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# Binomial Self-Inverse Sequences and Tangent Coefficients

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This paper treats the class of sequences $\{a_n\}$ that satisfy the recurrence relation $a_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} a_k d^{n-k}$, with $d$ constant, and shows that there is a relationship between the odd and even terms of $\{a_n\}$ that involves the coefficients of $\tan(t)$, namely

$$a_{2n+1} = \sum_{k=0}^{n} (-1)^k \binom{2n+1}{2k+1} T_k (d/2)^{2k+1} a_{2n-2k}.$$

A combinatorial setting is then provided to elucidate the appearance of the tangent coefficients in this equation.

## I. INTRODUCTION

Two sequences $f_n$ and $g_n$ are binomial inverses (cf. [4, p. 43]), if they satisfy the inverse relations

$$f_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} g_k, \qquad g_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f_k; \tag{1}$$

or equivalently,

$$f_n = (1-g)^n, \qquad g_n = (1-f)^n, \qquad f^n \equiv f_n, \text{ and } g^n \equiv g_n. \tag{2}$$

The sequence $f_n$ is then its own binomial inverse if it satisfies

$$f_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f_k = (1-f)^n, \qquad f^n \equiv f_n. \tag{3}$$

It will be convenient to generalize this slightly and say that if $a_n = d^n f_n$ for some constant $d$, so that

$$a_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} a_k d^{n-k} = (d-a)^n, \qquad a^n \equiv a_n, \tag{4}$$

then $a_n$ is *self-inverse of degree d.*

With the sequence $T_k$ defined by $e^{Tt} = \tan(t)$, $T^{2n+1} \equiv T_n$, Eq. (4), with $d = 2\delta$, implies the relations between even and odd terms:

$$a_{2n+1} = \sum_{k=0}^{n} (-1)^k \binom{2n+1}{2k+1} T_k \,\delta^{2k+1} a_{2n-2k}$$

and

$$a_{2n} = \frac{a_{2n+1}}{(2n+1)\,\delta} + \sum_{k=0}^{n-1} (-1)^k \binom{2n}{2k+1} \left[\frac{T_k \,\delta^{2k+1}}{2^{2k+2}-1}\right] a_{2n-2k-1} .$$

It is shown that every self-inverse sequence of degree $d$ satisfies the generating function relation

$$e^{at} = e^{dt} e^{-at}$$

and as a consequence $a_n$ is self-inverse of degree $d = 2\delta$ if it satisfies the relation

$$a_n = \sum_{k=0} \binom{n}{2k} f_k \,\delta^{n-2k}$$

for any arbitrary sequence $f_k$.

The particular instance

$$e_n = \sum_{k=0} \binom{n}{2k},$$

the number of even-subsets of a set, which satisfies the weighted inclusion-exclusion identity

$$e_{2n+1} = \sum_{k=0} (-1)^k \binom{2n+1}{2n-2k} T_k e_{2n-2k} ,$$

is used to determine the $T_k$'s combinatorially as the weightings required to insure that every even subset is counted exactly once in this summation.

The paper concludes with a number of examples of self-inverse sequences to give the subject some concreteness.

## II. Characterization of the Odd Terms

Expanding Eq. (4) and solving for the highest coefficient yields, for $n = 0(1)3$:

$$\begin{aligned}
a_0 &= a_0 & &\rightarrow & &\text{no information},\\
a_1 &= da_0 - a_1 & &\rightarrow & &2a_1 = da_0 ,\\
a_2 &= d^2a_0 - 2da_1 + a_2 & &\rightarrow & &\text{no new information},\\
a_3 &= d^3a_0 - 3d^2a_1 + 3da_2 - a_3 & &\rightarrow & &4a_3 = -d^3a_0 + 6da_2 .
\end{aligned}$$

In general Eq. (4), the defining identity for self-inverse sequences, imposes no constraints on the even terms but completely specifies the odd terms. Setting $d = 2\delta$, these specifying equations can be written, for $n = 1(2)7$, as:

$$a_1 = 1 \binom{1}{0} \delta a_0 ,$$

$$a_3 = -2 \binom{3}{0} \delta^3 a_0 + 1 \binom{3}{2} \delta a_2 ,$$

$$a_5 = 16 \binom{5}{0} \delta^5 a_0 - 2 \binom{5}{2} \delta^3 a_2 + 1 \binom{5}{4} \delta a_4 ,$$

$$a_7 = -272 \binom{7}{0} \delta^7 a_0 + 16 \binom{7}{2} \delta^5 a_2 - 2 \binom{7}{4} \delta^3 a_4 + 1 \binom{7}{6} \delta a_6 .$$

The numbers $T_0 = 1$, $T_1 = 2$, $T_2 = 16$, $T_3 = 272,\ldots$ [5, Sequence 829] appearing in these equations are the coefficients of

$$\tan(t) = \sum_{n=0}^{\infty} T_n t^{2n+1}/(2n+1)!. \tag{5}$$

This result is stated generally as

THEOREM 1 (MAIN THEOREM). *If $a_n$ is self-inverse of degree $d = 2\delta$, then with $T_k$ defined by Eq. (5),*

$$a_{2n+1} = \sum_{k=0}^{n} (-1)^k \binom{2n+1}{2k+1} T_k \, \delta^{2k+1} a_{2n-2k} . \tag{6}$$

*Proof.*

$$\begin{aligned} & a^n = (d-a)^n, \qquad a^k \equiv a_k \\ \Rightarrow\ & e^{at} = e^{(d-a)t}, \\ \Rightarrow\ & e^{(a-\delta)t} = e^{(\delta-a)t}, \qquad d = 2\delta \\ \Rightarrow\ & e^{iat}e^{-i\delta t} = e^{i\delta t}e^{-iat}, \qquad i^2 = -1 \\ \Rightarrow\ & \cos(\delta t)\sin(at) = \sin(\delta t)\cos(dt), \\ \Rightarrow\ & \sin(at) = \tan(\delta t)\cos(at). \end{aligned} \tag{7}$$

Equating the coefficients of $t^{2n+1}$ in (7) gives Eq. (6).

COROLLARY.

$$\sum_{k=0}^{n} (-1)^k \binom{2n+1}{2k+1} T_k = 1. \tag{8}$$

*Proof.* Eq. (8) is Eq. (6) with $a_n = \delta = 1$; and for this sequence Eq. (4) reduces to $1 = (2 - 1)^n$. In addition, Eq. (7) reduces to $\sin(t) = \tan(t)\cos(t)$.

## III. Extensions and Inverse of the Main Theorem

Let $b_n$ satisfy

$$b_n = -\sum_{k=0}^{n} (-1)^k \binom{n}{k} b_k d^{n-k} = -(d-b)^n, \qquad b^k \equiv b_k . \tag{9}$$

Then $b_n$ can be said to be anti-self-inverse since Eq. (9) differs from Eq. (4) only by a minus sign.

THEOREM 2. *If $b_n$ is anti-self-inverse of degree $d = 2\delta$, then with $T_k$ defined by Eq.* (5),

$$b_{2n+2} = \sum_{k=0}^{n} (-1)^k \binom{2n+2}{2k+1} T_k \, \delta^{2k+1} b_{2n+1-2k} . \tag{10}$$

*Proof.* Paralleling the proof of Theorem 1,

$$\begin{aligned} b^n &= -(d-b)^n, \qquad b^k \equiv b_k \\ \Rightarrow \quad e^{ibt}e^{-i\delta t} &= -e^{i\delta t}e^{-ibt}, \qquad i^2 = -1 \\ \Rightarrow \quad -\cos(\delta t)\cos(bt) &= \sin(\delta t)\sin(bt), \\ \Rightarrow \quad -\cos(bt) &= \tan(\delta t)\sin(bt). \end{aligned}$$

COROLLARY. *If $b_n$ satisfies*

$$b_n = \sum_{k=0}^{n} (-1)^k \binom{n+r}{k+r} b_k d^{n-k}$$

*for any positive integer $r$, then with $d = 2\delta$,*

$$b_{2n+1} = \sum_{k=0}^{n} (-1)^k \binom{2n+1+r}{2k+1} T_k \, \delta^{2k+1} b_{2n-2k} .$$

*Proof.* Relabeling $b_n \to b_{n+r}$ and $0 \to b_i$, $i = 0(1)r-1$, makes $b_n$ self-inverse for $r$ even and anti-self-inverse for $r$ odd.

The following theorem is the inverse of Theorem 1, and defines the even terms of $a_n$ as sums over the odd terms. It is included for completeness although it is perhaps too cumbersome to be of computational value.

THEOREM 3. *If $a_n$ is self-inverse of degree $d = 2\delta$, then with $T_k$ defined by Eq.* (5),

$$a_{2n} = \frac{a_{2n+1}}{(2n+1)\,\delta} + \sum_{k=0}^{n-1} (-1)^k \binom{2n}{2k+1}\left[\frac{T_k\,\delta^{2k+1}}{2^{2k+2}-1}\right] a_{2n-2k-1}\,.$$

*Proof.* From Eq. (7), $\cos(at) = \cot(\delta t)\sin(at)$. The coefficients of $\cot(t)$ are expressed here in terms of the $T_k$'s rather than the more usual Bernoulli numbers.

## IV. CHARACTERIZING SELF-INVERSE FUNCTIONS

THEOREM 4. (*Exponential generating function*): *The sequence $a_n$ is self-inverse of degree d if and only if*

$$e^{at} = e^{dt}e^{-at}, \qquad a^n \equiv a_n\,. \tag{11}$$

*Proof.*

$$\begin{aligned} a^n &= (d-a)^n, \qquad a^k \equiv a_k \\ \Leftrightarrow \quad e^{at} &= e^{(d-a)t} \\ &= e^{dt}e^{-at}. \end{aligned}$$

COROLLARY. *Every generating function of the form $e^{at} = e^{\delta t}L(t^2)$, with $L(t)$ any formal Laurent series, is the generating function of a self-inverse sequence of degree $d = 2\delta$.*

*Proof.*

$$\begin{aligned} e^{at} &= e^{\delta t}L(t^2), \\ \Rightarrow \quad e^{2\delta t}e^{-at} &= e^{2\delta t}e^{-\delta t}L((-t)^2) \\ &= e^{\delta t}L(t^2) = e^{at}, \end{aligned}$$

which is Eq. (11).

THEOREM 5. (*Construction*): *If $f_n$ is any sequence of numbers, and $\delta$ any constant, then*

$$a_n = \sum_{k=0}^{m} \binom{n}{2k} f_k\,\delta^{n-2k}, \qquad m = [n/2] \tag{12}$$

*is self-inverse of degree $d = 2\delta$.*

*Proof.* If

$$a_n = \sum_{k=0}^{m} \binom{n}{2k} f_k \, \delta^{n-2k}, \qquad m = [n/2],$$

then

$$e^{at} = e^{\delta t} \cosh(ft), \qquad f^{2k} \equiv f_k ,$$

and the result follows by the Corollary to Theorem 4 above.

## V. Combinatorial Setting of $T_k$

Let $a_n$ and $f_n$ be any two sequences related by Eq. (12) with $\delta = 1$, i.e., let $a_n$ and $f_n$ satisfy

$$a_n = \sum_{k=0}^{m} \binom{n}{2k} f_k , \qquad m = [n/2]. \tag{13}$$

Similarly define the particular instance

$$e_n = \sum_{k=0}^{m} \binom{n}{2k}, \qquad f_k \equiv 1. \tag{14}$$

Then $e_n$ counts the number of even cardinality subsets of an $n$ element set.

Letting $T_k$ be defined by Eq. (8), then,

$$\begin{aligned} e_{2n+1} &= \sum_{k=0}^{n} \binom{2n+1}{2k} \\ &= \sum_{k=0}^{n} \binom{2n+1}{2k} \left[ \sum_{j=0}^{n-k} (-1)^j \binom{2n+1-2k}{2j+1} T_j \right] \\ &= \sum_{j=0}^{n} (-1)^j \binom{2n+1}{2n-2j} T_j \left[ \sum_{k=0}^{n-j} \binom{2n-2j}{2k} \right] \\ &= \sum_{j=0}^{n} (-1)^j \binom{2n+1}{2n-2j} T_j e_{2n-2j} , \end{aligned} \tag{15}$$

which is an instance of Theorem 1 with $\delta = 1$.

The bottom equation in (15) asserts that the number of even cardinality subsets of an odd cardinality set $U$ can be found by inclusion-exclusion on the even-subset enumerators of the even subsets of $U$, appropriately weighted. The weighting factors $T_k$, defined by Eq. (8), ensure that every even cardinality subset of $U$ is counted in the inclusion-exclusion (15), exactly once.

The same method applied in general to $a_n$, defined in Eq. (13) with $f_k$ arbitrary, yields

$$\begin{aligned} a_{2n+1} &= \sum_{k=0}^{n} \binom{2n+1}{2k} f_k \\ &= \sum_{k=0}^{n} \binom{2n+1}{2k} f_k \left[\sum_{j=0}^{n-k} (-1)^j \binom{2n+1-2k}{2j+1} T_j\right] \\ &= \sum_{j=0}^{n} (-1)^j \binom{2n+1}{2n-2j} T_j \left[\sum_{k=0}^{n-j} \binom{2n-2j}{2k} f_k\right] \\ &= \sum_{j=0}^{n} (-1)^j \binom{2n+1}{2n-2j} T_j a_{2n-2j}, \end{aligned}$$

which is Theorem 1 with $\delta = 1$, and which provides an alternative proof of the theorem.

Interpreting the numbers $T_k$ as weighting factors for an inclusion-exclusion enumeration of even cardinality subsets of an odd cardinality set, Eq. (15), defines the $T_k$'s combinatorially. This is the way I originally computed their values. Only after I found, by using Sloan [5], that they were the coefficients of $\tan(t)$ did I search for a proof of this fact using generating functions.

## VI. Examples of Self-Inverse Sequences

1. $a_n = k^n$, $k$ constant. $[d = 2k]$. $e^{at} = e^{kt}$.

2. $a_0 = 1$, $a_n = 2^{n-1}k^n$, $n \geqslant 1$. $[d = 2k]$. The case $k = 1$ is the sequence $e_n$ defined by Eq. (14): $e_n = \sum_k \binom{n}{2k} = 2^{n-1}$. $e^{at} = \frac{1}{2} + e^{2kt}/2 = e^{kt}\cosh(kt)$.

3. $a_n = \binom{n}{2k}$, $k$ an integer. $[d = 2]$. $e^{at} = e^t t^{2k}/(2k)!$. Replacing $2k$ by $2k+1$ makes the sequence anti-self-inverse.

4. $a_n = c_{n+1} = \binom{2n+2}{n+1}/(n+2)$, the Catalan numbers [4]. $[d = 4]$. Equation (4) for the Catalan numbers is due to Touchard [6] (cf. also [4, p. 156]), and was my starting point for this work.

5. $a_n = m_n = \sum_k \binom{n}{2k} c_k$, $c_k$ the Catalan numbers ($m$ is for Th. Motzkin). $[d = 2]$. These numbers, as posed in [1], are the number of ways of selecting $n$ points on a circle either singly or in noncrossing pairs.

6. $a_0 = 1, a_n = a_{n-1} + (n-1)\, a_{n-2}$ . $[d = 2]$. $e^{at} = e^{(t+t^2/2)}$. The first terms are 1, 1, 2, 4, 10, 26, 76, 232, 764,... . This sequences enumerates the self-conjugate permutations of $\{1, 2,..., n\}$, that is, those permutations in which the number $i$ is in position $j$ if and only if $j$ is in position $i$ [2, p. 6]. It also enumerates the special switchboard problem; i.e., it enumerates the states of a telephone exchange with $n$ subscribers which is provided with means to connect subscribers in pairs only (no conference circuits and no outside lines) [3, p. 85].

7. $a_0 = 1,\quad a_n = 2a_{n-1} + (n-1)\, a_{n-1}$ . $[d = 4]$. $e^{at} = e^{(2t+t^2/2)}$. The first terms are 1, 2, 5, 14, 43, 142, 499, 1850. (Compare this with example 4 above; to wit: 1, 2, 5, 14, 42, 132, 429, 1430.) This sequence enumerates the general switchboard problem; i.e., it enumerates the states of a telephone exchange with $n$ subscribers which is provided with means to connect subscribers singly to outside lines and in pairs internally (no conference circuits). This result is new.

8. $a_0 = 1, a_n = 2a_{n-1} + 2(n-1)\, a_{n-2}$ . $[d = 4]$. [5, Sequence 645] $e^{at} = e^{(2t+t^2)}$.

9. $a_n = H_n(x)$, the Hermite polynomials [3, p. 86]. $[d = 4x]$. $e^{at} = e^{(2xt-t^2)}$.

Equation (4), and hence Theorem 1, applies to each of these examples, yielding two identities for each. Applying them to Example 9, for example, yields

$$H_n(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} H_k(x)\, 4^{n-k} x^{n-k}$$

and

$$H_{2n+1}(x) = \sum_{k=0}^{n} (-1)^k \binom{2n+1}{2k+1} T_k 4^{2k+1} x^{2k+1} H_{2n-2k}(x).$$


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