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Incomplete Generalized Jacobsthal and Jacobsthal-Lucas Numbers

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Abstract—In this paper, we present a systematic investigation of the incomplete generalized Jacobsthal numbers and the incomplete generalized Jacobsthal-Lucas numbers. The main results, which we derive here, involve the generating functions of these incomplete numbers. © 2005 Elsevier Ltd. All rights reserved.

Keywords—Incomplete generalized Jacobsthal numbers, Incomplete generalized Jacobsthal-Lucas numbers, Generating functions.

1. INTRODUCTION AND DEFINITIONS

Recently, Djordjević [1,2] considered four interesting classes of polynomials: the *generalized Jacobsthal polynomials* $J_{n,m}(x)$, the *generalized Jacobsthal-Lucas polynomials* $j_{n,m}(x)$, and their associated polynomials $F_{n,m}(x)$ and $f_{n,m}(x)$. These polynomials are defined by the following recurrence relations (cf., [1–3]):

$$J_{n,m}(x) = J_{n-1,m}(x) + 2xJ_{n-m,m}(x) \quad (n \geqq m;\ m,n \in \mathbb{N};\ J_{0,m}(x) = 0,\ J_{n,m}(x) = 1,\ \text{when } n = 1,\dots,m-1), \tag{1.1}$$

$$j_{n,m}(x) = j_{n-1,m}(x) + 2xj_{n-m,m}(x) \quad (n \geqq m;\ m,n \in \mathbb{N};\ j_{0,m}(x) = 2,\ j_{n,m}(x) = 1,\ \text{when } n = 1,\dots,m-1), \tag{1.2}$$

$$F_{n,m}(x) = F_{n-1,m}(x) + 2xF_{n-m,m}(x) + 3 \quad (n \geqq m;\ m,n \in \mathbb{N};\ F_{0,m}(x) = 0,\ F_{n,m}(x) = 1,\ \text{when } n = 1,\dots,m-1), \tag{1.3}$$

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$$f_{n,m}(x) = f_{n-1,m}(x) + 2x f_{n-m,m}(x) + 5 \quad (n \geqq m;\ m, n \in \mathbb{N};\ f_{0,m}(x) = 0;\ f_{n,m}(x) = 1, \text{ when } n = 1, \ldots, m-1), \tag{1.4}$$

$\mathbb{N}$ being the set of natural numbers and

$$\mathbb{N}_0 := \mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots\}.$$

Explicit representations for these four classes of polynomials are given by

$$J_{n,m}(x) = \sum_{r=0}^{[(n-1)/m]} \binom{n-1-(m-1)r}{r} (2x)^r, \tag{1.5}$$

$$j_{n,m}(x) = \sum_{k=0}^{[n/m]} \frac{n-(m-2)k}{n-(m-1)k} \binom{n-(m-1)k}{k} (2x)^k, \tag{1.6}$$

$$F_{n,m}(x) = J_{n,m}(x) + 3 \sum_{r=0}^{[(n-m+1)/m]} \binom{n-m+1-(m-1)r}{r+1} (2x)^r, \tag{1.7}$$

and

$$f_{n,m}(x) = J_{n,m}(x) + 5 \sum_{r=0}^{[(n-m+1)/m]} \binom{n-m+1-(m-1)r}{r+1} (2x)^r, \tag{1.8}$$

respectively. Tables for $J_{n,m}(x)$ and $j_{n,m}(x)$ are provided in [2].

By setting $x = 1$ in definitions (1.1)–(1.4), we obtain the *generalized Jacobsthal numbers*

$$J_{n,m} := J_{n,m}(1) = \sum_{r=0}^{[(n-1)/m]} \binom{n-1-(m-1)r}{r} 2^r, \tag{1.9}$$

and the *generalized Jacobsthal-Lucas numbers*

$$j_{n,m} := j_{n,m}(1) = \sum_{r=0}^{[n/m]} \frac{n-(m-2)r}{n-(m-1)r} \binom{n-(m-1)r}{r} 2^r, \tag{1.10}$$

and their associated numbers

$$F_{n,m} := F_{n,m}(1) = J_{n,m}(1) + 3 \sum_{r=0}^{[(n-m+1)/m]} \binom{n-m+1-(m-1)r}{r+1} 2^r \tag{1.11}$$

and

$$f_{n,m} := f_{n,m}(1) = J_{n,m}(1) + 5 \sum_{r=0}^{[(n-m+1)/m]} \binom{n-m+1-(m-1)r}{r+1} 2^r. \tag{1.12}$$

Particular cases of these numbers are the so-called *Jacobsthal numbers* J_n and the *Jacobsthal-Lucas numbers* j_n, which were investigated earlier by Horadam [4]. (See also a systematic investigation by Raina and Srivastava [5], dealing with an interesting class of numbers associated with the familiar Lucas numbers.)

Motivated essentially by the recent works by Filipponi [6], Pintér and Srivastava [7], and Chu and Vicenti [8], we aim here at introducing (and investigating the generating functions of) the analogously *incomplete* version of each of these four classes of numbers.

2. GENERATING FUNCTIONS OF THE INCOMPLETE GENERALIZED JACOBSTHAL AND JACOBSTHAL-LUCAS NUMBERS

We begin by defining the *incomplete* generalized Jacobsthal numbers $J_{n,m}^{k}$ by

$$J_{n,m}^{k} := \sum_{r=0}^{k} \binom{n-1-(m-1)r}{r} 2^{r} \qquad \left(0 \leqq k \leqq \left[\frac{n-1}{m}\right];\ m,n \in \mathbb{N}\right), \tag{2.1}$$

so that, obviously,

$$J_{n,m}^{[(n-1)/m(n-1)/m]} = J_{n,m}, \tag{2.2}$$

$$J_{n,m}^{k} = 0 \qquad (0 \leqq n < mk+1), \tag{2.3}$$

and

$$J_{mk+l,m}^{k} = J_{mk+l-1,m} \qquad (l = 1, \ldots, m). \tag{2.4}$$

The following known result (due essentially to Pintér and Srivastava [7]) will be required in our investigation of the generating functions of such incomplete numbers as the incomplete generalized Jacobsthal numbers $J_{n,m}^{k}$ defined by (2.1). For the theory and applications of the various methods and techniques for deriving generating functions of special functions and polynomials, we may refer the interested reader to a recent treatise on the subject of generating functions by Srivastava and Manocha [9].

LEMMA 1. *(See [7, p. 593].) Let $\{s_n\}_{n=0}^{\infty}$ be a complex sequence satisfying the following nonhomogeneous recurrence relation:*

$$s_n = s_{n-1} + 2s_{n-m} + r_n \qquad (n \geqq m;\ m,n \in \mathbb{N}), \tag{2.5}$$

where $\{r_n\}$ is a given complex sequence. Then the generating function $S(t)$ of the sequence $\{s_n\}$ is

$$S(t) = \left(s_0 - r_0 + \sum_{l=1}^{m-1} t^{l}\left(s_l - s_{l-1} - r_l\right) + G(t)\right)\left(1 - t - 2t^{m}\right)^{-1}, \tag{2.6}$$

where $G(t)$ is the generating function of the sequence $\{r_n\}$.

Our first result on generating functions is contained in Theorem 1 below.

THEOREM 1. *The generating function of the incomplete generalized Jacobsthal numbers $J_{n,m}^{k}$ $(k \in \mathbb{N}_0)$ is given by*

$$\begin{aligned} R_m^{k}(t) &= \sum_{r=0}^{\infty} J_{k,m}^{r} t^{r} \\ &= t^{mk+1}\left(\left[J_{mk,m} + \sum_{l=1}^{m-1} t^{l}\left(J_{mk+l,m} - J_{mk+l-1,m}\right)\right](1-t)^{k+1} - 2^{k+1}t^{m}\right) \\ &\quad \cdot \left[\left(1 - t - 2t^{m}\right)(1-t)^{k+1}\right]^{-1}. \end{aligned} \tag{2.7}$$

PROOF. From (1.1) (*with* $x = 1$) and (2.1), we get

$$\begin{aligned}
J^k_{n,m} - J^k_{n-1,m} - 2J^k_{n-m,m} &= \sum_{r=0}^{k} \binom{n-1-(m-1)r}{r} 2^r \\
&\quad - \sum_{r=0}^{k} \binom{n-2-(m-1)r}{r} 2^r - \sum_{r=0}^{k} \binom{n-1-m-(m-1)r}{r} 2^{r+1} \\
&= \sum_{r=0}^{k} \binom{n-1-(m-1)r}{r} 2^r - \sum_{r=0}^{k} \binom{n-2-(m-1)r}{r} 2^r \\
&\quad - \sum_{r=1}^{k+1} \binom{n-2-(m-1)r}{r-1} 2^r \\
&= \sum_{r=0}^{k} \binom{n-1-(m-1)r}{r} 2^r - \sum_{r=1}^{k} \binom{n-2-(m-1)r}{r} 2^r - 1 \\
&\quad - \sum_{r=1}^{k} \binom{n-2-(m-1)r}{r-1} 2^r - \binom{n-2-(m-1)(k+1)}{k} 2^{k+1} \\
&= -\sum_{r=1}^{k} \left[\binom{n-2-(m-1)r}{r} + \binom{n-2-(m-1)r}{r-1} \right] 2^r \\
&\quad - 1 - \binom{n-2-(m-1)(k+1)}{k} 2^{k+1} + \sum_{r=0}^{k} \binom{n-1-(m-1)r}{r} 2^r \\
&= \sum_{r=1}^{k} \binom{n-1-(m-1)r}{r} 2^r + 1 - \sum_{r=1}^{k} \binom{n-1-(m-1)r}{r} 2^r \\
&\quad - 1 - \binom{n-2-(m-1)(k+1)}{k} 2^{k+1} \\
&= -\binom{n-1-m-(m-1)k}{k} 2^{k+1} \\
&= -\binom{n-1-m-(m-1)k}{n-1-m-mk} 2^{k+1} \qquad (n \geqq m+1+mk;\ k \in \mathbb{N}_0).
\end{aligned} \tag{2.8}$$

Next, in view of (2.3) and (2.4), we set

$$s_0 = J^k_{mk+1,m}, s_1 = J^k_{mk+2,m}, \dots, s_{m-1} = J^k_{mk+m,m}$$

and

$$s_n = J^k_{mk+n+1,m}.$$

Suppose also that

$$r_0 = r_1 = \cdots = r_{m-1} = 0 \qquad \text{and} \qquad r_n = 2^{k+1} \binom{n-m+k}{n-m}.$$

Then, for the generating function $G(t)$ of the sequence $\{r_n\}$, we can show that

$$G(t) = \frac{2^{k+1} t^m}{(1-t)^{k+1}}.$$

Thus, in view of the above lemma, the generating function $S^k_m(t)$ of the sequence $\{s_n\}$ satisfies the following relationship:

$$S^k_m(t)(1-t-2t^m) + \frac{2^{k+1}t^m}{(1-t)^{k+1}} = J_{mk,m}(k) + \sum_{l=1}^{m-1} t^l (J_{mk+l,m} - J_{mk+l-1,m}) + \frac{2^{k+1}t^m}{(1-t)^{k+1}}.$$

Hence, we conclude that

$$R_m^k(t) = t^{mk+1} S_m^k(t).$$

This completes the proof of Theorem 1.

COROLLARY 1. *The incomplete Jacobsthal numbers* J_n^k $(k \in \mathbb{N}_0)$ *are defined by*

$$J_n^k := J_{n,2}^k = \sum_{r=0}^{k} \binom{n-1-r}{r} 2^r$$

$$\left(0 \leqq k \leqq \left[\frac{n-1}{2}\right];\ n \in \mathbb{N} \setminus \{1\}\right)$$

and the corresponding generating function is given by (2.7) when $m = 2$, *that is, by*

$$R_2^k(t) = t^{2k+1} \left[J_{2k} + t\left(J_{2k+1} - J_{2k}\right)(1-t)^{k+1} - 2^{k+1}t^2\right] \cdot \left[\left(1-t-2t^2\right)(1-t)^{k+1}\right]^{-1}. \quad (2.9)$$

3. INCOMPLETE GENERALIZED JACOBSTHAL-LUCAS NUMBERS

For the *incomplete* generalized Jacobsthal-Lucas numbers $j_{n,m}^k$ defined by [cf. equation (1.10)]

$$j_{n,m}^k := \sum_{r=0}^{k} \frac{n-(m-2)r}{n-(m-1)r} \binom{n-(m-1)r}{r} 2^r \qquad \left(0 \leqq k \leqq \left[\frac{n}{m}\right];\ m, n \in \mathbb{N}\right), \quad (3.1)$$

we now prove the following generating function.

THEOREM 2. *The generating function of the incomplete generalized Jacobsthal-Lucas numbers* $j_{n,m}^k$ $(k \in \mathbb{N}_0)$ *is given by*

$$\begin{aligned} W_m^k(t) &= \sum_{r=0}^{\infty} j_{k,m}^r t^r \\ &= t^{mk}\left[\left(j_{mk-1,m} + \sum_{l=1}^{m-1} t^l \left(j_{mk+l-1,m} - j_{mk+l-2,m}\right)\right)(1-t)^{k+1} - 2^{k+1}t^m(2-t)\right] \\ &\quad \cdot \left[\left(1-t-2t^m\right)(1-t)^{k+1}\right]^{-1}. \end{aligned} \quad (3.2)$$

PROOF. First of all, it follows from definition (3.1) that

$$j_{n,m}^{[n/m]} = j_{n,m}, \quad (3.3)$$

$$j_{n,m}^k = 0 \qquad (0 \leqq n < mk), \quad (3.4)$$

and

$$j_{mk+l,m}^k = j_{mk+l-1,m} \qquad (l = 1, \ldots, m). \quad (3.5)$$

Thus, just as in our derivation of (2.8), we can apply (1.2) and (1.10) (*with* $x = 1$) in order to obtain

$$j_{n,m}^k - j_{n-1,m}^k - 2j_{n-m,m}^k = -\frac{n-m+2k}{n-m+k} \binom{n-m+k}{n-m} 2^{k+1}. \quad (3.6)$$

Let

$$s_0 = j_{mk-1,m}, \qquad s_1 = j_{mk,m}, \dots, s_{m-1} = j_{mk+m,m},$$

and

$$s_n = j_{mk+n+1,m}.$$

Suppose also that

$$r_0 = r_1 = \cdots = r_{m-1} = 0 \qquad \text{and} \qquad r_n = \frac{n-m+2k}{n-m+k}\binom{n-m+k}{n-m}2^{k+1}.$$

Then, the generating function $G(t)$ of the sequence $\{r_n\}$ is given by

$$G(t) = \frac{2^{k+1}t^m(2-t)}{(1-t)^{k+1}}.$$

Hence, the generating function of the sequence $\{s_n\}$ satisfies relation (3.2), which leads us to Theorem 2.

COROLLARY 2. *For the incomplete Jacobsthal-Lucas numbers $j^k_{n,2}$, the generating function is given by (3.2) when $m = 2$, that is, by*

$$W_2^k(t) = t^{2k}\left[(j_{2k-1} + t(j_{2k} - j_{2k-1}))(1-t)^{k+1} - 2^{k+1}t^2(2-t)\right] \cdot \left[(1-t-2t^2)(1-t)^{k+1}\right]^{-1}.$$

4. TWO FURTHER PAIRS OF INCOMPLETE NUMBERS

For a natural number k, the *incomplete* numbers $F^k_{n,m}$ corresponding to the numbers $F_{n,m}$ in (1.11) are defined by

$$F^k_{n,m} := J^k_{n,m} + 3\sum_{r=0}^{k}\binom{n-m+1-(m-1)r}{r+1}2^r \qquad \left(0 \leqq k \leqq \left[\frac{n-1}{m}\right];\ m,n \in \mathbb{N}\right), \quad (4.1)$$

where

$$F^k_{n,m} = J^k_{n,m} = 0, \qquad (n < m + mk).$$

THEOREM 3. *The generating function of the incomplete numbers $F^k_{n,m}$ $(k \in \mathbb{N}_0)$ is given by $t^{mk+1}S^k_m(t)$, where*

$$S^k_m(t) = \left[F_{mk,m} + \sum_{l=1}^{m-1} t^l (F_{mk+l,m} - F_{mk+l-1,m})\right](1-t-2t^m)^{-1} + \frac{3t^m(1-t)^{k+1} - 2^{k+1}t^m(1-t+3t^{m-1})}{(1-t-2t^m)(1-t)^{k+2}}. \quad (4.2)$$

PROOF. Our proof of Theorem 3 is much akin to those of Theorems 1 and 2 above. Here, we let

$$s_0 = F^k_{mk+1,m} = F_{mk},$$
$$s_1 = F^k_{mk+2,m} = F_{mk-1,m}, \dots,$$
$$s_{m-1} = F^k_{mk+m,m} = F_{mk+m-1,m},$$

and

$$s_n = F^k_{mk+n+1,m}.$$

Suppose also that

$$r_0 = r_1 = \cdots = r_{m-1} = 0$$

and

$$r_n = \binom{n-m+k}{n-m} 2^{k+1} + 3\binom{n-m+2+k}{n-m+k} 2^{k+1}.$$

Then, by using the standard method based upon the above lemma, we can prove that

$$G(t) = \sum_{n=0}^{\infty} r_n t^n = \frac{2^{k+1} t^m \left(1 - t + 3t^{m-1}\right)}{(1-t)^{k+2}}.$$

Let $S_m^k(t)$ be the generating function of $F_{n,m}^k$. Then, it follows that

$$\begin{aligned} S_m^k(t) &= s_0 + ts_1 + \cdots + s_n t^n + \cdots, \\ tS_m^k(t) &= ts_0 + t^2 s_1 + \cdots + t^n s_{n-1} + \cdots, \\ 2t^m S_m^k(t) &= 2t^m s_0 + 2t^{m+1} s_1 + \cdots + 2t^n s_{n-m} + \cdots, \end{aligned}$$

and

$$G(t) = r_0 + r_1 t + \cdots + r_n t^n + \cdots.$$

The generating function $t^{mk+1} S_m^k(t)$ asserted by Theorem 3 would now result easily.

COROLLARY 3. *For the incomplete numbers $F_{n,2}^k$ defined by (4.1) with $m = 2$, the generating function is given by*

$$t^{2k+1} S_2^k(t) = t^{2k+1} \cdot \left(\frac{\left[F_{2k} + t\left(F_{2k+1} - F_{2k}\right)\right](1-t)^{k+2} + 3t^2 (1-t)^{k+2} - 2^{k+1} t^2 \left(1 - t + 3t^2\right)}{\left(1 - t - 2t^2\right)(1-t)^{k+2}} \right). \tag{4.3}$$

Finally, the incomplete numbers $f_{n,m}^k$ ($k \in \mathbb{N}_0$) corresponding to the numbers $f_{n,m}$ in (1.12) are defined by

$$f_{n,m}^k := J_{n,m}^k + 5 \sum_{r=0}^{k} \binom{n+1-m-(m-1)r}{r+1} 2^r \qquad \left(0 \leqq k \leqq \left[\frac{n-1}{m}\right];\ m, n \in \mathbb{N}\right). \tag{4.4}$$

THEOREM 4. *The incomplete numbers $f_{n,m}^k$ ($k \in \mathbb{N}_0$) have the following generating function:*

$$\begin{aligned} W_m^k(t) &= t^{mk+1} \left[f_{mk,m} + \sum_{l=1}^{m-1} t^l \left(f_{mk+l,m} - f_{mk+l-1,m} \right) \right] \left(1 - t - 2t^m\right)^{-1} \\ &\quad + t^{mk+1} \left(\frac{5t^m (1-t)^{k+1} - 2^{k+1} t^m \left(1 - t + 5t^{m-1}\right)}{\left(1 - t - 2t^m\right)(1-t)^{k+2}} \right). \end{aligned} \tag{4.5}$$

PROOF. Here, we set

$$\begin{aligned} s_0 &= f_{mk+1,m}^k = f_{mk,m}, \\ s_1 &= f_{mk+2,m}^k = f_{mk+1,m}, \\ &\vdots \\ s_{m-1,m} &= f_{mk+m,m}^k = f_{mk+m-1,m}, \end{aligned}$$

and

$$s_n = f^k_{mk+n+1,m} = f_{mk+n,m}.$$

We also suppose that

$$r_0 = r_1 = \cdots = r_{m-1} = 0$$

and

$$r_n = 2^{k+1}\binom{n-m+k}{n-m} + 5 \cdot 2^{k+1}\binom{n-2m+2+k}{n-2m+1}.$$

Then, by using the known method based upon the above lemma, we find that

$$G(t) = \frac{2^{k+1}t^m \left(1-t+5t^{m-1}\right)}{(1-t)^{k+2}}$$

is the generating function of the sequence $\{r_n\}$. Theorem 4 now follows easily.

In its special case when $m = 2$, Theorem 4 yields the following generating function for the incomplete numbers investigated in [6,7].

COROLLARY 4. *The generating function of the incomplete numbers* $f^k_{n,2}$ *is given by (4.5) when* $m = 2$, *that is, by*

$$W_2^k(t) = t^{2k+1} \cdot \left(\frac{\left[f_{2k} + t\left(f_{2k+1} - f_{2k}\right)\right](1-t)^{k+2} + 5t^2(1-t)^{k+1} - 2^{k+1}t^2(1+4t)}{\left(1-t-2t^2\right)(1-t)^{k+2}}\right). \tag{4.6}$$

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