The zeros of the Weierstrass $\wp$–function and hypergeometric series

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ABSTRACT. We express the zeros of the Weierstass $\wp$-function in terms of generalized hypergeometric functions. As an application of our main result we prove the transcendence of two specific hypergeometric functions at algebraic arguments in the unit disc. We also give a Saalschützian ${}_4F_3$–evaluation.

1. Introduction

The Weierstrass $\wp$-function is defined for $z \in \mathbb{C}$ and $\tau \in \mathcal{H}$, the upper half-plane, by

$$\wp(z,\tau) = z^{-2} + \sum_{\omega \neq 0} \big((z+\omega)^{-2} - \omega^{-2}\big),$$

where ω runs over the lattice $\mathbb{Z} + \tau\mathbb{Z}$. For τ fixed, $\wp$ and its derivative $\wp_z$ are the fundamental elliptic functions for $\mathbb{Z} + \tau\mathbb{Z}$. The fact that the zeros of $\wp_z$ in the torus $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ occur at the points of order 2, namely $1/2, \tau/2$ and $(1+\tau)/2$, is basic for the theory. On the other hand, the zeros of $\wp$ itself are not nearly as easy to describe. Since $\wp$ assumes every value in $\mathbb{C} \cup \{\infty\}$ exactly twice in $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$, it follows that $\wp$ has two zeros there which, $\wp$ being even, can be written in the form $\pm z_0$. Almost a century after Weierstrass' lectures on elliptic functions were published **[14]**, Eichler and Zagier **[6]** found the first explicit formula for z_0.

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This formula gives z_0 as a certain modular type integral of weight 3 in terms of τ. Here we will "deuniformize" their formula and express z_0 as a multi-valued function of the classical modular invariant

$$j(\tau) = q^{-1} + 744 + 196884q + \dots \qquad (q = e^{2\pi i\tau}), \tag{1}$$

although it is, in fact, better to work with

$$x = 1 - 1728/j. \tag{2}$$

Along these lines, it was already understood in the nineteenth century that τ can be written as the ratio of two solutions of the second order hypergeometric equation in x:

$$[\delta(\delta - \tfrac{1}{2}) - x(\delta + \tfrac{1}{12})(\delta + \tfrac{5}{12})]Y = 0 \quad \text{where} \quad \delta = x\tfrac{d}{dx}.$$

Similarly, we will show that z_0 can be expressed as the ratio of two solutions of the third order hypergeometric equation in x:

$$[\delta(\delta - \tfrac{1}{2})(\delta - \tfrac{1}{4}) - x(\delta + \tfrac{1}{12})(\delta + \tfrac{5}{12})(\delta + \tfrac{3}{4})]Y = 0.$$

To be more specific, we will use (generalized) hypergeometric series defined for $|x| < 1$ by

$$F(x) = F(a_1, \dots, a_m; b_1, \dots, b_{m-1} \mid x) = \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_m)_n}{(b_1)_n \dots (b_{m-1})_n} \frac{x^n}{n!}, \tag{3}$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$ and no $(b_k)_n = 0$. It is well known that for any fixed choice of $b \in \{1, b_1, \dots, b_{m-1}\}$, the function $x^{b-1}F(x)$ satisfies an m-th order hypergeometric equation and has an analytic continuation to a multi-valued function on the Riemann sphere punctured at $\{0, 1, \infty\}$. In terms of these functions it can be shown using the classical method of Fricke [**9**, I. p.329] (see also [**10**, p.159]) that

$$\tau = \frac{c_1\, F(\frac{1}{12}, \frac{5}{12}\,; \frac{1}{2} \mid x)}{F(\frac{1}{12}, \frac{5}{12}\,; 1 \mid 1 - x)} - i \qquad \Big(c_1 = \tfrac{2i\sqrt{\pi}}{\Gamma(7/12)\Gamma(11/12)}\Big). \tag{4}$$

Our main result, proved in the following section, gives a similar expression for z_0.

THEOREM 1. *The zeros of the $\wp$-function are given by $\pm z_0$, where*

$$z_0 = \frac{1+\tau}{2} + \frac{c_2\, x^{\frac{1}{4}}\, F\big(\frac{1}{3}, \frac{2}{3}, 1\,; \frac{3}{4}, \frac{5}{4} \mid x\big)}{F(\frac{1}{12}, \frac{5}{12}\,; 1 \mid 1 - x)} \qquad \Big(c_2 = -\tfrac{i\sqrt{6}}{3\pi}\Big). \tag{5}$$

Here τ is given in (4) and x in (2).

In addition to its basic interest for the theory of elliptic functions, Theorem 1 has some simple applications to hypergeometric series worth noting. One concerns the transcendence of their special values. Suppose that all parameters a_k and b_k of $F(x)$ given by (3) are rational. It is a well known problem to determine the set of algebraic x with $|x| < 1$ for which the value $F(x)$ of such an F is algebraic. When F is a Gauss hypergeometric series (m=2) this set is known to be finite unless F is an algebraic function or is one of a finite number of explicitly known exceptional functions (see [**1**] and its references, particularly [**5**]). For generalized hypergeometric functions ($m \geq 3$) there seem to be no nontrivial examples known where this question is settled. It is shown in §3 that Theorem 1 together with a classical result of Schneider provide two such examples.

COROLLARY 1. *For algebraic $x \neq 0$ with $|x| < 1$ the values*

$$F(\tfrac{1}{3}, \tfrac{2}{3}, 1; \tfrac{3}{4}, \tfrac{5}{4} \mid x) \quad \text{and} \quad F(\tfrac{3}{4}, \tfrac{5}{4}, 1; \tfrac{4}{3}, \tfrac{5}{3} \mid x)$$

are both transcendental.

Eichler and Zagier also gave an amusing corollary of their formula for $\tau = i$:

$$\sum_{n=1}^{\infty} \frac{A_n}{n^2}\, e^{-2\pi n} = \frac{\pi - \log(5 + 2\sqrt{6})}{72\sqrt{6}},$$

where $A_n = 1, 732, 483336, \ldots$ are defined through the q-series

$$\sum_{n=1}^{\infty} A_n q^n = \frac{q \prod_{n\geq 1}(1-q^n)^{24}}{(1 - 504 \sum_{n\geq 1} \sigma_5(n) q^n)^{3/2}} \qquad \Big(\sigma_s(n) = \sum_{d|n} d^s\Big).$$

In the same spirit, in the limiting case $\mathrm{Im}\ \tau \to \infty$ we present a hypergeometric counterpart.

COROLLARY 2. *We have*

$$F\big(\tfrac{3}{4}, \tfrac{5}{4}, 1, 1\ ; \tfrac{4}{3}, \tfrac{5}{3}, 2 \mid 1\big) = -\tfrac{128}{9} \log(2\sqrt{6} - 4).$$

This curious Saalschützian ${}_4F_3$–evaluation does not seem to follow easily from classical results [**3**]. It is derived in §4 from Theorem 1 and a delicate asymptotic formula discovered by Ramanujan.

2. The Eichler-Zagier formula

To state the Eichler-Zagier formula we need the Eisenstein series

$$E_4(\tau) = 1 + 240 \sum_{n\geq 1} \sigma_3(n) q^n \quad \text{and} \quad E_6(\tau) = 1 - 504 \sum_{n\geq 1} \sigma_5(n) q^n,$$

and the normalized discriminant function

$$\Delta(\tau) = \tfrac{1}{1728}(E_4^3(\tau) - E_6^2(\tau)) = q \prod_{n\geq 1}(1-q^n)^{24},$$

all familiar modular forms.

THEOREM (Eichler–Zagier). *The zeros of the Weierstrass $\wp$-function are given by*

$$z = m + \tfrac{1}{2} + n\tau \pm \left(\tfrac{\log(5+2\sqrt{6})}{2\pi i} + 144\pi i\sqrt{6} \int_{\tau}^{i\infty} (\sigma - \tau) \frac{\Delta(\sigma)}{E_6(\sigma)^{3/2}}\, d\sigma \right) \tag{6}$$

for all $m, n \in \mathbb{Z}$, where the integral is to be taken over the vertical line $\sigma = \tau + i\mathbb{R}^+$ in $\mathcal{H}$.

They gave two proofs of (6) in [**6**]. The first is based on the fact that if $z_0(\tau)$ is a zero of $\wp(z,\tau)$, then $z_0''(\tau)$ is a modular form of weight 3 that can be determined explicitly. The second proof uses elliptic integrals in a more direct manner.

Proceeding to the proof of Theorem 1, by analytic continuation it is enough to assume that $\tau = iy$ with $y \geq 1$. Any fractional powers that occur are assumed to be principal values. It is convenient to begin with the modular function $t = 1 - x = 1728/j$, where as before $j = E_4^3/\Delta$. We have the relations

$$t = \frac{1728\Delta}{E_4^3}, \qquad 1 - t = \frac{E_6^2}{E_4^3} \qquad \text{and} \qquad \frac{1}{t}\frac{dt}{d\tau} = 2\pi i \frac{E_6}{E_4}. \tag{7}$$

To obtain the last one we use the formulas of Ramanujan for derivatives with respect to τ [**11**, p.142.]:

$$E_4' = \tfrac{2\pi i}{3}(E_2E_4 - E_6),\ E_6' = \pi i(E_2E_6 - E_4^2),\ \Delta' = 2\pi i E_2 \Delta,$$

where

$$E_2(z) = 1 - 24\sum_{n\geq 1} \sigma_1(n) q^n.$$

It is a classical fact that a pair of linearly independent solutions to the hypergeometric equation

$$t(1-t)Y'' + (1 - \tfrac{3}{2}t)Y' - \tfrac{5}{144}Y = 0 \tag{8}$$

is given by

$$F_1(t) = F(\tfrac{1}{12}, \tfrac{5}{12}; 1 \mid t) \quad \text{and} \quad F_2(t) = \tau(t)F_1(t), \tag{9}$$

where $\tau(t)$ is the inverse of $t(\tau)$ ([**9**, I. p. 336.]). We need the remarkable identity of Fricke [**9**] (see also [**1**, p. 256]):

$$F_1(t(\tau)) = E_4^{1/4}(\tau). \tag{10}$$

Using (7) and (10) we obtain the Wronskian

$$\begin{vmatrix} F_1(t) & F_2(t) \\ F_1'(t) & F_2'(t) \end{vmatrix} = F_1(t)F_2'(t) - F_2(t)F_1'(t) = \tfrac{1}{2\pi i}\, t^{-1}(1-t)^{-1/2} \tag{11}$$

as well as the identity

$$\frac{1728\Delta(\tau)}{E_6^{3/2}(\tau)} = \tfrac{1}{2\pi i}(1-t)^{-5/4}F_1(t)\frac{dt}{d\tau}. \tag{12}$$

Write $u = t(\tau)$ and let

$$H(u) = 4\pi^2 F_1(u)\int_{i\infty}^{\tau(u)}(\sigma-\tau(u))\frac{1728\Delta(\sigma)}{E_6(\sigma)^{3/2}}\,d\sigma. \tag{13}$$

Changing variables $\sigma \mapsto t$ and we get using (12) and $\tau(u) = \frac{F_2(u)}{F_1(u)}$ that

$$H(u) = 2\pi i\int_0^u \big(F_1(t)F_2(u) - F_1(u)F_2(t)\big)(1-t)^{-5/4}\,dt.$$

Now apply the differential operator

$$L_u = u(1-u)\tfrac{d^2}{du^2} + (1-\tfrac{3}{2}u)\tfrac{d}{du} - \tfrac{5}{144}$$

to this integral to get

$$L_u H(u) = 2\pi i u(1-u)(F_1F_2' - F_2F_1')(1-u)^{-5/4},$$

where we are using that F_1 and F_2 satisfy (8). Thus by (11)

$$L_u H = (1-u)^{-3/4}$$

or, in other words, $H(u)$ satisfies an inhomogeneous hypergeometric equation. Letting $x = 1-u$ this equation can be written

$$x(1-x)Y'' + (\tfrac{1}{2} - \tfrac{3}{2}x)Y' - \tfrac{5}{144}Y = x^{-3/4}. \tag{14}$$

By using the method of Frobenius (see [**2**, p.201.]), it is easy to find a particular solution to (14) in the form

$$-16x^{1/4}F(\tfrac{1}{3},\tfrac{2}{3},1;\tfrac{3}{4},\tfrac{5}{4}\mid x) = -16x^{1/4}F(x), \tag{15}$$

say. Thus it follows from (13) that for some constants a and b we have

$$F_1(t)\int_{i\infty}^{\tau}(\tau-\sigma)\tfrac{\Delta(\sigma)}{E_6(\sigma)^{3/2}}\,d\sigma = \tfrac{4(1-t)^{1/4}}{(12)^3\pi^2}\,F(1-t) + a\tau F_1(t) + bF_1(t),$$

where $t = t(\tau) = \frac{1728\Delta(\tau)}{E_4^3(\tau)}$ and $\tau = \frac{F_2(t)}{F_1(t)}$, with $F_1(t) = F(\frac{1}{12},\frac{5}{12},1\mid t)$ from (9). Finally we get from (6) with $m = n = 0$ and the minus sign that for some other constants c and d the zeros can be represented by

$$z_0 = z_0(\tau) = \tfrac{-i\sqrt{6}(1-t)^{1/4}}{3\pi F_1(t)}F(1-t) + c\tau + d. \tag{16}$$

In order to compute the constant c, let $\tau = iy$ and take $y \to \infty$. The first term asymptotics of the zero-balanced hypergeometric series F from (15) is easily obtained:

$$F(1-t) = -\tfrac{\sqrt{6}}{8}(2\pi i\tau) + \mathrm{O}(1), \quad \text{as } \ y \to \infty. \tag{17}$$

When combined with (16) and (6) this shows that $c = 1/2$. Taking $d = 1/2$ gives $z_0(i) = (1+i)/2$, known to be a (double) zero of $\wp(z,i)$. Theorem 1 now follows. $\square$

We remark that Eichler and Zagier generalized their formula in **[6]** to equations of the form $\wp(z,\tau) = \phi(\tau)$ for any meromorphic $\phi(\tau)$ and also to the zeros of Jacobi forms in **[7]**. However, one finds in those cases where the above technique applies that a solution to the resulting inhomogeneous hypergeometic equation is not usually expressible in a simple way in terms of a hypergeometric function.

3. A theorem of Schneider

For arithmetic purposes it is best to define $\wp$ for any full lattice $\Lambda \subset \mathbb{C}$ as the sum over non-zero $\omega \in \Lambda$

$$\wp(z) = \wp(z,\Lambda) = z^{-2} + \sum_{\omega \neq 0} \big((z+\omega)^{-2} - \omega^{-2}\big).$$

As is well known, $\wp$ satisfies

$$\wp_z^2 = 4\wp^3 - g_2\wp - g_3, \tag{18}$$

where $g_2 = g_2(\Lambda) = 60\sum_{\omega\neq 0}\omega^{-4}$ and $g_3 = g_3(\Lambda) = 140\sum_{\omega\neq 0}\omega^{-6}$. It is a fundamental fact that

$$g_2^3 - 27g_3^2 \neq 0 \tag{19}$$

and that, given any pair of complex numbers g_2, g_3 satisfying (19), there is a (unique) lattice Λ whose $\wp$-function satisfies (18). Of course, $\wp(z,\Lambda) = \omega_1^{-2}\wp(z/\omega_1,\tau)$ when $\Lambda = \omega_1(\mathbb{Z}+\tau\mathbb{Z})$ for a non-zero $\omega_1 \in \mathbb{C}$ and $\tau \in \mathcal{H}$, which is always possible to arrange. In this case we have the identities

$$g_2 = \tfrac{4\pi^4}{3\omega_1^4}E_4(\tau) \quad \text{and} \quad g_3 = \tfrac{8\pi^6}{27\omega_1^6}E_6(\tau). \tag{20}$$

Turning now to the proof of Corollary 1, we need the following classical result of Schneider **[13]**.

THEOREM (Schneider). *If g_2 and g_3 are algebraic, then, for any algebraic $z \neq 0$, $\wp(z,\Lambda)$ is transcendental.*

A short proof of this result can be found in [**4**, Chapter 6]. Theorem 1 and (10) together with (20) imply that

$$w = 2i\tfrac{\sqrt{g_3}}{g_2}F\left(\tfrac{1}{3},\tfrac{2}{3},1;\tfrac{3}{4},\tfrac{5}{4}\mid\tfrac{27g_3^2}{g_2^3}\right)$$

differs from a zero of $\wp(z,\Lambda)$ by a point of order 2 in $\mathbb{C}/\Lambda$. The duplication formula for the $\wp$-function [**9**, II. p.184] applied at this zero yields the evaluation

$$\wp(2w) = -\frac{g_2^2}{16g_3}.$$

Thus for the lattice Λ with invariants $g_2 = g_3 = \frac{27}{x}$ we have that

$$\wp\left(\tfrac{4}{9}\sqrt{-3x}\; F(\tfrac{1}{3},\tfrac{2}{3},1;\tfrac{3}{4},\tfrac{5}{4}\mid x)\right) = -\tfrac{27}{16x}.$$

Schneider's theorem now gives the first statement of Corollary 1.

A parallel treatment of the proof of Theorem 1, but starting with the modular function $v = 1-1/x$ and the identity

$$F(\tfrac{1}{12},\tfrac{7}{12},1\mid v(\tau)) = E_6^{1/6}(\tau),$$

(see [**1**, p. 256]) shows that

$$w = \tfrac{i}{48}\tfrac{g_2^2}{\sqrt{g_3^3}}\,F\left(\tfrac{3}{4},\tfrac{5}{4},1;\tfrac{4}{3},\tfrac{5}{3}\mid\tfrac{g_2^3}{27g_3^2}\right)$$

differs from a zero of $\wp(z,\Lambda)$ by a point of order 3 in $\mathbb{C}/\Lambda$. Now the triplication formula for $\wp$ [**9**, II. p.184] applied at this zero yields the evaluation

$$\wp(3w) = \frac{8g_3}{g_2} - \frac{2^8g_3^3}{g_2^4}.$$

Thus for the lattice with invariants $g_2 = g_3 = 27x$ we see that

$$\wp\left(\tfrac{3}{16}\sqrt{-3x}\; F(\tfrac{3}{4},\tfrac{5}{4},1;\tfrac{4}{3},\tfrac{5}{3}\mid x)\right) = 8-\tfrac{2^8}{27x}.$$

As before, the second statement of Corollary 1 now follows from Schneider's Theorem. □

4. A result of Ramanujan

It is instructive to compare Theorem 1 with the corresponding result for the degenerate $\wp$–function

$$\lim_{\mathrm{Im}\,\tau\to\infty}\wp(z,\tau) = \frac{\pi^2}{\sin^2\pi z} - \frac{\pi^2}{3}.$$

The zeros of this function are given by $\pm z_0 + \mathbb{Z}$, where

(21) $$z_0 = \tfrac{1}{2} + \tfrac{i}{2\pi}\log(5+2\sqrt{6}).$$

In order to compare this with Theorem 1, we need to determine explicitly the constant term in the asymptotic formula (17). Such a result was found by Ramanujan and appears in his notebook [**12**, p.132] without proof.

THEOREM (Ramanujan). *If* $a+b+c=d+e$ *and* $\mathrm{Re}(c)>0$ *then*

$$\lim_{x\to 1^-} \tfrac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(d)\Gamma(e)} F(a,b,c\,;d,e\mid x)+\log(1-x)=L, \quad \text{where} \tag{22}$$

$$L=2\psi(1)-\psi(a)-\psi(b)+\sum_{n=1}^{\infty}\frac{(d-c)_n(e-c)_n}{(a)_n(b)_n\, n},$$

with $\psi(a)=\Gamma'(a)/\Gamma(a)$.

Ramanujan's method of deriving this is unknown. In 1984 Evans and Stanton [**8**] gave a proof of it in a more precise form; their proof is rather intricate. To derive Corollary 2, specialize (22) to

$$\lim_{x\to 1^-} \tfrac{4\sqrt{6}}{3} F(\tfrac{1}{3},\tfrac{2}{3},1;\tfrac{3}{4},\tfrac{5}{4}\mid x)+2\pi i\tau+\log 1728=L, \tag{23}$$

after using (1), (2) and the duplication and triplication formulas

$$\begin{aligned}\Gamma(2z)&=\tfrac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma(z+\tfrac{1}{2}) \quad \text{and}\\ \Gamma(3z)&=\tfrac{3^{3z-1/2}}{2\pi}\Gamma(z)\Gamma(z+\tfrac{1}{2})\Gamma(z+\tfrac{2}{3}).\end{aligned} \tag{24}$$

By Theorem 1 and its proof we have from (23)

$$L=4\pi i(z_0-\tfrac{1}{2})+\log 1728,$$

where z_0 in (21) is the correct degenerate zero, as follows from the discussion above (16). Thus

$$L=-2\log(5+2\sqrt{6})+3\log 12, \tag{25}$$

By (24) we get easily that

$$2\psi(1)-\psi(1/3)-\psi(2/3)=3\log 3.$$

Now Corollary 2 follows from (25) after shifting indices $n\mapsto n+1$ in the sum in L and using that $1/(n+1)=(1)_n/(2)_n$. $\square$

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