The vector space of polynomials with coefficients in some fixed field of characteristic zero is denoted by $\mathcal{P}$. We refer to Van Hamme (1992) and Verdoodt (1996) for a version of the Umbral Calculus where the field is of non-zero characteristic

2 Shift-invariant operators and polynomials of binomial type

In this section we present the basic definitions and results of the Umbral Calculus. We follow the approach of Di Bucchianico (1997).

2.1 Shift-invariant operators

Definition 2.1.1 *The shift-operator E^a is the operator on $\mathcal{P}$ defined by $(E^a p)(x) := p(x+a)$ $(p \in \mathcal{P})$.*

Definition 2.1.2 *An operator T on $\mathcal{P}$ is called* **shift-invariant** *if $E^a T = T E^a$ for all a.*

Examples 2.1.3 Examples of shift-invariant operators are:

a) the identity operator I.

b) the differentiation operator D.

c) the operators E^a of Definition 2.1.1.

d) the forward difference operator $E^1 - I$.

e) the backward difference operator $I - E^{-1}$.

f) the Abel operators DE^a.

g) the Laguerre operator L , defined by

$$(Lp)(x) := -\int_0^\infty e^{-t}\, p'(x+t)\, dt$$

h) the Bernoulli operator J , defined by

$$(Jp)(x) := \int_x^{x+1} p(t)\, dt$$

Remark 2.1.4 If S is an invertible shift-invariant operator on $\mathcal{P}$, then its inverse S^{-1} is also shift-invariant, since $S^{-1}E^a = S^{-1}E^a S S^{-1} = S^{-1} S E^a S^{-1} = E^a S^{-1}$ for all a.

Definition 2.1.5 *A linear operator Q on $\mathcal{P}$ is called a* **delta operator** *if Q is shift-invariant and Qx is a nonzero constant.*

Examples 2.1.6 Examples of delta operators include b, d, e, f and g from Examples 2.1.3, but not a, c and h.

It is a remarkable fact that every linear shift-invariant operator has a Taylor-like expansion in terms of an arbitrary delta operator (see Theorem 2.3.9).
We start by proving this expansion theorem for the differentiation operator D, because this yields simple proofs for properties of shift-invariant operators.

Theorem 2.1.7 *Let D be the differentiation operator and define $q_n(x) = \frac{x^n}{n!}$ for all $n \in \mathbb{N}$. Then T is a linear shift-invariant operator on $\mathcal{P}$ if and only if*

$$T = \sum_{k=0}^{\infty} (Tq_k)(0)\, D^k.$$

Proof: '$\Leftarrow$' Note that the infinite sum is in fact a finite sum when applied to a polynomial and thus is a well-defined operator on $\mathcal{P}$. Shift-invariance of T follows from shift-invariance of D.
'$\Rightarrow$' Since $(q_n)_{n\in\mathbb{N}}$ is a basis for $\mathcal{P}$, it suffices to verify that both operators coincide when applied to q_n for all $n \in \mathbb{N}$. Using the Binomial Theorem, we obtain

$$(T\,q_n)(a) = (E^a T\,q_n)\,(0) = (T E^a\,q_n)(0) = \left(\sum_{k=0}^{n} q_{n-k}(a)\,T q_k\right)(0) = \left(\sum_{k=0}^{\infty} (T q_k)(0) D^k q_n\right)(a)$$

for all $n \in \mathbb{N}$ and all a. $\square$

Examples 2.1.8
a) Consider the shift-invariant operator E^a. Theorem 2.1.7 yields $E^a = e^{aD}$. Hence,

$$p(x+a) = (E^a p)(x) = \sum_{k=0}^{\infty} (D^k\,p)(x)\,\frac{a^k}{k!}$$

for all $p \in \mathcal{P}$, which is Taylor's Formula.
b) Consider the Laguerre operator of Example 2.1.3e. Since for $k \geq 1$ we have

$$L\left(\frac{x^k}{k!}\right)(0) = -\int_0^{\infty} e^{-t}\,\frac{t^{k-1}}{(k-1)!}\,dt \;=\; -1,$$

it follows that $L = -\sum_{k=0}^{\infty} D^k = D(D-I)^{-1}$.
c) Consider the Bernoulli operator of Example 2.1.3f. Since

$$J\left(\frac{x^k}{k!}\right)(0) = \int_0^1 \frac{t^k}{k!}\,dt = \frac{1}{(k+1)!},$$

it follows that $J = \sum_{k=0}^{\infty} \frac{D^k}{(k+1)!}$.

We now derive some corollaries from Theorem 2.1.7. Recall that the degree of a nonzero constant is defined to be zero and that the degree of the zero polynomial is defined to be -1.

Corollary 2.1.9 *a) If T is a linear shift-invariant operator on $\mathcal{P}$, then there exists a non-negative integer $n(T)$ such that* $\deg Tp = \max\{-1, \deg p - n(T)\}$ *for all $p \in \mathcal{P}$. The null space of T equals the set of polynomials with degree less than $n(T)$.*
b) If Q is a delta operator, then $\deg Qp = \max\{-1, \deg(p) - 1\}$ *and the null space of Q equals the set of constant polynomials.*

Proof: a) By Theorem 2.1.7, we have $T = \sum_{k=0}^{\infty} a_k D^k$ for some sequence $(a_n)_{n\in\mathbb{N}}$. It follows from $\deg D^k p = \max\{-1, \deg(p) - k\}$ that if we set $n(T) := \min\{k \in \mathbb{N} \;:\; a_k \neq 0\}$, then $\deg Tp = \max\{-1, \deg(p) - n(T)\}$ for all $p \in \mathcal{P}$. Thus $Tp = 0$ if and only if $\deg p < n(T)$.
b) By definition, $Q\,x$ is a nonzero constant. Thus a) implies that $\deg Qp = \max\{-1, \deg(p) - 1\}$ for all polynomials $p \in \mathcal{P}$. $\square$

The converse of Corollary 2.1.9a is not true. Fix $m \in \mathbb{N}$. The linear operator T on $\mathcal{P}$ defined by $Tx^k := 0$ if $k < m$, $Tx^m := 1$, $Tx^{m+1} := \frac{1}{2}x$ and $Tx^k := x^{k-m}$ if $k \geq m+2$. Then T is not shift-invariant, but $\deg Tp = \max\{-1, \deg(p) - m\}$ for all $p \in \mathcal{P}$.

Corollary 2.1.10 *Let T be a linear shift-invariant operator on $\mathcal{P}$. Then the following are equivalent:*

a) T is invertible.

b) $T1 \neq 0$.

c) $\deg p = \deg Tp$ for all $p \in \mathcal{P}$.

Proof: 'a $\Rightarrow$ b' The null space of an invertible linear operator consists of 0 only, so $T1 \neq 0$.
'b $\Rightarrow$ c' Since $T1 \neq 0$, it follows from Corollary 2.1.9a that $\deg p = \deg Tp$ for all $p \in \mathcal{P}$.
'c $\Rightarrow$ a' It suffices to prove that T is injective and surjective. If $p, q \in \mathcal{P}$ and $p \neq q$, then $T(p-q) \neq 0$ since $\deg(p-q) \geq 0$. Moreover, $\deg p = \deg Tp$ implies that $(Tx^n)_{n \in N}$ is a basis for $\mathcal{P}$. Hence, T is surjective. □

Corollary 2.1.11 *Any two linear shift-invariant operators on $\mathcal{P}$ commute.*

Proof: All linear shift-invariant operators can be represented as a formal power series in the differentiation operator D by Theorem 2.1.7. Since the action of these operators on a polynomial only involves finitely many terms of their expansions, the result follows. □

2.2 Polynomials of convolution type

The polynomials $q_n(x) = \frac{x^n}{n!}$ appeared in the proof of Theorem 2.1.7. These polynomials satisfy a binomial-like formula. We now study a general class of polynomials satisfying a binomial-like formula.
We first need some definitions.

Definition 2.2.1 *A sequence $(q_n)_{n \in \mathbb{N}}$ of polynomials is a* **sequence of polynomials of convolution type** *if*

$$q_n(x+y) = \sum_{k=0}^{n} q_k(x)\, q_{n-k}(y) \qquad (n = 0, 1, \ldots), \tag{6}$$

Note that if $(q_n)_{n \in \mathbb{N}}$ is of convolution type, then $(p_n)_{n \in \mathbb{N}} := (n!\, q_n)_{n \in \mathbb{N}}$ satisfies

$$p_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} p_k(x)\, p_{n-k}(y) \qquad (n = 0, 1, \ldots). \tag{7}$$

Such sequences are called polynomials of binomial type in Mullin and Rota (1970). Since the theory runs somewhat more smoothly when using polynomials of convolution type, they are used in these lecture notes. Before we continue to give a full description of the class of polynomials of convolution type, let us note some simple properties that follow directly from the defining equations (6).

Lemma 2.2.2 *If $(q_n)_{n \in \mathbb{N}}$ is a sequence of polynomials of convolution type such that $q_0 \neq 0$, then*

a) $q_0 = 1$

b) $\deg q_n \le n$

c) $q_n(0) = 0$ *for* $n \ge 1$.

Proof: a) Since $q_0 \neq 0$, we may write $q_0(x) = \sum_{k=0}^{N} a_k\, x^k$ $(a_N \neq 0)$. Let y be arbitrary. Then

$$\sum_{k=0}^{N} a_k\,(x+y)^k = q_0(x+y) = q_0(x)\,q_0(y) = q_0(y)\sum_{k=0}^{N} a_k\,x^k.$$

Comparing coefficients of x^N, we see that $a_N = q_0(y)\,a_N$, hence $q_0(y) = 1$. This proves the result, since y was arbitrary.
b) We proceed by induction on n. It follows from a) that the result is true for $n = 0$. Suppose $\deg q_n \le n$ for all $n < m$ $(m \ge 1)$. Suppose that $\deg q_m > m$, so $q_m(x) = \sum_{k=0}^{M} a_k\, x^k$ $(M > m, a_M \neq 0)$. It follows from (6) that

$$\sum_{k=0}^{M} a_k (2x)^k = q_m(2x) = \sum_{k=0}^{M} q_k(x)\, q_{m-k}(x).$$

Using the induction hypothesis, we see that the coefficient of x^M on the left-hand side equals $a_M\, 2^M$, whereas the coefficient of x^M on the right-hand side equals $2\,a_M$. This leads to $M = 1$, which is in contradiction with $M > m \ge 1$.
c) First note that $q_1(0) = 2\,q_1(0)\,q_0(0) = 2\,q_1(0)$. Hence, $q_1(0) = 0$. We now proceed by induction on n. Suppose $q_n(0) = 0$ for $1 \le n < m$. Then $q_n(0) = \sum_{k=0}^{n} q_k(0)\,q_{n-k}(0) = 0$. Hence, $q_n(0) = 2\,q_n(0)$ by the induction hypothesis, which implies that $q_n(0) = 0$. □

Definition 2.2.3 *Let* $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ *and* $\beta = (\beta_n)_{n\in\mathbb{N}}$ *be sequences in a commutative ring* $\mathcal{R}$. *The convolution* $\alpha * \beta$ *is the sequence defined by* $(\alpha * \beta)_n := \sum_{k=0}^{n} \alpha_k\,\beta_{n-k}$.
If $k \in \mathbb{N}$, *then* α^{k*} *is defined recursively as follows:* $\alpha^{0*} := (\delta_{0n})_{n\in\mathbb{N}}$ *(*δ_{0n} *is the Kronecker delta) and* $\alpha^{(k+1)*} := \alpha^{k*} * \alpha$.
For sake of brevity, we will write α_n^{k*} *instead of* $(\alpha^{k*})_n$.

If $\sum_{n=0}^{\infty} \alpha_n z^n$ is a formal power series, then α_n^{k*} is the coefficient of z^n in $(\sum_{n=0}^{\infty} \alpha_n\, z^n)^k$. In other words,

$$\alpha_n^{k*} = \sum_{i_1+\cdots+i_k=n} \alpha_{i_1}\,\ldots\,\alpha_{i_k}. \tag{8}$$

Note that the convolution operation is commutative and associative. Hence,

$$\sum_{k=0}^{n} \alpha_k^{i*}\,\alpha_{n-k}^{j*} = \alpha_n^{(i+j)*}. \tag{9}$$

Lemma 2.2.4 *Let* $(\alpha_n)_{n\in\mathbb{N}}$ *be a sequence in a commutative ring such that* $\alpha_0 = 0$. *Then:*

a) $\alpha_n^{k*} = 0$ *if* $k > n$ $(k, n \in \mathbb{N})$.

b) $\alpha_n^{n*} = (\alpha_1)^n$ *for all* $n \in \mathbb{N}$.

c) α_n^{k*} *is a polynomial in* $\alpha_1, \ldots, \alpha_{n-k+1}$ *for* $2 \le k \le n$ $(k, n \in \mathbb{N})$.

Proof: a) Since $\alpha_0 = 0$, $\left(\sum_{n=0}^{\infty} \alpha_n z^n\right)^k$ only has terms z^j with $j \ge k$.
b) Since $\alpha_0 = 0$, the only contribution to x^n in $\left(\sum_{n=0}^{\infty} \alpha_n z^n\right)^n$ comes from the coefficient α_1 of z.
c) This follows from (8), since $\alpha_0 = 0$ implies that the indices in the summation are at least 1. □

Theorem 2.2.5 *A sequence* $(q_n)_{n\in\mathbb{N}}$ *of polynomials is sequence of polynomials of convolution type if and only if there exists a scalar sequence* $(g_n)_{n\in\mathbb{N}}$ *with* $g_0 = 0$*, such that*

$$q_n(x) = \sum_{k=0}^{n} g_n^{k*} \, \frac{x^k}{k!}$$

for all $n \in \mathbb{N}$ *and all* x*. In both cases, the following formal generating function relation holds:*

$$\sum_{n=0}^{\infty} q_n(x)\, t^n = e^{x\, g(t)}, \tag{10}$$

where $g(t) = \sum_{n=0}^{\infty} g_n\, t^n$*.*

Proof: '⇐' This follows from direct substitution and (9).
'⇒' We use induction on n. If $n = 0$, then $q_0(x) = 1 = g_0^{0*} \frac{x^0}{0!}$ for all x. Since $q_1(x+y) = q_0(x)\, q_1(y) + q_1(x)\, q_0(y) = q_1(x) + q_1(y)$, it follows that $q_1(x) = q_1(1)\, x$. So $q_1(x) = g_1^{0*} \frac{x^0}{0!} + g_1^{1*} \frac{x^1}{1!} = g_1 x$, if we set $g_0 := 0$ and $g_1 := q_1(1)$.
Suppose that we have $g_0, g_1, \ldots, g_{n-1}$ $(n > 1)$ such that $g_0 = 0$ and $q_m(x) = \sum_{k=0}^{m} g_m^{k*} \frac{x^k}{k!}$ for $m < n$. It follows from (6) that q_n is a solution of the following linear functional equation in p:

$$p(x+y) - p(x) - p(y) = \sum_{k=1}^{n-1} p_k(x)\, p_{n-k}(y)$$

for all x, y.
It follows from Lemma 2.2.4c and (9 that for any choice of g_n, the polynomial p, defined by $p(x) := \sum_{k=0}^{n} g_n^{k*} \frac{x^k}{k!}$, is a well-defined solution of this functional equation. Thus $(q_n - p)(x + y) = (q_n - p)(x) + (q_n - p)(y)$ for all x, y. Hence, there exists c such that $(q_n - p)(x) = cx$ for all x. Differentiating at 0, we obtain that $c = q_n'(0) - p'(0) = q_n'(0) - g_n$. Hence, if we choose $g_n = q_n'(0)$, then $q_n(x) = \sum_{k=0}^{n} g_n^{k*} \frac{x^k}{k!}$.

The formal generating function follows since

$$\sum_{n=0}^{\infty} q_n(x)\, t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} g_n^{k*} \, \frac{x^k}{k!}\, t^n \; = \sum_{k=0}^{\infty} \left(\sum_{n=0}^{\infty} g_n^{k*} \; t^n \right) \frac{x^k}{k!} = e^{x\, g(t)}.$$

□

Definition 2.2.6 *Let* $(q_n)_{n\in\mathbb{N}}$ *be a sequence of polynomials of convolution type. The* **coefficient sequence** *of* $(q_n)_{n\in\mathbb{N}}$ *is the sequence* $(g_n)_{n\in\mathbb{N}}$ *such that* $q_n(x) = \sum_{k=0}^{n} g_n^{k*} \frac{x^k}{k!}$*.*

2.3 Basic sequences

In this subsection we show the relation between delta operators and polynomials of convolution type.

Definition 2.3.1 *Let Q be a delta operator. A sequence $(q_n)_{n\in\mathbb{N}}$ of polynomials is a* **basic sequence** *for Q if:*

1. $q_0 = 1$
2. $q_n(0) = 0$ *if* $n \geq 1$
3. $Qq_n = q_{n-1}$ *if* $n \geq 1$.

Remark 2.3.2 It follows from (1), (3) and Corollary 2.1.9b that $\deg q_n = n$ for all $n \in \mathbb{N}$.

Theorem 2.3.3 *There exists a unique basic sequence for every delta operator.*

Proof: Let Q be an arbitrary delta operator. It follows from Theorem 2.1.7 and Corollary 2.1.9b that there exists a sequence $(\alpha_n)_{n\in\mathbb{N}}$ with $\alpha_1 \neq 0$ such that $Q = \sum_{k=1}^{\infty} \alpha_k D^k$. By Remark 2.3.2, we must construct polynomials q_n of degree n. By (1) of Definition 2.3.1, $q_0 = 1$. Suppose by induction that $q_{n-1} = \sum_{k=0}^{n-1} a_{n-1,k} x^k$ has been constructed. Since $\deg q_n = n$, q_n must be of the form $\sum_{k=0}^{n} a_{n,k} x^k$. Because $q_n(0) = 0$ by (3) of Definition 2.3.1, $a_{n,0}$ must be zero. Substitution of $Q = \sum_{k=1}^{\infty} \alpha_k D^k$ into $Qq_n = q_{n-1}$ and comparing coefficients yields the following system of equations:

$$
\begin{aligned}
a_{n-1,n-1} &= \alpha_1 n a_{n,n} \\
a_{n-1,n-2} &= \alpha_1 (n-1) a_{n,n-1} + \alpha_2 n(n-1) a_{n,n} \\
\vdots \quad &\quad \vdots \\
a_{n-1,1} &= \alpha_1 . 2 a_{n,2} + \alpha_2 . 2.3 a_{n,3} + \cdots + \alpha_{n-1} n! a_{n,n}
\end{aligned}
$$

Because $\alpha_1 \neq 0$ this system of equations has a unique solution. This proves uniqueness and existence. □

Examples 2.3.4

a) The differentiation operator D has basic sequence $\left(\frac{x^n}{n!}\right)_{n\in\mathbb{N}}$.

b) The forward difference operator $E^1 - I$ has basic sequence $\left(\binom{x}{n}\right)_{n\in\mathbb{N}}$, where

$$\binom{x}{n} := \frac{x(x-1)\ldots(x-n+1)}{n!}$$

are the lower factorials.

c) The backward difference operator $I - E^{-1}$ has basic sequence $\left(\binom{x+n-1}{n}\right)_{n\in\mathbb{N}}$, where

$$\binom{x+n-1}{n} := \frac{x(x+1)\ldots(x+n-1)}{n!}$$

are the upper factorials .

d) The Abel operator DE^a has basic sequence $\left(\frac{x(x-na)^{n-1}}{n!}\right)_{n\in\mathbb{N}}$, the Abel polynomials.

Theorem 2.3.5 *The basic sequence of a delta operator is a sequence of polynomials of convolution type.*

Proof: Let Q be a delta operator with basic sequence $(q_n)_{n\in\mathbb{N}}$. According to Definition 2.2.1 we have to prove

$$q_n(x+y) = \sum_{k=0}^{n} q_k(x)\, q_{n-k}(y) \tag{11}$$

for all $n \in \mathbb{N}$ and all x, y We proceed by induction on n. The case $n = 0$ is trivial because $q_0 = 1$ by Lemma 2.2.2a.
Suppose by induction that (11) has been proved for $m < n$. Fix y. It follows from Definition 2.1.5 that $QE^y q_n = E^y Q q_n = E^y q_{n-1}$. Hence,

$$Q\left(E^y q_n - \sum_{j=0}^{n} q_j\, q_{n-j}(y)\right) = E^y q_{n-1} - \sum_{j=1}^{n} q_{j-1}\, q_{n-j}(y) =$$

$$E^y q_{n-1} - \sum_{k=0}^{n} q_k\, q_{n-1-k}(y) = 0.$$

Corollary 2.1.9b implies that $E^y q_n - \sum_{k=0}^{n} q_k\, q_{n-k}(y)$ is a constant. So $q_n(x+y) = c + q_k(x)\, q_{n-k}(y)$. Evaluating at $x = 0$ we obtain $c = 0$, since $q_n(0) = 1$ for $n \geq 1$. Because y was arbitrary, we obtain $q_n(x+y) = \sum_{k=0}^{n} q_k(x)\, q_{n-k}(y)$ for all x, y. $\square$

Remark 2.3.6 Theorem 2.3.5 shows that the polynomials appearing in Examples 2.3.4 are of convolution type. This yields the following formulas:

a)

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

(the well-known Binomial Formula).

b)

$$\binom{x+y}{n} = \sum_{k=0}^{n} \binom{x}{k}\binom{y}{n-k}$$

(this is the Vandermonde convolution formula, see e.g. Riordan (1968, p. 8))

c)

$$\binom{x+y+n-1}{n} = \sum_{k=0}^{n} \binom{x+n-1}{k}\binom{y+n-k-1}{n-k}$$

(another form of the Vandermonde convolution formula, since $\binom{x+k-1}{k} = (-1)^k\binom{-x}{k}$)

d)

$$(x+y)(x+y-na)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} x\,(x-ka)^{k-1}\, y\,(y-(n-k)a)^{n-k-1}$$

(this is the Abel generalization of the Binomial Formula, see e.g. Riordan (1968, p. 18)).

The following theorem is a converse to Theorem 2.3.5.

Theorem 2.3.7 *Let $(q_n)_{n\in\mathbb{N}}$ be a sequence of polynomials of convolution type such that* $\deg q_n = n$. *Then there exists a unique delta operator Q with basic sequence $(q_n)_{n\in\mathbb{N}}$.*

Proof: Since $(q_n)_{n\in\mathbb{N}}$ is a basis for $\mathcal{P}$, there exists a unique linear operator Q on $\mathcal{P}$ such that $Qq_n = q_{n-1}$ $(n \geq 1)$ and $Q\,q_0 = 0$. Since $\deg q_1 = 1$, it follows that Qx is a nonzero constant. Shift-invariance of Q follows from

$$QE^y q_n = Q\left(\sum_{k=0}^{n} q_{n-k}(y)\,q_k\right) = \sum_{k=1}^{n} q_{n-k}(y)\,q_{k-1} = \sum_{h=0}^{n-1} q_{n-1-h}(y)\,q_h \;= E^y\,q_{n-1} \;= E^y Q q_n.$$

Hence, $TE^y = E^y T$, since $(q_n)_{n\in\mathbb{N}}$ is a basis for $\mathcal{P}$. □

Theorem 2.3.8 (Polynomial Expansion Theorem) *Let Q be a delta operator with basic sequence $(q_n)_{n\in\mathbb{N}}$. Then*

$$p = \sum_{k=0}^{\infty} (Q^k p)(0)\,q_k$$

for all $p \in \mathcal{P}$.

Proof : Let $p \in \mathcal{P}$ be arbitrary and let n be the degree of p. By Remark 2.3.2, there exist constants c_k such that $p = \sum_{k=0}^{n} c_k\,q_k$. It follows that $Q^r p = \sum_{k=r}^{n} c_k\,q_{k-r}$ for $0 \leq r \leq n$. Evaluating at zero yields $c_r = (Q^r p)(0)$ since $q_k(0) = 0$ for $k \geq 1$. Hence, $p = \sum_{k=0}^{\infty} (Q^k p)(0)\,q_k$. □

The following theorem generalizes Theorem 2.1.7.

Theorem 2.3.9 (Operator Expansion Theorem) *Let T be a linear shift-invariant operator on $\mathcal{P}$ and let Q be a delta operator with basic sequence $(q_n)_{n\in\mathbb{N}}$. Then:*

a) $T = \sum_{k=0}^{\infty} (T q_k)(0) Q^k$

b) In particular, if $(g_n)_{n\in\mathbb{N}}$ is the coefficient sequence of $(q_n)_{n\in\mathbb{N}}$, then $D = \sum_{n=0}^{\infty} g_n Q^n$ and $Q = \sum_{n=0}^{\infty} \overline{g}_n D^n$ where $\sum_{n=0}^{\infty} \overline{g}_n t^n$ is the composition inverse of the formal power series $\sum_{n=0}^{\infty} g_n t^n$.

Proof: a) Let $p \in \mathcal{P}$ be arbitrary with degree n. Applying Lemma 2.3.8 to $E^y p$, we obtain

$$T E^y p = \sum_{k=0}^{n} (Q^k E^y p)(0)\, T q_k = \sum_{k=0}^{n} (Q^k p)(y)\, T q_k.$$

Hence,

$$(Tp)(y) = (E^y T p)(0) = (T E^y p)(0) = \sum_{k=0}^{n} (T q_k)(0) \left(Q^k p\right)(y) = \sum_{k=0}^{\infty} (T q_k)(0) \left(Q^k p\right)(y)$$

for all y. This completes the proof, since p is arbitrary.
b) It follows from $q_n(x) = \sum_{k=0}^{n} g_n^{k*} \frac{x^k}{k!}$ that $(Dq_n)(0) = g_n$ for all $n \in \mathbb{N}$. Thus a) yields $D = \sum_{n=0}^{\infty} (Dq_n)(0) Q^n = \sum_{n=0}^{\infty} g_n Q^n$. Since $g_0 = 0$, the formal power series $\sum_{n=0}^{\infty} g_n\, t^n$ has a compositional inverse. □

There also exist operator expansions in terms of arbitrary degree reducing operators. The coefficients of these expansions are polynomials in x rather than constants (see Di Bucchianico and Loeb (1996a) and Kurbanov and Maksimov (1986)).

Examples 2.3.10 a) We want to expand the differentiation operator D in powers of the forward difference operator $E^1 - I$. The basic sequence of $E^1 - I$ is $\left(\binom{x}{n}\right)_{n\in\mathbb{N}}$, so

$$D = \sum_{k=0}^{\infty} \left(D\binom{x}{k}\right)(0)\, (E^1 - I)^k = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (E^1 - I)^k.$$

This is a classical formula for numerical differentiation.
b) Consider the shift operator E^a. Expanding E^a in powers of $E^1 - I$ yields

$$E^a = \sum_{k=0}^{\infty} \binom{a}{k} (E^1 - I)^k.$$

This is Newton's forward difference interpolation formula.

2.4 Explicit formulas for polynomials of convolution type

In this subsection we show how to compute the basic sequence of a delta operator. and how to compute connection coefficients. For this we need to introduce a derivation on the algebra of shift-invariant operators.

Definition 2.4.1 *If T is a linear operator on $\mathcal{P}$, then its Pincherle derivative T' is defined by $T' := T\,\underline{\mathbf{x}} - \underline{\mathbf{x}}\,T$ where the linear operator $\underline{\mathbf{x}}$ is defined by $(\underline{\mathbf{x}}\, p)(x) := x\, p(x)$ for all x and all polynomials $p \in \mathcal{P}$.*

The Pincherle derivative was introduced by Pincherle in Pincherle (1897, Section 56).

The following lemma lists some elementary properties of the Pincherle derivative.

Lemma 2.4.2 *a) If $T = \sum_{k=0}^{\infty} a_k\, D^k$, then $T' = \sum_{k=0}^{\infty} k\, a_k\, D^{k-1}$.*

b) *The Pincherle derivative of a linear shift-invariant operator on $\mathcal{P}$ is a linear shift-invariant operator on $\mathcal{P}$.*

c) *The Pincherle derivative of a delta operator is an invertible shift-invariant operator on $\mathcal{P}$.*

d) *If T and S are linear shift-invariant operators on $\mathcal{P}$, then $(T\,S)' = T'\,S + T\,S'$.*

Proof: a) Since $\underline{\mathbf{x}}$ is a linear operator on $\mathcal{P}$, it suffices to prove a) for the polynomials $\frac{x^n}{n!}$. We have

$$T'\,\frac{x^n}{n!} := (T\,\underline{\mathbf{x}} - \underline{\mathbf{x}}\,T)\,\frac{x^n}{n!} = (n+1)\sum_{k=0}^{\infty} a_k\,D^k\,\frac{x^{n+1}}{(n+1)!} - x\sum_{k=0}^{\infty} a_k\,D^k\,\frac{x^n}{n!} =$$

$$(n+1)\sum_{k=0}^{n+1}\frac{x^{n+1-k}}{(n+1-k)!} - \sum_{k=0}^{n}(n+1-k)\,a_k\,\frac{x^{n+1-k}}{(n+1-k)!} =$$

$$\sum_{k=0}^{n+1} k\,a_k\,\frac{x^{n+1-k}}{(n+1-k)!} = \sum_{i=0}^{n}(i+1)\,a_{i+1}\,D^i\,\frac{x^n}{n!}.$$

Hence, $T' = \sum_{i=0}^{\infty}(i+1)\,a_{i+1}\,D^i$, since $\underline{\mathbf{x}}$ is a linear operator on $\mathcal{P}$.
b) This follows directly from a) and Theorem 2.1.7.
c) By Theorem 2.1.7 and Corollary 2.1.9b we have $Q = \sum_{k=0}^{\infty} b_k\,D^k$ with $b_1 \neq 0$. We get from a) that $Q' = \sum_{i=0}^{\infty}(i+1)\,b_{i+1}\,D^i$. Hence, Q' is invertible by Corollary 2.1.10.
d) This follows from $(T\,S)' = T\,S\,\underline{\mathbf{x}} - \underline{\mathbf{x}}\,T\,S = (T\,S\,\underline{\mathbf{x}} - T\,\underline{\mathbf{x}}\,S) + (T\,\underline{\mathbf{x}}\,S - \underline{\mathbf{x}}\,T\,S) = T\,S' + T'\,S$. □

Lemma 2.4.3 *For every delta operator Q there exists a unique invertible shift-invariant operator U on $\mathcal{P}$ such that $Q = DU$.*

Proof: By Theorem 2.1.7 and Corollary 2.1.9b, we have $Q = \sum_{k=1}^{\infty} b_k\,D^k$ with $b_1 \neq 0$. Define U by $U := \sum_{k=0}^{\infty} b_{k+1}\,D^k$, so $Q = DU$. The invertibility of U follows from Corollary 2.1.10, since $b_1 \neq 0$. Uniqueness of U follows from the expansion of Q and U in powers of D. □

We now are ready explicit formulas for basic sequences of delta operators. Formulas a) through d) of Theorem 2.4.4 were already known to Steffensen (see Steffensen (1941, Sections 2 and 3); see also Rota, Kahaner, and Odlyzko (1973, Theorem 4)). The operator U that appears in the statement of Theorem 2.4.4 is the operator whose existence is assured by Lemma 2.4.3.

Theorem 2.4.4 *Let Q be a delta operator with basic sequence $(q_n)_{n\in\mathbb{N}}$ and let $(g_n)_{n\in\mathbb{N}}$ be the coefficient sequence of $(q_n)_{n\in\mathbb{N}}$. Let U be the unique invertible shift-invariant operator such that $Q = DU$. Then the following formulas hold for $n \geq 1$:*

a) $n!\,q_n = (Q'\,U^{-n-1})\,(x^n)$

b) $n!\,q_n = (U^{-n})\,(x^n) - (U^{-n})'\,(x^{n-1})$

c) $n!\,q_n = (\underline{\mathbf{x}}\,U^{-n})\,(x^{n-1})$ (**Transfer Formula**)

d) $n\,q_n = (x(Q')^{-1})\,q_{n-1}$ (**Rodrigues Formula**) *e)* $n\, q_n(x) = x\, \sum_{k=0}^{n} k\, g_k\, q_{n-k}(x)$.

Proof: Since $D' = I$, we have $Q'\, U^{-n-1}\, x^n = (D\, U)'\, U^{-n-1}\, x^n = ((D'\, U + D\, U')\, U^{-n-1})\, x^n = ((U + D\, U')\, U^{-n-1})\, x^n = (U^{-n} + D\, U'\, U^{-n-1})\, x^n = U^{-n}\, x^n + U'\, U^{-n-1}\, D\, x^n = U^{-n}\, x^n - (U^{-n})'\, x^{n-1} = U^{-n}\, x^n - (U^{-n}\, \underline{\mathbf{x}} - \underline{\mathbf{x}}\, U^{-n})\, x^{n-1} = (\underline{\mathbf{x}}\, U^{-n})\, x^{n-1}$, so the right-hand sides of a), b) and c) are identical. Since Q has a unique basic sequence by Theorem 2.3.3, it suffices to note that $\left(\underline{\mathbf{x}}\, U^{-n}\, x^{n-1}\right)(0) = 0$ and

$$\left(Q\, Q'\, U^{-n-1}\right)\, \frac{x^n}{n!} = \left(D\, U\, Q'\, U^{-n-1}\right)\, \frac{x^n}{n!} = \left(Q'\, U^{-n}\, D\right)\, \frac{x^n}{n!} = \left(Q'\, U^{-n}\right)\, \frac{x^{n-1}}{(n-1)!}$$

for $n \geq 1$. This proves a), b) and c).
By Lemma 2.4.2c, Q' is invertible. Thus it follows from a) that $\frac{x^{n-1}}{(n-1)!} = \left((Q')^{-1}\, U^n\right)\, q_{n-1}(x)$ for $n \geq 2$. By c),

$$nq_n(x) = \left(\underline{\mathbf{x}}\, U^{-n}\right)\, \frac{x^{n-1}}{(n-1)!} = \left(\underline{\mathbf{x}}\, U^{-n}(Q')^{-1}\, U^n\right)\, q_{n-1}(x) = \left(\underline{\mathbf{x}}\, (Q')^{-1}\right)\, q_{n-1}(x)$$

for $n \geq 2$. This proves d), since the case $n = 1$ follows from Lemma 2.4.2a and Theorem 2.3.9b. In order to prove e) we write $n\frac{q_n(x)}{x} = \sum_{k=0}^{n-1} c_k\, q_k(x)$. Using d) and Lemma 2.3.8 we obtain that

$$c_k = \left(Q^k\, n\, \frac{q_n(x)}{x}\right)\, (0) = \left(Q^k\, (Q')^{-1}\, q_{n-1}\right)\, (0) = ((Q')^{-1}\, q_{n-1-k})(0) =$$

$$(n-k)\, \left(\frac{q_{n-k}(x)}{x}\right)\, (0) = (n-k)\, g_{n-k}.$$

This completes the proof. □

The name Rodrigues Formula comes from the theory of orthogonal polynomials (see e.g. Chihara (1978, Rasala (1981)). An example of a classical Rodrigues Formula can be found in Example 2.4.5e.

Examples 2.4.5 We consider the delta operators of Examples 2.1.6 and use Theorem 2.4.4 to calculate the corresponding basic sequences (cf. Examples 2.3.4).
a) Consider the differentiation operator D. It is clear that $D' = I$ and that $U = I$, since $D = DI$. Thus Theorem 2.4.4a yields $q_n(x) = \frac{x^n}{n!}$.
b) Consider the forward difference operator $E^1 - I$. Then $\left(E^1 - I\right)' = \left(E^1\right)' = \left(e^D\right)'$ (use Theorem 2.3.9a) $= e^D$ (use Lemma 2.4.2a) $= E^1$. Thus Theorem 2.4.4d yields $q_n(x) = \frac{x}{n}\, E^{-1}\, q_{n-1}(x)$. Since $q_0 = 1$, induction on n yields

$$q_n(x) = \binom{x}{n} := \frac{x(x-1)\dots(x-n+1)}{n!}.$$

c) Consider the backward difference operator $I - E^{-1}$. In the same way as in b) we now find that

$$q_n(x) = \binom{x+n-1}{n} := \frac{x(x+1)\dots(x+n-1)}{n!}.$$

d) Consider the Abel operator DE^a for some fixed a. Obviously $U = E^a$, so $U^{-n} = E^{-na}$ for all $n \in \mathbb{N}$. Thus Theorem 2.4.4c yields

$$q_n(x) = \frac{x(x-na)^{n-1}}{n!}.$$

e) Consider the Laguerre operator L of Example 2.1.3g. We will show that the basic sequence of the Laguerre operator is the sequence of Laguerre polynomials $L_n^{(-1)}$. We know from Example 2.1.8b that $L = -\sum_{k=0}^{\infty} D^k = D\,(D-I)^{-1}$, hence $U = (D-I)^{-1}$ in this case. Thus Theorem 2.4.4c yields

$$q_n(x) = \frac{x^n}{n!}(D-I)^n\, x^{n-1} = \sum_{k=1}^{n} (-1)^k \binom{n-1}{k-1} \frac{x^k}{k!}.$$

Since

$$e^x\, D\left(e^{-x}\, p\right) = e^x\, \left(e^{-x}\, p' - e^{-x} p\right) = (D-I)\,(p),$$

we may write

$$q_n(x) = \frac{x^n}{n!}\, e^x\, D^n\, \left(e^{-x}\, x^{n-1}\right)$$

which is the classical Rodrigues formula for the Laguerre polynomials $L_n^{(-1)}$.
The formula $L\,q_n = q_{n-1}$ is the recurrence formula $q_n' = q_{n-1}' - q_{n-1}$, since $L = D\,(D-I)^{-1}$. Since $L' = -(D-I)^{-2}$, Theorem 2.4.4d yields $nq_n(x) = -x\,(D-I)^2\, q_{n-1}(x)$.

We conclude this section with a discussion of umbral operators. Umbral operators play an important role in the connection-constant problem which will be discussed below.

Definition 2.4.6 *An* **umbral operator** T *is a linear operator on* $\mathcal{P}$ *such that there exist basic sequences* $(r_n)_{n\in\mathbb{N}}$ *and* $(v_n)_{n\in\mathbb{N}}$ *with* $T\,r_n = v_n$ *for all* $n \in \mathbb{N}$.

If $(v_n)_{n\in\mathbb{N}} = \frac{x^n}{n!}$, then T "raises powers" (cf. Section 1), whereas if $(r_n)_{n\in\mathbb{N}} = \frac{x^n}{n!}$, then Tp is the umbral composition of p with $(v_n)_{n\in\mathbb{N}}$, i.e. it replaces powers of x by the corresponding member of $(v_n)_{n\in\mathbb{N}}$.

It is important to have basic sequences in Definition 2.4.6, since this implies deg $r_n =$ deg $v_n = n$ for all $n \in \mathbb{N}$ by Remark 2.3.2. Hence, both $(r_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are bases for $\mathcal{P}$.
Some important properties of umbral operators are listed in Theorem 2.4.8, which is an extension of Rota, Kahaner, and Odlyzko (1973, Proposition 1)). The following theorem is important for our proof of Theorem 2.4.8.

Theorem 2.4.7 *Let* Q *be a delta operator with basic sequence* $(q_n)_{n\in\mathbb{N}}$ *and let the sequence* $(p_n)_{n\in\mathbb{N}}$ *be given by* $p_n = \sum_{k=0}^{n} a_{n,k}\, q_k$. *Suppose* $(p_n)_{n\in\mathbb{N}}$ *is a sequence of polynomials with* $\deg p_n = n$ *for all* n.*Then* $(p_n)_{n\in\mathbb{N}}$ *is of convolution type if and only if there exists a sequence* $(\gamma_n)_{n\in\mathbb{N}}$ *such that* $\gamma_0 = 0$ *and* $a_{n,k} = \gamma_n^{k*}$ *for all* k *and* n. *Moreover,* $\gamma_n = (Q\,p_n)(0)$.

Proof: '$\Leftarrow$' This follows by direct computation using (6) and (9).
'$\Rightarrow$' We construct the sequence $(\gamma_n)_{n\in\mathbb{N}}$ by induction. Set $\gamma_0 = 0$. Since $p_1(0) = 0$ and $\deg p_1 = \deg q_1 = 1$, there is a unique γ_1 such that $p_1 = \gamma_1\, q_1$. Suppose by induction that γ_k has been constructed for $k < n$ such that $p_m = \gamma_m^{k*}\, q_k$ for $m < n$. Since $\gamma_0 = 0$, Lemma 2.2.4c yields that γ_n^{k*} is a polynomial in $\gamma_1, \ldots, \gamma_{n-1}$ for $2 \le k \le n$. Thus we can choose γ_n such that $p_n(1) = \sum_{k=0}^{n} \gamma_n^{k*}\, q_k(1)$. It follows from the associativity of convolution that $p_n(m) = \sum_{k=0}^{n} \gamma_n^{k*}\, q_k(m)$ for all $m \in \mathbb{N}$. Thus $p_n = \sum_{k=0}^{n} \gamma_n^{k*}\, q_n$, since p_n and $\sum_{k=0}^{n} \gamma_n^{k*}\, q_k$ are polynomials.
The last statement follows from $Q\, q_n = q_{n-1}$ and $q_n(0) = 0$ for $n \ge 1$. □

The following theorem describes the basic properties of umbral operators. As an introduction to parts d), e) and f), we let Q and P be delta operators with basic sequence $(q_n)_{n\in\mathbb{N}}$, $(p_n)_{n\in\mathbb{N}}$ respectively. Let T be the umbral operator that maps p_n to q_n for all $n \in \mathbb{N}$. This leads to the following commutative diagram.

$$\begin{tikzcd} q_n \arrow[r, "Q"] & q_{n-1} \arrow[d, "T"] \\ p_n \arrow[u, "T^{-1}"] \arrow[r, "P"] & p_{n-1} \end{tikzcd}$$

We immediately read off that $P = T\, Q\, T^{-1}$.

Theorem 2.4.8 *Let T be an umbral operator. Then:*

a) T is invertible.

b) T is shift-invariant if and only if $T = I$.

c) If $(p_n)_{n\in\mathbb{N}}$ is an arbitrary sequence of polynomials of convolution type, then $(T\, p_n)_{n\in\mathbb{N}}$ is also of convolution type.

d) If $(q_n)_{n\in\mathbb{N}}$ is the basic sequence of the delta operator Q, then $(T\, q_n)_{n\in\mathbb{N}}$ is the basic sequence of the delta operator $T\, Q\, T^{-1}$.

e) If Q is a delta operator with basic sequence $(q_n)_{n\in\mathbb{N}}$, then $T\, Q^n\, T^{-1} = P^n$, where P is the delta operator of the basic sequence $(T\, q_n)_{n\in\mathbb{N}}$.

f) If $Q = q(D)$ is a delta operator, $p_n := T\, \frac{x^n}{n!}$ and P is the delta operator of $(p_n)_{n\in\mathbb{N}}$, then $T\, Q\, T^{-1} = q(P)$.

Proof: Let $(r_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ be basic sequences such that $T\, r_n = v_n$. Let R and V be the delta operators of $(r_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}}$ respectively.
a) Since $\deg r_n = \deg v_n = n$ for all $n \in \mathbb{N}$, T is invertible by Corollary 2.1.10.
b) If T is shift-invariant, then Corollary 2.1.11 yields $R\, v_n = R\, T\, r_n = T\, R\, r_n = T\, r_{n-1} = v_{n-1}$ for $n \ge 1$. Hence, $r_n = v_n$ for all $n \in \mathbb{N}$, since both $(r_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are basic sequences for R.
c) By Theorem 2.4.7, there exists a sequence $(\gamma_n)_{n\in\mathbb{N}}$ such that $\gamma_0 = 0$ and $p_n = \sum_{k=0}^{n} \gamma_n^{k*}\, r_k$. Thus $T\, p_n = \sum_{k=0}^{n} \gamma_n^{k*}\, v_k$ and Theorem 2.4.7 implies that $(T\, p_n)_{n\in\mathbb{N}}$ is of convolution type.
d) We know from c) that $(T\, q_n)_{n\in\mathbb{N}}$ is of convolution type. Since $\deg q_1 = 1$, it follows from a) and Corollary 2.1.10 that $\deg (T\, q_n) = n$. Thus $(T\, q_n)_{n\in\mathbb{N}}$ is a basic sequence by Lemma 2.2.2

and Theorem 2.3.7. Because $(T\,Q\,T^{-1})\,(T\,q_n) = T\,q_{n-1}$ for $n \geq 1$, the same argument as in the proof of Theorem 2.3.7 yields that $T\,Q\,T^{-1}$ is the delta operator of $(T\,p_n)_{n\in\mathbb{N}}$.
e) This follows from d) and $(T\,Q\,T^{-1})^n = T\,Q^n\,T^{-1}$.
f) Since $(p_n)_{n\in\mathbb{N}}$ is a basis for $\mathcal{P}$ by a), it suffices to note that the Operator Expansion Theorem yields

$$T\,Q\,T^{-1}\,p_n = T\,Q\frac{x^n}{n!} = T\sum_{k=0}^{n}\left(Q\,\frac{x^k}{k!}\right)(0)\,\frac{x^{n-k}}{n-k!} = \sum_{k=0}^{n}\left(Q\,\frac{x^k}{k!}\right)(0)\,p_{n-k}$$

and

$$q(p(D)) = \sum_{k=0}^{\infty}\left(Q\,\frac{x^k}{k!}\right)(0)\,P^k\,p_n = \sum_{k=0}^{n}\left(Q\,\frac{x^k}{k!}\right)(0)\,p_{n-k}.$$

□

For a probabilistic interpretation of umbral operators in terms of subordination we refer to Di Bucchianico (1994).

Now that we know how to calculate basic sequences, we are ready to discuss the problem of connection coefficients. The problem of connection coefficients consists of finding numbers $a_{n,k}$ such that $p_n = \sum_{k=0}^{n} a_{n,k}\,q_k$ where $(p_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ are sequences of polynomials with $\deg p_n = \deg q_n = n$ for all $n \in \mathbb{N}$. Note that the connection coefficients are the coefficients of the basis change $(p_n)_{n\in\mathbb{N}}$ to $(q_n)_{n\in\mathbb{N}}$.

If both $(p_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ are sequences of polynomials of convolution type, then the Umbral Calculus gives the following elegant answer.

Theorem 2.4.9 *Let P and Q be delta operators with basic sequences $(p_n)_{n\in\mathbb{N}}$, $(q_n)_{n\in\mathbb{N}}$ respectively. Let T be the umbral operator defined by $T\,q_n := \frac{x^n}{n!}$ for all $n \in \mathbb{N}$. Then the constants $a_{n,k}$ $(k, n \in \mathbb{N})$, defined by $p_n := \sum_{k=0}^{n} a_{n,k}\,q_k$, are uniquely determined as follows. The polynomials r_n, defined by $r_n(x) := \sum_{k=0}^{n} a_{n,k}\,\frac{x^k}{k!}$, are the basic polynomials of the delta operator $T\,P\,T^{-1}$. Moreover, if $P = \sum_{i=1}^{\infty} a_i\,Q^i$, then $T\,P\,T^{-1} = \sum_{i=1}^{\infty} a_i\,D^i$.*

Proof: It follows from Theorem 2.4.8d that $T\,P\,T^{-1}$ is a delta operator with basic sequence $(T\,p_n)_{n\in\mathbb{N}}$. Since $r_n = T\,p_n$ for all $n \in \mathbb{N}$, $(r_n)_{n\in\mathbb{N}}$ is the basic sequence of $T\,P\,T^{-1}$. For the last statement, note that $T\,P\,T^{-1} = D$ by Theorem 2.4.8d, since $T\,q_n = \frac{x^n}{n!}$. The last statement follows from Theorem 2.4.8e. □

Examples 2.4.10 a) We want to express the lower factorials in terms of upper factorials of Example 2.3.4c, i.e. we want to calculate coefficients $a_{n,k}$ such that $\binom{x}{n} = \sum_{k=0}^{n} a_{n,k}\binom{x+k-1}{k}$. We apply Theorem 2.4.9 with $P = E^1 - I$, $Q = I - E^{-1}$. Let T be the umbral operator defined by $T\binom{x+n-1}{n} = \frac{x^n}{n!}$ for all $n \in \mathbb{N}$. Theorem 2.3.9a yields $P = \sum_{n=0}^{\infty}\left(P\binom{x+n-1}{n}\right)(0)\,Q^n = \sum_{n=1}^{\infty} Q^n$. Hence, it follows from Theorems 2.4.8d and 2.4.8e that

$$T\,P\,T^{-1} = \sum_{n=1}^{\infty}(T\,QT^{-1})^n = \sum_{n=1}^{\infty} D^n = D\,(I-D)^{-1}.$$

Thus the coefficients $a_{n,k}$ are the coefficients of x^k of the polynomials $q_n(-x)$, where $(q_n)_{n\in\mathbb{N}}$ are the Laguerre polynomials of Example 2.4.5e.

b) We want to derive duplication formulas for the Laguerre polynomials q_n of Example 2.4.5e. Fix α and define polynomials p_n by $p_n(x) := q_n(\alpha\, x)$ for all x. Let W be the umbral operator defined by $W\,x^n := \alpha^n\,x^n$. Note that $W\,q_n = p_n$. It follows from Theorem 2.4.8d that $(p_n)_{n\in\mathbb{N}}$ is the basic sequence of the delta operator P, defined by $P := W\,L\,W^{-1} = \alpha^{-1}\,D\,\left(\alpha^{-1}\,D - I\right)^{-1}$. Theorem 2.4.9 yields that the connection coefficients of $(p_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ are the coefficients of the basic sequence of the delta operator $T\,P\,T^{-1}$, where T is the umbral operator defined by $T\,q_n := \frac{x^n}{n!}$ for all $n\in\mathbb{N}$. By Theorem 2.3.9a,

$$D = \sum_{k=0}^{\infty} \left(D\,q_k(0)\right)\,L^k = \sum_{k=1}^{\infty} (-1)^k\,L^k = L\,(L-I)^{-1},$$

since $q_n(x) = \sum_{k=1}^{n} (-1)^k \binom{n-1}{k-1} \frac{x^k}{k!}$ (see Example 2.4.5e). Hence,

$$P = \alpha^{-1}\,L\,\left(I - (1-\alpha^{-1})\,L\right)^{-1} = L\,\left(\alpha I + (1-\alpha)\,L\right)^{-1}.$$

and the last statement of Theorem 2.4.9 yields

$$T\,P\,T^{-1} = D\,\left(\alpha I + (1-\alpha)\,D\right)^{-1}.$$

It follows from Theorem 2.4.4c that the basic sequence $(r_n)_{n\in\mathbb{N}}$ of $T\,P\,T^{-1}$ is given by

$$\frac{x}{n}\,\left(\alpha I\,+\,(1-\alpha)\,D\right)^n\,\frac{x^{n-1}}{n-1!} = \frac{x}{n}\sum_{k=0}^{n}\binom{n}{k}\,\alpha^k\,(1-\alpha)^{n-k}\,D^{n-k}\,\frac{x^{n-1}}{n-1!} =$$

$$\sum_{k=1}^{n}\binom{n}{k}\,\alpha^k\,(1-\alpha)^{n-k}\,\frac{x^{k-1}}{k-1!} = \sum_{k=1}^{n}\binom{n-1}{k-1}\,\alpha^k\,(1-\alpha)^{n-k}\,\frac{x^k}{k!}.$$

Putting everything together yields the following duplication formula for the Laguerre polynomials of Example 2.4.5e:

$$q_n(\alpha\,x) = \sum_{k=1}^{n}\binom{n-1}{k-1}\,\alpha^k\,(1-\alpha)^{n-k}\,q_k(x) \qquad\qquad (n\geq 1).$$

2.5 Combinatorial applications

In this section we mention some applications to combinatorics.

A powerful tool in combinatorics is Lagrange inversion. We will now show that Lagrange inversion formulas can be derived from the Transfer Formula (Theorem 2.4.4c).

Let $q(t) = \sum_{n=1}^{\infty} a_n\,t^n$ be a formal power series in t with $a_1 \neq 0$. Let $(q_n)_{n\in\mathbb{N}}$ be the basic sequence of $Q = q(D)$ with coefficient sequence $(g_n)_{n\in\mathbb{N}}$. We know from Theorem 2.3.9b that $\sum_{n=0}^{\infty} g_n\,t^n$ is the compositional inverse of q. Since $g_0 = 0$, it follows from Theorem 2.2.5 that

$$\left.\frac{q_n(x)}{x}\right|_{x=0} = g_n.$$

Now note that $(q(D)\, x^n)(0) = n!\, a_n$. Combining this with the Transfer Formula, we obtain the following simple form of the Lagrange Inversion Formula:

$$\langle \overline{q}(t) |\, t^n \rangle = \left\langle \left(\frac{q(t)}{t} \right)^{-n} \middle|\, x^{n-1} \right\rangle .$$

If instead we use Theorem 2.4.4a and $g_n = (D\, q_n)(0)$, then we obtain the following form of the Lagrange Inversion Formula:

$$\langle \overline{q}(t) \mid t^n \rangle = \left\langle t\, q'(t) \left(\frac{q(t)}{t} \right)^{-n-1} \middle|\, x^n \right\rangle .$$

More general Lagrange inversion formulas can be obtained similarly if we use more theory of umbral operators. (see e.g. Roman (1984)). Various forms of Lagrange inversion using operators can be found in Barnabei (1985), Hofbauer (1979), Joni (1978), Krattenthaler (1988), Niederhausen (1986b), Niederhausen (1992) and Verde-Star (1985).

Our next application deals with counting labeled trees. Recall that a tree is a connected graph that has no cycles. A rooted tree is a tree with a distinguished vertex (the root).

We first need a lemma.

Lemma 2.5.1 *Let U be an invertible shift-invariant operator. If $(q_n)_{n\in\mathbb{N}}$ is a basic sequence satisfying*

$$\left.\frac{q_n(x)}{x}\right|_{x=0} = (T^{-1}\, q_{n-1})(0),$$

then $D\, U$ is the delta operator of $(q_n)_{n\in\mathbb{N}}$.

Proof: Define the linear operator Q by $Qq_0 = 0$ and $Q\, q_n = q_{n-1}$ for $n \geq 1$. As in the proof of Theorem 2.3.7, we conclude that Q is shift-invariant. Then we have

$$\left.\frac{q_n(x)}{x}\right|_{x=0} = (U^{-1}\, Q\, q_n)(0).$$

Since $(q_n)_{n\in\mathbb{N}}$ is a basis for $\mathcal{P}$, this identity also holds when q_n is replaced by an arbitrary polynomial. In particular, it holds for $\frac{x^n}{n!}$. Now expand the shift-invariant operator $U^{-1}\, Q$ in powers of D by Theorem 2.1.7:

$$U^{-1}\, Q = \sum_{n=0}^{\infty} U^{-1}\, Q \left.\frac{x^n}{n!}\right|_{x=0} D^n .$$

By Theorem 2.1.10, the operator $U^{-1}\, Q$ is a delta operator. Combining this with the identity for $(U^{-1}\, Q\, \frac{x^n}{n!})(0)$, we see that only the term with $n = 1$ contributes to the expansion in powers of D. Hence, $U^{-1}\, Q = D$ and $Q = D\, U$. $\square$

Theorem 2.5.2 *Let $t_{n,k}$ be the number of forests of rooted labeled trees with n vertices and k trees, then*

$$A_n(x) = \sum_{k=0}^{n} t_{n,k}\, x^k = x\,(x+n)^{n-1}. \tag{12}$$

Proof: Define $q_n = A_n/n!$. We want to show that $(q_n)_{n\in\mathbb{N}}$ is the basic sequence of the Abel operator $D\,E^1$. The first step is to show that $(q_n)_{n\in\mathbb{N}}$ is of convolution type. Define $c_{n,k} = \frac{k!}{n!}\,t_{n,k}$. An easy combinatorial argument shows that

$$k\,t_{n,k} = \sum_{j=0}^{n} \binom{n}{j}\, t_{j,1}\, t_{n-j,k-1}.$$

Hence, the numbers $c_{n,k}$ satisfy

$$c_{n,k} = \sum_{j=0}^{\infty} c_{j,1}\, c_{n-j,k-1}.$$

A simple induction argument shows that $c_{n,k}$ is the k-fold convolution of $(c_{n,1})_{n\in\mathbb{N}}$. Hence, $(q_n)_{n\in\mathbb{N}}$ is a basic sequence by Theorems 2.2.5 and 2.3.7. Now each rooted labeled tree on n vertices may be obtained from a forest on $n-1$ vertices by

- adding a new vertex v
- adding edges between the new vertex and the roots of the trees of the forest root
- rooting the new tree at v (with n possibilities for labeling!)

Cdonversely, removing the root of a rooted tree on n vertices results in a forest on $n-1$ vertices. Hence, we have $t_{n,1} = n\,A_{n-1}(1)$ and thus

$$\left.\frac{q_n(x)}{x}\right|_{x=0} = c_{n,1} = \frac{t_{n,1}}{n!} = \frac{1}{n-1}\,A_{n-1}(1) = q_{n-1}(1).$$

Now Lemma 2.5.1 shows that $D\,E^1$ is the delta operator of $(q_n)_{n\in\mathbb{N}}$, since $q_{n-1}(1) = (E^1\,q_{n-1})(0)$. The result now follows from Example 2.4.5d. $\square$

Corollary 2.5.3 (Cayley) *The number of rooted labeled trees on n vertices equals n^{n-1}.*

Proof: The number of rooted labeled trees on n vertices is the coefficient of x of the polynomial A_n of Theorem 2.5.2, which is easily seen to be n^{n-1}. $\square$

For an interesting comparison with other proofs of these results, see Wilf (1990, Sections 3.12 and 3.17).

3 Recurrences and Umbral Calculus

Most properties of a basic sequence $(q_n)_{n\in\mathbb{N}}$ essentially depend only on the property $Q\,q_n = q_{n-1}$, the other definining properties being normalization conditions. Thus it seems plausible that the theory of basic sequences can be extended under weaker conditions. This is indeed the case, as the theory of Sheffer sequences shows in the next subsection. Sheffer polynomials are useful for solving recurrences as shown in subsection 3.2.

3.1 Sheffer polynomials

Definition 3.1.1 *Let Q be a delta operator. A sequence of polynomials $(s_n)_{n\in\mathbb{N}}$ is called a* **Sheffer sequence for** *Q if:*

1. *s_0 is a nonzero constant*
2. *$Q\, s_n = s_{n-1}$, $n = 1, 2, \ldots$.*

Note that if $(s_n)_{n\in\mathbb{N}}$ is a Sheffer sequence then, by Corollary 2.1.9b, deg $s_n = n$ for all $n \in \mathbb{N}$.

Examples 3.1.2 a) Sheffer polynomials for the differentiation operator D are called Appell polynomials. They were studied by Appell in Appell (1880). Examples of Appell polynomials include the Hermite polynomials H_n, defined by

$$\sum_{n=0}^{\infty} H_n(x)\, z^n = \exp\left(xz - \frac{1}{2}\, z^2\right),$$

and the Bernoulli polynomials B_n, defined by

$$\sum_{n=0}^{\infty} B_n(x)\, z^n = \frac{z}{e^z - 1}\, e^{x\,z}.$$

It follows directly from their generating functions or from Theorem 3.1.3d that these polynomials are Appell polynomials, i.e. $D\, H_n = H_{n-1}$ and $D\, B_n = B_{n-1}$ for $n \geq 1$.
Note that the numbers $B_n(0)$ are the Bernoulli numbers that we encountered in Section 1.
b) The Laguerre polynomials of order α are Sheffer sequences for the Laguerre operator of Example 2.1.3g. The Laguerre polynomials of Example 2.4.5e are the Laguerre polynomials of order $\alpha = -1$ (cf. (Roman 1984, p. 108)).

Sheffer sequences satisfy a convolution-like equation (see Theorem 3.1.3b below).

Theorem 3.1.3 *Let Q be a delta operator with basic sequence $(q_n)_{n\in\mathbb{N}}$. Then the following are equivalent:*

a) $(s_n)_{n\in\mathbb{N}}$ is a Sheffer sequence for Q.

b) s_0 is a nonzero constant and $s_n(x+y) = \sum_{k=0}^{n} s_k(x)\, q_{n-k}(y)$ for all $n \in \mathbb{N}$ and all x, y.

c) s_0 is a nonzero constant and $s_n = \sum_{k=0}^{n} s_k(0)\, q_{n-k}$ for all $n \in \mathbb{N}$.

d) there exists a sequence $(a_n)_{n\in\mathbb{N}}$ such that $a_0 \neq 0$ and $s_n = \sum_{k=0}^{n} a_k\, q_{n-k}$ for all $n \in \mathbb{N}$.

Proof: a $\Rightarrow$ b' Fix an arbitrary x. Applying the Polynomial Expansion Theorem 2.3.8 to $E^x s_n$ we obtain

$$E^x s_n = \sum_{i=0}^{\infty} \left(Q^i E^x s_n\right)(0)\, q_i = \sum_{i=0}^{n} s_{n-i}(x)\, q_i,$$

since $\deg s_n = n$. Hence, it follows that

$$s_n(x+y) = \sum_{i=0}^{n} s_{n-i}(x)\, q_i(y) = \sum_{k=0}^{n} s_k(x)\, q_{n-k}(y)$$

for all x and y, since x is arbitrary.
'b $\Rightarrow$ c' This follows by setting $x = 0$.
'c $\Rightarrow$ d' Take $a_k := s_k(0)$.
'd$\Leftarrow$a' Note that s_0 is constant because $s_0 = a_0\, q_0 = a_0$. If $n \geq 1$, then

$$Q\, s_n = Q \left(\sum_{k=0}^{n} a_k\, q_{n-k}\right) = \sum_{k=0}^{n-1} a_k\, q_{n-1-k} = s_{n-1}.$$

Hence, $(s_n)_{n\in\mathbb{N}}$ is a Sheffer sequence for Q. $\Box$

Corollary 3.1.4 *Let $(w_n)_{n\in\mathbb{N}}$ be a Sheffer sequence for the delta operator Q with basic sequence $(q_n)_{n\in\mathbb{N}}$. Let $(g_n)_{n\in\mathbb{N}}$ be the coefficient sequence of $(q_n)_{n\in\mathbb{N}}$. Then the following formal generating function identity holds:*

$$\sum_{n=0}^{\infty} s_n(x)\, t^n = \left(\sum_{n=0}^{\infty} s_n(0)\, t^n\right)\ exp\ \left(x \sum_{n=0}^{\infty} g_n\, t^n\right).$$

Proof: This follows directly from Theorems 2.2.5 and 3.1.3c. $\Box$

The following theorem describes Sheffer sequences (of a delta operator Q with basic sequence $(q_n)_{n\in\mathbb{N}}$) in terms of the linear operator A on $\mathcal{P}$, defined by $A\, q_n := s_n$. It follows directly from Theorem 2.3.9a that $A = \sum_{k=0}^{\infty} s_k(0)\, Q^k$ (cf. the proof of (Rota, Kahaner, and Odlyzko 1973, Corollary 1)). We also give a description of Sheffer sequences in terms of delta operators and functionals in the style of Roman and Rota (1978) and Roman (1984). We first need a lemma.

Lemma 3.1.5 *Let Λ be a linear functional such that $\Lambda\, 1 \neq 0$ and let Q be a delta operator on $\mathcal{P}$. There exists a unique sequence of polynomials $(p_n)_{n\in\mathbb{N}}$ with $deg\, p_n = n$ for all $n \in \mathbb{N}$ such that $\Lambda\, Q^k\, p_n = \delta_{nk}$ for all $k, n \in \mathbb{N}$, where δ_{nk} denotes the Kronecker delta.*

Proof: Existence follows in the same way as in the proof of Theorem 2.3.3. In order to prove uniqueness, consider another sequence $(\widetilde{p}_n)_{n\in\mathbb{N}}$ such that $\Lambda\, Q^k\, p_n = \Lambda\, Q^k\, \widetilde{p}_n$ for all $k, n \in \mathbb{N}$. Suppose there is an $n \in \mathbb{N}$ such that $p_n \neq \widetilde{p}_n$. Let ℓ be the degree of $p_n - \widetilde{p}_n$. Then $Q^\ell\,(p_n - \widetilde{p}_n)$ is a non-zero constant, which contradicts $\Lambda\, Q^k\,(p_n - \widetilde{p}_n) = 0$. $\Box$

Theorem 3.1.6 *Let Q be a delta operator with basic sequence $(q_n)_{n\in\mathbb{N}}$. Let $(s_n)_{n\in\mathbb{N}}$ be a sequence of polynomials and define the linear operator A on $\mathcal{P}$ by $A\, q_n := s_n$ for all $n \in \mathbb{N}$. Then:*

a) $(s_n)_{n\in\mathbb{N}}$ *is a Sheffer sequence for* Q *if and only if* A *is shift-invariant and invertible.*

b) $(s_n)_{n\in\mathbb{N}}$ *is a Sheffer sequence for* Q *if and only if there exists a linear functional* Λ *on* $\mathcal{P}$ *such that* $\Lambda\, 1 \neq 0$ *and* $\Lambda\, Q^k\, s_n = \delta_{nk}$ *for all* $k, n \in \mathbb{N}$*, where* δ_{nk} *denotes the Kronecker delta.*

c) If $(s_n)_{n\in\mathbb{N}}$ *is a Sheffer sequence, then* $\Lambda p = A^{-1} p(0)$ *for all* $p \in \mathcal{P}$*, where* A *is as in a).*

Proof: a) '$\Rightarrow$' Since $(q_n)_{n\in\mathbb{N}}$ is of convolution type, we have for all y

$$A\, E^y\, q_n = A\left(\sum_{k=0}^{n} q_k(y)\, q_{n-k}\right) = \sum_{k=0}^{n} q_k(y)\, s_{n-k} = E^y\, s_n = E^y\, A\, q_n.$$

Hence, by linearity, $A\, E^y = E^y\, A$ for all y. By Corollary 2.1.9b, deg $s_n = n$ for all $n \in \mathbb{N}$. Hence, A is invertible by Corollary 2.1.10.
'$\Leftarrow$' Corollary 2.1.9a and $s_0 = A\, q_0 = A\, 1$ together imply that s_0 is constant. Using Corollary 2.1.11 we see that $Q\, s_n = Q\, A\, q_n = A\, Q\, q_n = A\, q_{n-1} = s_{n-1}$ for $n \geq 1$. Moreover, since A is invertible and $s_0 = A\, q_0$, Corollary 2.1.10 yields that $s_0(0) \neq 0$.
b) '$\Rightarrow$' Define the linear functional Λ by $\Lambda\, s_n = \delta_{0n}$. Because s_0 is a nonzero constant, we have $\Lambda\, 1 \neq 0$. Moreover, since shift-invariant operators commute by Corollary 2.1.11, it follows that $\Lambda\, Q^k\, s_n = \Lambda\, Q^k\, A\, q_n = \Lambda\, A\, Q^k\, q_n = \delta_{0,n-k} = \delta_{nk}$.
'$\Leftarrow$' Define the polynomials r_n by $r_n := Q\, s_{n+1}$ ($n \in \mathbb{N}$). Then $\Lambda Q^k\, (Q\, s_{n+1}) = \delta_{k+1,n+1} = \delta_{k,n}$. By the uniqueness part of Lemma 3.1.5, we have $Q\, s_{n+1} = s_n$ for all $n \in \mathbb{N}$. Thus $(s_n)_{n\in\mathbb{N}}$ is a Sheffer sequence.
It follows from $\Lambda\, Q^k\, s_n = \Lambda\, s_{n-k} = \delta_{nk}$ with $k = 0$ that $\Lambda\, s_n = \delta_{0n}$. Since $A^{-1}\, s_n(0) = q_n(0) = \delta_{0n}$ by Definition 2.3.1 and deg $s_n = n$ for all $n \in \mathbb{N}$, the results follows.
d) Since $(s_n)_{n\in\mathbb{N}}$ is a basis for $\mathcal{P}$, it suffices to note that $\Lambda\, s_n = \delta_{0n} = q_n(0) = A^{-1}\, s_n(0)$. □

The operator A of the above theorem is called **invertible operator**.

Corollary 3.1.7 *Let* $(s_n)_{n\in\mathbb{N}}$ *be a strict sense Sheffer sequence for the delta operator* Q *with basic sequence* $(q_n)_{n\in\mathbb{N}}$ *and invertible operator* A*. Let* $(g_n)_{n\in\mathbb{N}}$ *be the coefficient sequence of* $(q_n)_{n\in\mathbb{N}}$ *and let* g *be the formal power series defined by* $g(t) := \sum_{n=0}^{\infty} g_n\, t^n$*. Then the following formal generating function identity holds:*

$$\sum_{n=0}^{\infty} s_n(x)\, t^n = f(g(t))\, e^{x\, g(t)},$$

where $A = f(D)$*.*

Proof: Define $s(t) := \sum_{n=0}^{\infty} s_n(x)\, t^n$. It follows from Theorem 2.3.9a that $A = \sum_{k=0}^{\infty} s_k(0)\, Q^k$. Hence, we have $A = s(Q)$. Since $g_0 = 0$, the formal power series is invertible (w.r.t. to composition, cf. Niven (1969)). Hence, there exists a formal power series f such that $s = f \circ g$. By Theorem 2.3.9b, we have $A = f(g(Q)) = f(D)$. The result now follows from Corollary 3.1.4. □

Corollary 3.1.8 *Let* Q *be a delta operator with basic sequence* $(q_n)_{n\in\mathbb{N}}$*.*

a) The sequence $(s_n)_{n\in\mathbb{N}}$ defined by $s_n(x) := (n+1)\,\dfrac{q_{n+1}(x)}{x}$ $(x \neq 0)$ and $s_n(0) := (n+1)\,(q_{n+1})'(0)$ is a Sheffer sequence.

b) The sequence $(s_n)_{n\in\mathbb{N}}$ defined by $s_n := (q_{n+1})'$ is a Sheffer sequence.

Proof: a) Recall that $q_n(0) = 0$ for $n \geq 1$. Then $(s_n)_{n\in\mathbb{N}}$ is a Sheffer sequence by Theorem 3.1.6b, since $s_n = (Q')^{-1}\,q_n$ by Theorem 2.4.4d.
b) By Theorem 2.3.9b, $D\,q_{n+1} = \sum_{k=0}^{n+1} g_k\,q_{n-k}$. Thus $s_0 = g_1 \neq 0$ and $(s_n)_{n\in\mathbb{N}}$ is a Sheffer sequence by Theorem 3.1.3. □

We now extend the Expansion Theorems 2.3.8 and 2.3.9 to Sheffer sequences.

Theorem 3.1.9 *Let $(s_n)_{n\in\mathbb{N}}$ be a Sheffer sequence with delta operator Q and let A be the linear operator on $\mathcal{P}$ defined by $A\,q_n := s_n$.*

a) For all $p \in \mathcal{P}$, we have

$$p = \sum_{k=0}^{\infty} (A^{-1}\,Q^k p)(0)\,s_k.$$

b) If T is a linear shift-invariant operator, then

$$T = \sum_{k=0}^{\infty} (T s_k(0))\,A^{-1}\,Q^k$$

Proof: a) Apply Theorem 2.3.8 to $p = A\left(A^{-1}\,p\right)$ and use shift-invariance.
b) Apply Theorem 2.3.9 to $T = A^{-1}\,(A\,T)$ and use shift-invariance. □

Theorem 3.1.6 enables us to generalize Theorem 2.4.7 to Sheffer sequences.

Theorem 3.1.10 *Let Q be a delta operator with basic sequence $(q_n)_{n\in\mathbb{N}}$. Let $(s_n)_{n\in\mathbb{N}}$ be a Sheffer sequence for Q and let A be the linear operator on $\mathcal{P}$ defined by $A\,q_n := s_n$. The following are equivalent for a sequence $(r_n)_{n\in\mathbb{N}}$ of polynomials:*

a) $(r_n)_{n\in\mathbb{N}}$ is a Sheffer sequence and there exists a basic sequence $(p_n)_{n\in\mathbb{N}}$ such that $r_n = A\,p_n$ for all $n \in \mathbb{N}$.

b) there exists a sequence $(\gamma_n)_{n\in\mathbb{N}}$ with $\gamma_0 = 0$ and $\gamma_1 \neq 0$ such that $r_n = \sum_{k=0}^{n} \gamma_n^{k*}\,s_k$ for all $n \in \mathbb{N}$.

Proof: 'a $\Rightarrow$ b' Since $(p_n)_{n\in\mathbb{N}}$ is a basic sequence, Theorem 2.4.7 yields the existence of a sequence $(\gamma_n)_{n\in\mathbb{N}}$ with $\gamma_0 = 0$ such that $p_n = \sum_{k=0}^{n} \gamma_n^{k*}\,q_k$ for all $n \in \mathbb{N}$. Since $\deg p_1 = 1$ (Remark 2.3.2), we have $\gamma_1 \neq 0$. Since $A\,q_n = s_n$ for all $n \in \mathbb{N}$, it follows that $r_n = A\,p_n = \sum_{k=0}^{n} \gamma_n^{k*}\,s_k$ for all $n \in \mathbb{N}$.
'b $\Rightarrow$ a' Define polynomials p_n ($n \in \mathbb{N}$) by $p_n := \sum_{k=0}^{n} \gamma_n^{k*}\,q_k$. Since $\gamma_0 = 0$ and $\gamma_1 \neq 0$, it follows from Theorem 2.4.7 and Theorem 2.3.7 that $(p_n)_{n\in\mathbb{N}}$ is a basic sequence. Moreover, it is obvious that $A\,p_n = r_n$ for all $n \in \mathbb{N}$ since $A\,q_n = s_n$. It follows from Theorem 3.1.6b that $(r_n)_{n\in\mathbb{N}}$ is a strict sense Sheffer sequence. □

As a corollary to Theorem 3.1.10, we now derive a Rodrigues Formula for Sheffer sequences (cf. Theorem 2.4.4d). This form of the Rodrigues Formula is due to Avramjonok (see Avramjonok (1977)).

Theorem 3.1.11 (Avramjonok) *Let Q be a delta operator with basic sequence $(q_n)_{n\in\mathbb{N}}$. Let $(s_n)_{n\in\mathbb{N}}$ be a Sheffer sequence for Q and let A be the linear operator on $\mathcal{P}$ defined by $A\,q_n := s_n$ for all $n \in \mathbb{N}$. Then we have*

$$n\,s_n(x) = \left(x\ (Q')^{-1} + (Q')^{-1}\ A'\,A^{-1}\right)\,s_{n-1}(x). \tag{13}$$

Proof: By Theorem 2.4.4d, we have $n\,q_n(x) = x\ (Q')^{-1}\ q_{n-1}(x)$ for all $n \geq 1$ and all x. Writing $q_k = A^{-1}\,Aq_k$ $(k = n-1, n)$, we obtain $n\,s_n(x) = A\,x\ (Q')^{-1}\ A^{-1}\,s_{n-1}(x)$. By the definition of Pincherle derivative, we may write $A\,x = x\,A + A'$. Substituting this into the expression for $n\,s_n(x)$, we obtain the result. □

3.2 Lattice path counting

In this subsection we show a glimpse of the powerful umbral methods developed by Niederhausen for solving recurrences and lattice path counting. For overviews of his results, we refer to Niederhausen (1986a) and Niederhausen (1997).

Many recursions can be written in the form $Q\,s_n = s_{n-1}$, where Q is a delta operator. As a toy example, let $r(n,m)$ be the number of lattice paths from $(0,0)$ to (n,m) with unit steps in the direction $(1,0)$ or $(0,1)$. Of course, a standard combinatorial argument yields that $r(n,m) = \binom{m+n}{n}$. Let us see how this fits in with Sheffer polynomials. Suppose that there exist a Sheffer sequence $(s_n)_{n\in\mathbb{N}}$ such that $s_n(m) = r(n,m)$. If we can construct such a sequence, then this assumption is justified. The standard recursion

$$r(n,m) = r(n,m-1) + r(n-1,m)$$

translates into

$$(I - E^{-1})\,s_n = s_{n-1}.$$

It follows from Example 2.4.5c that $q_n(x) = \binom{x+n-1}{n}$ are the basic polynomials of the delta operator $I - E^{-1}$. Obviously, $r(n,0) = 1$ for all n, i.e. $s_n(0) = 1$ for all n. Let A the invertible operator of $(s_n)_{n\in\mathbb{N}}$, i.e. $A\,q_n = s_n$. The Operator Expansion Theorem yields that

$$A = \sum_{k=0}^{\infty} s_k(0)\ \left(I - E^{-1}\right)^k = \left(I - (I - E^{-1}\right)^{-1} = E^1.$$

Thus, $s_n(m) = E^1\,q_n(m) = \binom{n+m}{n}$.

If the lattice paths are required to satisfy bounds as in the ballot problem, then further techniques (e.g. reflection principles) are needed. Niederhausen has shown that the Umbral Calculus provides tools that yield closed expressions for the number of lattice paths with complicated boundaries. The simplest case is the case of an affine boundary.

Theorem 3.2.1 *If $(q_n)_{n\in\mathbb{N}}$ is the basic sequence of a delta operator Q, then the Sheffer sequence $(s_n)_{n\in\mathbb{N}}$ defined by*

$$s_n(x) := \sum_{k=0}^{n} y_k\,\frac{x - an - c}{x - ak - c}\,q_{n-k}(x - ak - c)$$

has the initial values $s_n an + c = y_n$ for all n.

Proof: First note since the shift-operators are invertible, a shifted Sheffer sequence is again Sheffer for the same delta operator. By Example 3.1.8, the polynomials $(n+1)\,\frac{q_{n+1}(x)}{x}$ form a Sheffer sequence for Q. Combining this, we see that the polynomials r_n defined by

$$r_n(x) := \frac{x - an - c}{x - c}\, q_n(x - c) = q_n(x - c) - a\, n\, \frac{q_n(x - c)}{x - c}$$

is Sheffer for Q. Moreover, direct computation yields that $r_n(an + c) = \delta_{0n}$. Now note that $\frac{x-an-c}{x-ak-c}\, q_{n-k}(x - ak - c) = r_{n-k}(x - ak)$. Hence,

$$s_n = \sum_{k=0}^{n} y_k\, r_{n-k}(x - ak).$$

It is now straightforward to verify that $(s_n)_{n\in\mathbb{N}}$ is Sheffer with initial values $s_n an + c = y_n$ for all n. $\square$

Repeated application of the above theorem yields expressions for the case where the boundary is piecewise affine. For two-sided boundaries, Niederhausen introduces Sheffer splines (functions that are piecewise Sheffer polynomials); for details we refer to Niederhausen (1986a).

Niederhausen also shows how to solve more general operator equations. For details, we refer to Niederhausen (1986a) and Niederhausen (1997).

4 Extensions of the Umbral Calculus

The setting of the Umbral Calculus that we have studied is a theory of polynomials and shift-invariant operators. However, the main ideas of the Umbral Calculus (basic sequences, delta operators, expansion theorems, umbral operators, etc.) can be extended to other settings. In this section we briefly mention some of these extensions. For a overview we refer to Di Bucchianico and Loeb (1995).
The Umbral Calculus is built on shift-invariant operators. In fact, one can prove that this is the class of linear operators on $\mathcal{P}$ that commute with the differentiation operator. Since the basic theory of Umbral Calculus does not use analytic properties of differentiation, it is not surprising that there exists versions of the Umbral Calculus based on other operators that the differentiation operator. In fact, an Umbral Calculus exists for every linear operator that reduces the degree of a polynomial by one as shown in the papers Kreid (1990a), Kreid (1990b), Markowsky (1978) and Viskov (1978). However, these papers lack explicit examples. An interesting class of operators that yields many explicit examples is the class of generalized differentiation operators. The idea goes back to Ward (1936).

Let $(c_n)_{n\in\mathbb{N}}$ be a sequence of nonzero numbers. Define the generalized differentiation operator D_c by $D_c c = 0$ and

$$D_c \frac{x^n}{c_n} = \frac{x^{n-1}}{c_{n-1}}$$

for $n \geq 1$. Important choices of $(c_n)_{n\in\mathbb{N}}$ are:

- $c_n = n!$: this yields the ordinary differentiation operator

- $c_n = 1$: this yields the divided difference calculus with $D_c p(x) = (p(x) - p(0)/x$ and shift operators
$$E_c^y p(x) = \frac{x\,p(x) - y\,p(y)}{x - y}.$$
See Hirschhorn and Raphael (1992) and Verde-Star (1988) for more details.

- $c_n = \binom{-\lambda}{n}^{-1}$: this yields an Umbral Calculus for Gegenbauer polynomials (see Section 6.3 of Roman (1984)).

- $c_n = (2n)!$: this Umbral Calculus is coined the Hyperbolic Umbral Calculus in Di Bucchianico and Loeb (1996b), since the shift operators can be expressed as $E_c^y p(x) = \cosh\left(\sqrt{y\,D_c}\right)$, with $D_c = D\,x^2 + D$, where D denotes the ordinary differentiation operator.

- $c_n = \frac{(1-q)(1-q^2)\ldots(1-q^n)}{(1-q)^n}$: this leads to a q-Umbral Calculus (see Roman (1985) and Section 6.4 of Roman (1984)).

5 Epilogue: Classical Umbral Calculus

These lectures started with a mysterious 19th century manipulation rule in which a sequence of scalars $a_0, a_1, a_2, \ldots$ is treated as a sequence of powers $1, \alpha, \alpha^2, \ldots$ of a variable α called an umbra. Using a 20th century approach to linear algebra, we were able to replace this manipulation rule with operator calculations. Although rigorous and powerful, the feeling of 'witchcraft' that surrounds the 19th Umbral Calculus is lost in this 20th century operator version. It would be nice if there would be a rigorous version of the Umbral Calculus that is as close as possible to the original version. Recently, Rota and collaborators have succeeded in finding the key to a rigorous version of the 19th century Umbral Calculus. Their results (the so-called "Classical Umbral Calculus") can be found in the papers Rota and Taylor (1993), Rota and Taylor (1994), Di Crescenzo and Rota (1994) and Cerasoli (1995).

The main idea is to consider a linear functional on polynomials, called **eval** below, so that $\mathbf{eval}(\alpha^n) = a_n$.

A classical umbral calculus consists of three types of data:

1. a polynomial ring $\mathbf{k}[\mathcal{A}, x]$. The variables belonging to the set $\mathcal{A}$ will be denoted by Greek letters, and are called *umbrae*.

2. a linear functional $\mathbf{eval} : \mathbf{k}[\mathcal{A}, x] \to k[x]$ such that

   (a) $\mathbf{eval}(1) = 1$,

   (b) $\mathbf{eval}(\alpha^i \beta^j \cdots \gamma^k x^n) = x^n \mathbf{eval}(\alpha^i)\mathbf{eval}(\beta^j)\cdots\mathbf{eval}(\gamma^k)$,
   for distinct umbrae $\alpha, \beta, \ldots, \gamma \in \mathcal{A}$ and $i, j, \cdots, k \geq 0$.

3. A distinguished umbra, $\varepsilon \in \mathcal{A}$, satisfying $\mathbf{eval}(\varepsilon^i) = \delta_{0,i}$ where δ is the Kronecker delta.

A polynomial $p \in \mathbf{k}[A]$ is called an **umbral polynomial**. In order to avoid the troubles of the 19th century Umbral Calculus, we now introduce two(!) different type of "equality".

Definition 5.0.2 *Two umbral polynomials $p, q \in \mathbf{k}[\mathcal{A}]$ are said to be*

- **umbrally equivalent**, *written $p \simeq q$, when* $\mathbf{eval}(p) = \mathbf{eval}(q)$.
- **exchangeable**, *written $p \equiv q$, when* $\mathbf{eval}(p^n) = \mathbf{eval}(q^n)$ *for all n.*

These are the basic definitions; for advanced calculations there are more definitions (see the papers cited above).

Let us perform some calculations to get a feeling for these definitions. If α and β are exchangeable umbrae representing the sequence $(a_n)_{n\in\mathbb{N}}$, then

$$(\alpha+\alpha)^2 \simeq 4\,a_2,$$

but

$$(\alpha+\beta)^2 \simeq \alpha^2 + 2\alpha\,\beta + \beta^2 \simeq 2\,a_2 + 2a_1^2.$$

Thus $\alpha+\beta$ is umbrally equivalent, but in general not exchangeable to 2α. This explains the major source of incorrect umbral manipulations.

We now conclude these lecture notes by returning to the Bernoulli example that we started with. Let β be an umbra representing the Bernoulli numbers B_n as defined by the generating function (1), i.e. $\beta^n \simeq B_n$. It follows that

$$e^{\beta\,x} \simeq \frac{x}{e^x - 1}.$$

The umbral derivation of the recursion that we gave in Section 1 is now perfectly legal. For more Bernoulli computations (with different starting definitions), we refer to Rota and Taylor (1994) and Di Crescenzo and Rota (1994). An umbral treatment of generalized Bernoulli numbers can be found in Cerasoli (1995).

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