#A54 INTEGERS 13 (2013)

GENERALIZING DELANNOY NUMBERS VIA COUNTING WEIGHTED LATTICE PATHS

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Received: 11/12/12, Revised: 5/15/13, Accepted: 7/10/13, Published: 9/26/13

Abstract

The aim of this paper is to introduce a generalization of Delannoy numbers. The standard Delannoy numbers count lattice paths from $(0,0)$ to (n,k) consisting of horizontal $(1,0)$, vertical $(0,1)$, and diagonal $(1,1)$ steps called segments. We assign weights to the segments of the lattice paths, and we sum weights of all lattice paths from any (a,b) to (n,k). Generating functions for the generalized Delannoy numbers and weighted central Delannoy numbers are delivered and some consequences of them are drawn. Several identities, summations, and convolution-like formulas for the general case are given. We show how our approach may be used in counting lattice paths with certain restrictions. Namely, we remove certain classes of segments from the complete lattice and then we ask about the number of paths in such modified lattices. In the last section we show how these numbers generalize, among others, binomial, q-binomial coefficients, ordinary Stirling numbers of both kinds, and p,q-Stirling numbers. Applying the results of generalized Delannoy numbers we deliver a q-Vandermonde identity and a generalized Carlitz-like formula for p,q-binomial coefficients.

1. Introduction

Let $L(n,k)$ denote the family of lattice paths from $(0,0)$ to (n,k), where $n,k \geq 0$, consisting of vertical $(0,1)$, horizontal $(1,0)$, and diagonal $(1,1)$ steps (segments). An example of a lattice path from $L(4,2)$ is given in Fig. 1. Let $D(n,k)$ denote the size of $L(n,k)$. The numbers $D(n,k)$ satisfy

$$D(n,k) = D(n,k-1) + D(n-1,k) + D(n-1,k-1)$$

with initial values $D(n,0) = D(0,k) = D(0,0) = 1$.

From the recurrence relation one can obtain the generating function

$$\sum_{n,k\geq 0} D(n,k)x^k y^n = \frac{1}{1-x-y-xy}. \tag{1.1}$$

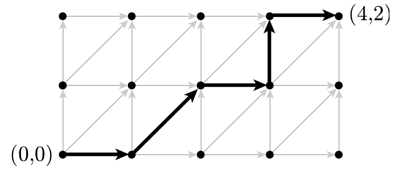

Figure 1: A lattice path from the set $L(4,2)$.

Recall that $D(n,k)$ are called *Delannoy numbers* from the name of Henri Delannoy [8] and $D(n) = D(n,n)$ *central Delannoy numbers*. These numbers are denoted in OEIS [23] by A152250 and A001850, respectively. See Comtet [6, p. 81] and Stanley [24, p. 185]. Delannoy numbers appear also in Stanton and Cowan [25]. We refer the reader to Banderier and Schwer [3]. From the generating function it follows that

$$D(n,k) = \sum_{i=0}^{k} \binom{n}{i} \binom{n+k-i}{n}. \tag{1.2}$$

There are several ways of generalizing Delannoy numbers in the literature. Wagner [28] defines some statistics on lattice paths. Autebert and Schwer [2] extend this 2-dimensional case to the d-dimensional space $\mathbb{Z}^d$, where $d > 2$. See also Caughman, Haithcock and Veerman [5]. The main idea of our approach is to assign weights to the segments of a lattice and to sum weights of paths. This idea has been previously used by Fray and Roselle [13], Hetyei [17] and Loehr and Savage [21, Sec. 3.2].

Let $\mathfrak{L}$ denote the 4-tuple $\langle \mathbf{v}, \mathbf{h}, \mathbf{d}, \zeta \rangle$ of three sequences of complex numbers $\mathbf{v} = (v_0, v_1, \ldots)$, $\mathbf{h} = (h_0, h_1, \ldots)$, $\mathbf{d} = (d_0, d_1, \ldots)$, and a complex number ζ. A *step* is a pair of two lattice points. We consider only three types of steps: horizontal $((a,b),(a+1,b))$, vertical $((a,b),(a,b+1))$, and diagonal $((a,b),(a+1,b+1))$. We represent a path $\pi \in L(n,k)$ by the sequence $(\pi_1, \pi_2, \ldots, \pi_s)$ of steps π_i from the origin $(0,0)$ to (n,k). By the *weight* of a step $\pi_i = ((a,b),(c,e))$ we mean the value of $w^{\mathfrak{L}}(\pi_i)$ defined as follows

$$w^{\mathfrak{L}}(\pi_i) = \begin{cases} v_c, & \text{if } \pi_i \text{ is a vertical step;} \\ h_c \zeta^e, & \text{if } \pi_i \text{ is a horizontal step;} \\ d_c \zeta^{e-1}, & \text{if } \pi_i \text{ is a diagonal step.} \end{cases}$$

Incoming and outcoming steps of a lattice point with corresponding weights are given in Fig. 2. We define the *weight* of a path π, denoted by $w^{\mathfrak{L}}(\pi)$, to be $w^{\mathfrak{L}}(\pi_1) \cdots w^{\mathfrak{L}}(\pi_s)$. By the (n,k) $\mathfrak{L}$*-Delannoy number*, expressed in angular brackets, we mean the sum

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{\mathfrak{L}} = \sum_{\pi \in L(n,k)} w^{\mathfrak{L}}(\pi). \tag{1.3}$$

The $\mathfrak{L}$-Delannoy numbers generalize a wide class of combinatorial numbers,

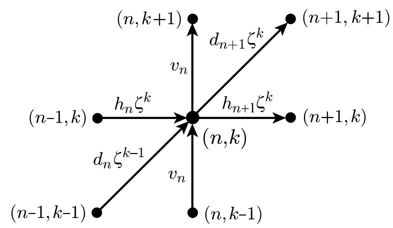

Figure 2: Weights of incoming and outcoming steps.

including binomial, q-binomial coefficients, Stirling numbers of both kinds, p,q-Stirling numbers [27], ζ-analogues [12] (see Section 8). For $\mathfrak{B} = \langle \mathbf{1}, \mathbf{1}, \mathbf{1}, 1\rangle$, where $\mathbf{1} = (1, 1, \ldots)$, the $\mathfrak{B}$-Delannoy numbers reduce to the ordinary Delannoy numbers, i.e.,

$$\left\langle {n \atop k} \right\rangle_{\mathfrak{B}} = D(n, k). \tag{1.4}$$

In Section 3 we show that the generating function for the general case of the $\mathfrak{L}$-Delannoy numbers is

$$\sum_{k\geq 0} \left\langle {n \atop k} \right\rangle_{\mathfrak{L}} z^k = \frac{1}{(1 - v_0\zeta^n z)} \prod_{i=1}^{n} \frac{h_i + d_i\zeta^{n-i}z}{1 - v_i\zeta^{n-i}z}.$$

In Section 4 we consider central weighted Delannoy numbers studied by Fray and Roselle [13], and Hetyei [17]. Namely, let $\mathfrak{W} = \langle \mathbf{v}^c, \mathbf{h}^c, \mathbf{d}^c, 1\rangle$, where $\mathbf{v}^c = (v, v, \ldots), \mathbf{h}^c = (h, h, \ldots), \mathbf{d}^c = (d, d, \ldots)$ are constant sequences of $v, h, d \in \mathbb{C}$. We show that

$$\sum_{n\geq 0} \left\langle {n \atop n} \right\rangle_{\mathfrak{W}} z^n = \frac{1}{\sqrt{d^2\, z^2 - z(2\, d + 4\, h\, v) + 1}}.$$

We define two reduced cases of the generalized Delannoy numbers. Namely, for a given 4-tuple $\mathfrak{L} = \langle \mathbf{v}, \mathbf{h}, \mathbf{d}, \zeta\rangle$ we set $\mathfrak{L}_1 = \langle \mathbf{0}, \mathbf{h}, \mathbf{d}, \zeta\rangle$ and $\mathfrak{L}_2 = \langle \mathbf{v}, \mathbf{1}, \mathbf{0}, \zeta\rangle$, where $\mathbf{0} = (0, 0, \ldots)$. By the $\mathfrak{L}$-*Delannoy numbers of the first kind* (expressed in square brackets) we mean $\mathfrak{L}_1$-Delannoy numbers, and by the $\mathfrak{L}$-*Delannoy numbers of the second kind* (expressed in curly brackets) we mean $\mathfrak{L}_2$-Delannoy numbers, i.e.,

$$\left[{n \atop k} \right]_{\mathfrak{L}} \equiv \left\langle {n \atop k} \right\rangle_{\mathfrak{L}_1}, \qquad \left\{ {n \atop k} \right\}_{\mathfrak{L}} \equiv \left\langle {n \atop k} \right\rangle_{\mathfrak{L}_2}.$$

It is shown in Section 3 that

$$\sum_{k\geq 0} \left[{n \atop k} \right]_{\mathfrak{L}} z^k = \prod_{i=1}^{n} \left(h_i + d_i\zeta^{n-i}z\right),$$

$$\sum_{k\geq 0} \left\{ {n \atop k} \right\}_{\mathfrak{L}} z^k = \prod_{i=0}^{n} \frac{1}{(1 - v_i\zeta^{n-i}z)},$$

It turns out that (1.2) has a natural generalization for $\mathfrak{L}$-Delannoy numbers of the first and second kind, that is,

$$\left\langle {n \atop k} \right\rangle_{\mathfrak{L}} = \sum_{i=0}^{k} \left[{n \atop i} \right]_{\mathfrak{L}} \left\{ {n \atop k-i} \right\}_{\mathfrak{L}},$$

which follows from the generating function. However, a combinatorial proof is given in Section 2. The generalization of the Carlitz formula is delivered in Section 3 (Theorem 10) for the $\mathfrak{L}$-Delannoy numbers of the first and second kind. That is,

$$\left[{n \atop k} \right]_{\mathfrak{H}} = \sum_{j=0}^{k} \binom{n-k+j}{j} (\alpha_2 \zeta^n)^j \left[{n \atop k-j} \right]_{\mathfrak{H}^*},$$

$$\left\{ {n \atop k} \right\}_{\mathfrak{H}} = \sum_{j=0}^{k} \binom{n+k}{j} (\alpha_1 \zeta^n)^j \left\{ {n \atop k-j} \right\}_{\mathfrak{H}^*},$$

where $\mathfrak{H} = \langle \mathbf{v}, \mathbf{1}, \mathbf{d}, \zeta \rangle$ and $\mathfrak{H}^* = \langle \mathbf{v}^*, \mathbf{1}, \mathbf{d}^*, \zeta \rangle$ such that $v_i = v_i^* + \alpha_1 \zeta^i$ and $d_i = d_i^* + \alpha_2 \zeta^i$, where $\alpha_1, \alpha_2 \in \mathbb{C}$.

In Section 5 we study the case where the sequences $\mathbf{v}, \mathbf{h}$, and $\mathbf{d}$ take values over the set $\{0, 1\}$ which allows to count Delannoy paths with certain restrictions. Namely, paths whose steps have weight equal to zero are omitted in the sum. Therefore, by setting certain elements of $\mathbf{v}, \mathbf{h}$, and $\mathbf{d}$ to zero we remove corresponding steps of the lattices. Let $H = H(\sigma, \tau, \eta)$ be a family of steps (see Section 5), we denote by $D(n, k; H)$ the number of Delannoy paths from $(0, 0)$ to (n, k) which consist of steps from H. We show that

$$D(n, k; H) = \sum_{j=0}^{k} \binom{n - |\sigma| - |\tau|}{j - |\sigma|} \binom{n - |\eta| + k - j}{n - |\eta|}, \tag{1.5}$$

where $\sigma, \tau \subseteq \{1, 2, \ldots, n\}$ such that $\tau \cap \sigma = \emptyset$, and $\eta \subseteq \{0, 1, \ldots, n\}$.

In Section 6 we generalize previous results to the case where weighted Delannoy paths begin at any lattice point (m, l). Let x, y be any two lattice points, and let us denote by $L[x, y]$ the set of all Delannoy paths from x to y. Let

$$\left\langle {n \atop k} \right\rangle_{\mathfrak{L}}^{(m,l)} = \sum_{\pi \in L[x,y]} w^{\mathfrak{L}}(\pi),$$

where $x = (m, l)$ and $y = (m+n, l+k)$. Although, this generalization does not give essentially new results but provides a convenient notation for some new convolution-like formulae given in Section 6.

In Section 7 we show the connection between these generalized Delannoy numbers and partitions of numbers. We show, for instance, that for certain 4-tuples $\mathfrak{P}(H)$, the $\mathfrak{P}(H)$-Delannoy numbers of the first and second kind are generating functions of certain sequences counting partitions of numbers, i.e.,

$$\begin{aligned}\begin{bmatrix} m \\ k \end{bmatrix}_{\mathfrak{P}(H)} &= \sum_{n\geq 0} |\mathcal{D}(H;n,k)|\zeta^n, \\ \begin{Bmatrix} m \\ k \end{Bmatrix}_{\mathfrak{P}(H)} &= \sum_{n\geq 0} |\mathcal{R}(H;n,k)|\zeta^n,\end{aligned}$$

where $\mathcal{D}(H;n,k)$, and $\mathcal{R}(H;n,k)$ are the families of partitions of n into k distinct parts from the set H without and with repetition of parts allowed, respectively.

In Section 8 we show how $\mathfrak{L}$-Delannoy numbers generalize binomial and q-binomial coefficients, ordinary Stirling numbers of both kinds, and p,q-Stirling numbers.

2. Counting Lattice Paths with Diagonals

For $n,k \geq 0$, the $\mathfrak{L}$-Delannoy numbers satisfy the following recurrence relation

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{\mathfrak{L}} = v_n \left\langle \begin{matrix} n \\ k-1 \end{matrix} \right\rangle_{\mathfrak{L}} + h_n\zeta^k \left\langle \begin{matrix} n-1 \\ k \end{matrix} \right\rangle_{\mathfrak{L}} + d_n\zeta^{k-1} \left\langle \begin{matrix} n-1 \\ k-1 \end{matrix} \right\rangle_{\mathfrak{L}} \tag{2.1}$$

with initial value $\left\langle \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right\rangle_{\mathfrak{L}} = 1$. We assume that $\left\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rangle_{\mathfrak{L}} = 0$ for $n < 0$ or $k < 0$. We denote by $L_1(n,k)$ the set of all lattice paths from $(0,0)$ to (n,k) consisting of only horizontal and diagonal steps, and by $L_2(n,k)$ the set of lattice paths from $(0,0)$ to (n,k) consisting of vertical and horizontal steps (see Fig. 3). With this notation, we have

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\mathfrak{L}} = \sum_{\pi\in L_1(n,k)} w^{\mathfrak{L}_1}(\pi), \qquad \begin{Bmatrix} n \\ k \end{Bmatrix}_{\mathfrak{L}} = \sum_{\pi\in L_2(n,k)} w^{\mathfrak{L}_2}(\pi).$$

Let $D_1(n,k)$ and $D_2(n,k)$ denote the cardinalities of the sets $L_1(n,k)$ and $L_2(n,k)$, respectively. It is well-known that

$$D_1(n,k) = \binom{n}{k}, \qquad D_2(n,k) = \binom{n+k}{k}.$$

The $\mathfrak{L}$-Delannoy numbers of the first and second kind satisfy the following recurrence relations

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\mathfrak{L}} = h_n\zeta^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_{\mathfrak{L}} + d_n\zeta^{k-1} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{\mathfrak{L}}, \tag{2.2a}$$

$$\begin{Bmatrix} n \\ k \end{Bmatrix}_{\mathfrak{L}} = v_n \begin{Bmatrix} n \\ k-1 \end{Bmatrix}_{\mathfrak{L}} + \zeta^k \begin{Bmatrix} n-1 \\ k \end{Bmatrix}_{\mathfrak{L}}, \tag{2.2b}$$

with initial values $\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right]_{\mathfrak{L}} = \left\{ \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right\}_{\mathfrak{L}} = 1$.

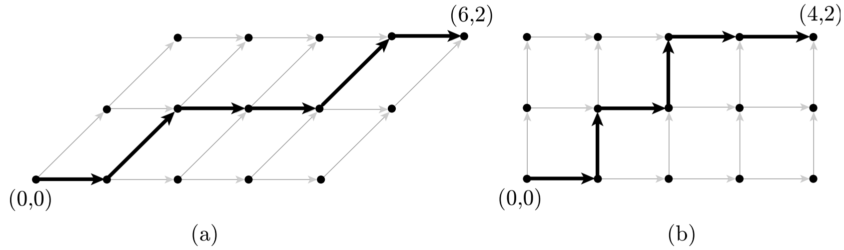

Figure 3: Two lattice paths: (a) from $L_1(6,2)$, and (b) from $L_2(4,2)$.

Let us denote by $L(n,k,j)$ the set of those paths from $L(n,k)$ which contain exactly j diagonal steps. Observe that any path contains at most $\min(n,k)$ diagonal steps, and thus

$$D(n,k) = \sum_{j=0}^{\min(n,k)} |L(n,k,j)|.$$

One shows that

$$|L(n,k,j)| = \binom{n}{j}\binom{n+k-j}{k-j},$$

which immediately implies (1.2). In fact, we show a generalization of this result for the $\mathfrak{L}$-Delannoy numbers.

Theorem 1. *For $n,k \geq 0$, we have*

$$\left\langle {n \atop k} \right\rangle_{\mathfrak{L}} = \sum_{i=0}^{k} \left[{n \atop i} \right]_{\mathfrak{L}} \left\{ {n \atop k-i} \right\}_{\mathfrak{L}}. \tag{2.3}$$

Proof. We divide the proof into three lemmas. Let us consider the left-hand side of (2.3). By Lemma 2 and Lemma 4, we show that

$$\begin{aligned}
\left\langle {n \atop k} \right\rangle_{\mathfrak{L}} &= \sum_{i=0}^{\min(n,k)} \sum_{\pi \in L(n,k,i)} w^{\mathfrak{L}}(\pi) \\
&= \sum_{i=0}^{\min(n,k)} \sum_{\substack{\alpha \in L_1(n,i) \\ \beta \in L_2(n,k-i)}} w^{\mathfrak{L}_1}(\alpha) w^{\mathfrak{L}_2}(\beta) \\
&= \sum_{i=0}^{\min(n,k)} \left[{n \atop i} \right]_{\mathfrak{L}} \left\{ {n \atop k-i} \right\}_{\mathfrak{L}}.
\end{aligned}$$

$\square$

Note that the ith summand of the right-hand side of (2.3) is the sum of weights of these Delannoy paths which consist of exactly i diagonal steps.

Let $\sigma = ((a,b),(c,d))$ and $\phi = ((e,f),(g,h))$ be two lattice segments. We write $\sigma \sim \phi$ if $a = e$ and $c = g$. We set $\mathtt{x}(\sigma) = c$, and $\mathtt{y}(\sigma) = b$. We define two operations on segments: $\mathtt{up}(\sigma) = ((a,b+1),(c,d+1))$, and if σ is horizontal, then $\mathtt{dg}(\sigma) = ((a,b),(c,d+1))$, and $\mathtt{dg}(\sigma) = \sigma$ otherwise.

Lemma 2. *There is a bijection $f : L_1(n,j) \times L_2(n,k-j) \to L(n,k,j)$.*

Proof. The proof consists in the construction of the bijection $f : L_1(n,j) \times L_2(n,k-j) \to L(n,k,j)$.

($\to$) Let us take two paths $\alpha \in L_1(n,j)$ and $\beta \in L_2(n,k-j)$. On one hand, the path α contains j diagonal and $n-j$ horizontal segments, respectively. On the other hand, the path β contains n horizontal and $k-j$ vertical steps, respectively. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_{n+k-j})$. Now, we describe the construction of the path $\pi = f(\alpha,\beta) \in L(n,k,j)$ in $j+1$ consecutive steps. Every path from $L(n,k,j)$ consists of exactly $n+k-j$ segments: j diagonal, $n-j$ horizontal and $k-j$ vertical, respectively. Let $\pi = (\pi_1, \pi_2, \ldots, \pi_{n+k-j})$.

Step 0. We set $\pi := \beta$. Step i for $i = 1, 2, \ldots, j$. Suppose that the ith diagonal segment of α is α_l (counting from the left-hand side). There is only one segment π_r of π such that $\alpha_l \sim \pi_r$. We apply the operation $\mathtt{dg}$ on π_r, i.e., $\pi_r := \mathtt{dg}(\pi_r)$. Next, for all next segments $\pi_{r+1}, \ldots, \pi_{n+k-j}$ of π, we apply the operation $\mathtt{up}$, i.e., for all π_m such that $m = r+1, r+2, \ldots, n+k-j$, we set $\pi_m := \mathtt{up}(\pi_m)$. It is easy to check that after these $j+1$ steps the path π is one of the set $L(n,k,j)$.

($\leftarrow$) Simply executing f in reverse order gives f^{-1}. $\square$

Two paths $\alpha \in L_1(3,2)$, $\beta \in L_2(3,2)$ and corresponding $f(\alpha,\beta) \in L(3,4)$ are given in Fig. 4.

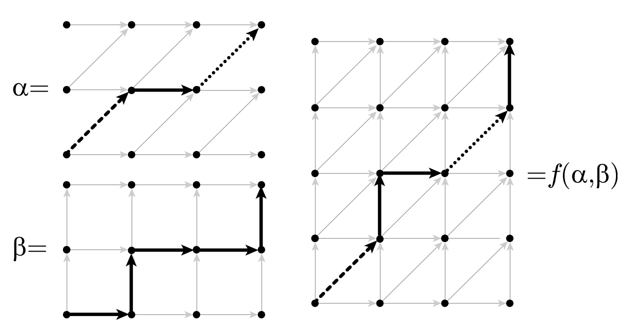

Figure 4: Illustration of the bijection f from Lemma 2.

Lemma 3. *Let $\pi \in L(n,k,j)$ and $\alpha \in L_1(n,j)$, $\beta \in L_2(n,k-j)$ such that $\pi = f(\alpha,\beta)$. For $i = 1,2,\ldots,n$, if α_i is the ith segment of α, and β_l, π_l are segments of β and π, respectively, such that $\alpha_i \sim \beta_l \sim \pi_l$, then*

$$\mathtt{y}(\pi_l) = \mathtt{y}(\alpha_i) + \mathtt{y}(\beta_l).$$

Proof. Consider the construction of f given in the proof of Lemma 2. From the 0th step we see that $\mathtt{y}(\pi_l) = \mathtt{y}(\beta_l)$. In the next j consecutive steps we raise up, by the operation up, certain segments of π such that the first segment of π begins at $(0,0)$ and the last one ends at (n,k).

Observe that the value of $\mathtt{y}(\alpha_i)$ determines the number of diagonal segments $\alpha_b \in \alpha$ for which we have $\mathtt{x}(\alpha_b) < \mathtt{x}(\alpha_i)$. Thus the value of y for π_l is increased from $\mathtt{y}(\beta_l)$ by the number $\mathtt{y}(\alpha_i)$ which is equivalent to the number of up operations applied to π_l. □

Lemma 4. *Let $\pi \in L(n,k,j)$ and $\alpha \in L_1(n,j)$, $\beta \in L_2(n,k-j)$ such that $\pi = f(\alpha,\beta)$ for $1 \le j \le \min(n,k)$. We have*

$$w^{\mathfrak{L}}(\pi) = w^{\mathfrak{L}_1}(\alpha) w^{\mathfrak{L}_2}(\beta). \tag{2.4}$$

Proof. Let $\pi^v = (\pi_1^v, \ldots, \pi_{k-j}^v)$, $\pi^h = (\pi_1^h, \ldots, \pi_{n-j}^h)$, and $\pi^d = (\pi_1^d, \ldots, \pi_j^d)$ be the sequences of all vertical, horizontal, and diagonal segments of π, respectively, in the ascending order of x. In the same manner we define the sets α^h, α^d, and β^h, β^v. With this notation, we have

$$w^{\mathfrak{L}}(\pi) = \prod_{i=1}^{k-j} w^{\mathfrak{L}}(\pi_i^v) \prod_{i=1}^{n-j} w^{\mathfrak{L}}(\pi_i^h) \prod_{i=1}^{j} w^{\mathfrak{L}}(\pi_i^d).$$

We rewrite (2.4) as

$$w^{\mathfrak{L}}(\pi) = \left(\prod_{i=1}^{n-j} w^{\mathfrak{L}_1}(\alpha_i^h) \prod_{i=1}^{j} w^{\mathfrak{L}_1}(\alpha_i^d) \right) \left(\prod_{i=1}^{n} w^{\mathfrak{L}_2}(\beta_i^h) \prod_{i=1}^{k-j} w^{\mathfrak{L}_2}(\beta_i^v) \right).$$

The proof is divided into three parts.

(a) First, we show that $\prod_{i=1}^{k-j} w^{\mathfrak{L}}(\pi_i^v) = \prod_{i=1}^{k-j} w^{\mathfrak{L}_2}(\beta_i^v)$. To see this, observe that the weight function $w^{\mathfrak{L}}$ on vertical segments depends only on the sequence $\{v_n\}_{n\ge 0}$. For $i = 1,2,\ldots,k-j$, denote $x_i = \mathtt{x}(\pi_i^v)$. By the definition of $w^{\mathfrak{L}}$, we see that

$$\prod_{i=1}^{k-j} w^{\mathfrak{L}}(\pi_i^v) = v_{x_1} v_{x_2} \cdots v_{x_{k-j}}.$$

Let us observe that the corresponding vertical segments β^v of the path β and π^v of π share a common value of x, i.e., $\mathtt{x}(\pi_i^v) = \mathtt{x}(\beta_i^v)$ for $i = 1,2,\ldots,k-j$. Indeed, operations up and dg do not change the first coordinate of the segments' points.

Let us divide $\beta^h = (\beta^h_1, \ldots, \beta^h_n)$ into two sequences $B^d = (b^d_1, \ldots, b^d_j)$ and $B^h = (b^h_1, \ldots, b^h_{n-j})$ as follows. For $i = 1, 2, \ldots, n$, let $\alpha_l \in \alpha$, be a segment such that $\alpha_l \sim \beta^h_i$, if α_l is diagonal then $\beta^h_i \in B^d$, and $\beta^h_i \in B^h$ otherwise.

(b) Next, we show $\prod_{i=1}^{n-j} w^{\mathfrak{L}}(\pi^h_i) = \prod_{i=1}^{n-j} w^{\mathfrak{L}_1}(\alpha^h_i)w^{\mathfrak{L}_2}(b^h_i)$. For $i = 1, 2, \ldots, n-j$, denote $x_i = \mathtt{x}(\pi^h_i)$ and $y_i = \mathtt{y}(\pi^h_i)$. By definition, we see that

$$\prod_{i=1}^{n-j} w^{\mathfrak{L}}(\pi^h_i) = h_{x_1} \cdots h_{x_{n-j}} \zeta^{y_1 + \cdots + y_{n-j}}. \tag{2.5}$$

Let us denote $y'_i = \mathtt{y}(\alpha^h_i)$, $y''_i = \mathtt{y}(b^h_i)$, and observe that for $i = 1, \ldots, n-j$, we have $x_i = \mathtt{x}(\pi^h_i) = \mathtt{x}(\alpha^h_i) = \mathtt{x}(b^h_i)$. Thus by the definitions of $w^{\mathfrak{L}_1}$ and $w^{\mathfrak{L}_2}$, we have

$$\begin{aligned}\prod_{i=1}^{n-j} w^{\mathfrak{L}_1}(\alpha^h_i)w^{\mathfrak{L}_2}(b^h_i) &= \left(h_{x_1} \cdots h_{x_{n-j}} \zeta^{y'_1 + \cdots y'_{n-j}}\right)\left(\zeta^{y''_1 + \cdots + y''_{n-j}}\right) \\ &= h_{x_1} \cdots h_{x_{n-j}} \zeta^{y'_1 + y''_1 + \cdots + y'_{n-j} + y''_{n-j}}.\end{aligned}$$

Now, observe that $\pi^h_i \sim \alpha^h_i \sim b^h_i$ for $i = 1, \ldots, n-j$. Thus we apply Lemma 3 to the values of $\mathtt{y}$ for these segments to get $y_i = y'_i + y''_i$, for $i = 1, 2, \ldots, n-j$. Thus

$$\prod_{i=1}^{n-j} w^{\mathfrak{L}_1}(\alpha^h_i)w^{\mathfrak{L}_2}(b^h_i) = h_{x_1} \cdots h_{x_{n-j}} \zeta^{y_1 + \cdots + y_{n-j}},$$

which is exactly (2.5).

(c) In the same manner we can show the equivalent formula for diagonal segments, i.e., $\prod_{i=1}^{j} w^{\mathfrak{L}}(\pi^d_i) = \prod_{i=1}^{j} w^{\mathfrak{L}_1}(\alpha^d_i)w^{\mathfrak{L}_2}(b^d_i)$. If we set $x_i = \mathtt{x}(\pi^d_i)$ and $y_i = \mathtt{y}(\pi^d_i)$, then the left-hand side of the above is equal to

$$\prod_{i=1}^{j} w^{\mathfrak{L}}(\pi^d_i) = d_{x_1} \cdots d_{x_j} \zeta^{y_1 + \cdots + y_j}.$$

The rest of the proof of (c) goes much the same way as in (b).

What is left is to observe that the product of the left-hand sides of (a), (b), and (c) is equal to

$$w^{\mathfrak{L}}(\pi) = \prod_{i=1}^{k-j} w^{\mathfrak{L}}(\pi^v_i) \prod_{i=1}^{n-j} w^{\mathfrak{L}}(\pi^h_i) \prod_{i=1}^{j} w^{\mathfrak{L}}(\pi^d_i),$$

and the product of the right-hand sides of (a), (b), and (c) is equal to

$$w^{\mathfrak{L}_1}(\alpha)w^{\mathfrak{L}_2}(\beta) = \prod_{i=1}^{k-j} w^{\mathfrak{L}_2}(\beta^v_i) \prod_{i=1}^{n-j} w^{\mathfrak{L}_1}(\alpha^h_i)w^{\mathfrak{L}_2}(b^h_i) \prod_{i=1}^{j} w^{\mathfrak{L}_1}(\alpha^d_i)w^{\mathfrak{L}_2}(b^d_i),$$

which is equivalent to $w^{\mathfrak{L}}(\pi) = w^{\mathfrak{L}_1}(\alpha)w^{\mathfrak{L}_2}(\beta)$. $\square$

Proposition 5 (Vertical summation). *For $n, k \in \mathbb{N}$, we have*

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{\mathfrak{L}} = (v_n h_n + d_n) \sum_{i=0}^{k-1} \left\langle \begin{matrix} n-1 \\ i \end{matrix} \right\rangle_{\mathfrak{L}} v_n^{k-i-1} \zeta^i + h_n \zeta^k \left\langle \begin{matrix} n-1 \\ k \end{matrix} \right\rangle_{\mathfrak{L}}. \tag{2.6}$$

Proof. Let us consider the left-hand side of the equation, and observe that the family $L(n,k)$ can be partitioned into classes $A_0, \ldots, A_k$, such that A_i contains paths which have the diagonal segment $\sigma_i^d = ((n-1,i),(n,i+1))$ or horizontal segment $\sigma_i^h = ((n-1,i),(n,i))$.

(a) Let us consider a path π from the class A_k. Note that the last segment of π is horizontal $\sigma_k^h = ((n-1,k),(n,k))$. Thus the weight of π is equal to the product of its segments' weights from $(0,0)$ to $(n-1,k)$ times the weight of the segment σ_k^h. Therefore, by definition, we have

$$\sum_{\pi \in A_k} w^{\mathfrak{L}}(\pi) = h_n \zeta^k \left\langle \begin{matrix} n-1 \\ k \end{matrix} \right\rangle_{\mathfrak{L}}.$$

(b) For $i = 0, 1, \ldots, k-1$, let π be a path of A_i which contains σ_i^d. Let $\pi = (\pi_1, \ldots, \pi_l, \sigma_i^d, \pi_{l+2}, \ldots, \pi_s)$. The weight of π is equal to the product

$$w^{\mathfrak{L}}(\pi) = w^{\mathfrak{L}}(\pi_1) \cdots w^{\mathfrak{L}}(\pi_l) w^{\mathfrak{L}}(\sigma_i^d) w^{\mathfrak{L}}(\pi_{l+2}) \cdots w^{\mathfrak{L}}(\pi_s).$$

All the remaining segments $\pi_{l+2}, \ldots, \pi_s$ are vertical, and their number equals $(k-i-1)$. Thus, by definition, we have

$$w^{\mathfrak{L}}(\pi) = \left\langle \begin{matrix} n-1 \\ i \end{matrix} \right\rangle_{\mathfrak{L}} w^{\mathfrak{L}}(\sigma_i^d) v_n^{k-i-1}.$$

(c) In the same manner we compute the weight of a path π from the class A_i which contains the horizontal segment σ_i^h, where $i = 0, 1, \ldots, k-1$. We need to observe that the number of vertical segments in this case is equal to $(k-i)$, and thus

$$w^{\mathfrak{L}}(\pi) = \left\langle \begin{matrix} n-1 \\ i \end{matrix} \right\rangle_{\mathfrak{L}} w^{\mathfrak{L}}(\sigma_i^h) v_n^{k-i}.$$

The rest of the proof is straightforward. □

Proposition 6 (Horizontal summation). *For $n, k \in \mathbb{N}$, we have*

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{\mathfrak{L}} = \sum_{i=0}^{n-1} H_{i+2,n} \left\langle \begin{matrix} i \\ k-1 \end{matrix} \right\rangle_{\mathfrak{L}} \zeta^{k(n-i)-1} \left(d_{i+1} + v_i h_{i+1} \zeta\right) + v_n \left\langle \begin{matrix} n \\ k-1 \end{matrix} \right\rangle_{\mathfrak{L}}, \tag{2.7}$$

where $H_{i,n} = h_i h_{i+1} \cdots h_n$, and $H_{i,n} = 1$ for $i > n$.

Proof. We only outline the proof. We partition the set of paths $L(n,k)$ into $A_0, A_1, \ldots, A_n$ defined as follows. Take a path π from $L(n,k)$. If π contains the vertical segment $\sigma_i^v = ((i,k-1),(i,k))$ or the diagonal segment $\sigma_i^d = ((i,k-1),(i+1,k))$, then $\pi \in A_i$ for $i = 0,1,\ldots,n$. Note that the class A_n contains paths whose last segment is vertical σ_k^v, and $w^{\mathfrak{L}}(\sigma_k^v) = v_n$. The union of $A_0,\ldots,A_n$ is the whole $L(n,k)$. Therefore,

$$\left\langle {n \atop k} \right\rangle_{\mathfrak{L}} = \sum_{i=0}^{n-1} \sum_{\pi \in A_i} w^{\mathfrak{L}}(\pi) + \sum_{\pi \in A_n} w^{\mathfrak{L}}(\pi).$$

The rest of the proof can be handled in much the same way as the proof of Proposition 5. $\square$

3. Generating Functions

We denote by $\mathcal{P}_k(n)$ the family of subsets $X \subseteq \{1,2,\ldots,n\}$ such that $|X| = k$. If $\pi \in \mathcal{P}_k(n)$ then we denote by $\overline{\pi}$ the set $\{1,2,\ldots,n\} \setminus \pi$. We denote by $\mathcal{P}(n)$ the union $\mathcal{P}_0(n) \cup \mathcal{P}_1(n) \cup \cdots \cup \mathcal{P}_n(n)$. For $n \geq 0$, let

$$F_n(z) = \sum_{k \geq 0} \left\langle {n \atop k} \right\rangle_{\mathfrak{L}} z^k. \tag{3.1}$$

Theorem 7. *$F_0(z) = 1/(1 - v_0 z)$, and for $n = 1,2,\ldots,$ we have*

$$F_n(z) = \frac{1}{(1 - v_0 \zeta^n z)} \prod_{i=1}^{n} \frac{h_i + d_i \zeta^{n-i} z}{1 - v_i \zeta^{n-i} z}. \tag{3.2}$$

Proof. Using the recurrence relation we see that $F_n(z) = (h_n + d_n z)/(1 - v_n z) F_{n-1}(\zeta z)$. Solving this recurrence with initial value $F_0(z) = 1/(1 - v_0 z)$ we obtain the formula. $\square$

The corresponding generating functions for the $\mathfrak{L}$-Delannoy numbers of the first and second kind are denoted by $C_n(z)$ and $S_n(z)$, respectively. We have $C_0(z) = 1$,

$$C_n(z) = \sum_{k \geq 0} \left[{n \atop k} \right]_{\mathfrak{L}} z^k = \prod_{i=1}^{n} \left(h_i + d_i \zeta^{n-i} z \right), \qquad \text{for } n \geq 1, \tag{3.3a}$$

$$S_n(z) = \sum_{k \geq 0} \left\{ {n \atop k} \right\}_{\mathfrak{L}} z^k = \prod_{i=0}^{n} \frac{1}{(1 - v_i \zeta^{n-i} z)}, \qquad \text{for } n \geq 0. \tag{3.3b}$$

Note that there is another, very simple proof of Theorem 1 using the generating functions. Indeed, for every $n \geq 0$, we have $F_n(z) = C_n(z) S_n(z)$.

Proposition 8 (Symmetric-like functions). *For $n, k \geq 0$, we have*

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\mathfrak{L}} = \sum_{\pi \in \mathcal{P}_k(n)} \prod_{i=1}^{k} d_{\pi_i} \zeta^{n-\pi_i} \prod_{i=1}^{n-k} h_{\overline{\pi}_i}, \tag{3.4a}$$

$$\begin{Bmatrix} n \\ k \end{Bmatrix}_{\mathfrak{L}} = \sum_{0 \leq \sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_k \leq n} \prod_{i=1}^{k} v_{\sigma_i} \zeta^{n-\sigma_i}, \tag{3.4b}$$

where $\pi = \{\pi_1, \ldots, \pi_k\}$, and $\overline{\pi} = \{\overline{\pi}_1, \ldots, \overline{\pi}_{n-k}\}$.

Proof. From (3.3a) and (3.3b), we obtain (3.4a) and (3.4b), respectively. □

Corollary 9. *For $n, k \geq 0$, we have*

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{\mathfrak{L}} = \sum_{j=0}^{k} \sum_{\substack{1 \leq \pi_1 < \cdots < \pi_j \leq n \\ 0 \leq \sigma_1 \leq \cdots \leq \sigma_{k-j} \leq n}} \zeta^{kn-S} \prod_{i=1}^{j} d_{\pi_i} \prod_{i=1}^{n-j} h_{\overline{\pi}_i} \prod_{i=1}^{k-j} v_{\sigma_i}, \tag{3.5}$$

where $S = \pi_1 + \cdots + \pi_j + \sigma_1 + \cdots + \sigma_{k-j}$.

Proof. Combining Proposition 8 with (2.3) we obtain the formula. □

Theorem 10 (Generalization of the Carlitz formula). *Let $\mathfrak{H} = \langle \mathbf{v}, \mathbf{1}, \mathbf{d}, \zeta \rangle$ and $\mathfrak{H}^* = \langle \mathbf{v}^*, \mathbf{1}, \mathbf{d}^*, \zeta \rangle$ such that $v_i = v_i^* + \alpha_1 \zeta^i$ and $d_i = d_i^* + \alpha_2 \zeta^i$, where $\alpha_1, \alpha_2 \in \mathbb{C}$. Then*

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\mathfrak{H}} = \sum_{j=0}^{k} \binom{n-k+j}{j} (\alpha_2 \zeta^n)^j \begin{bmatrix} n \\ k-j \end{bmatrix}_{\mathfrak{H}^*}, \tag{3.6a}$$

$$\begin{Bmatrix} n \\ k \end{Bmatrix}_{\mathfrak{H}} = \sum_{j=0}^{k} \binom{n+k}{j} (\alpha_1 \zeta^n)^j \begin{Bmatrix} n \\ k-j \end{Bmatrix}_{\mathfrak{H}^*}. \tag{3.6b}$$

Proof. (a) Substituting $d_i = d_i^* + \alpha_2 \zeta^i$ and $h_i = 1$ to (3.4a) we get

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\mathfrak{H}} = \sum_{1 \leq \pi_1 < \cdots < \pi_k \leq n} \prod_{i=1}^{k} (d_{\pi_i}^* \zeta^{n-\pi_i} + \alpha_2 \zeta^n). \tag{3.7}$$

Let us consider a power series of the product $\prod_{i=1}^{k} (d_{\pi_i}^* \zeta^{n-\pi_i} + \alpha_2 \zeta^n)$ in the parameter $\alpha_2 \zeta^n$. That is,

$$\sum_{j=0}^{k} \left(\sum_{1 \leq s_1 < \cdots < s_{k-j} \leq k} \prod_{i=1}^{k-j} d_{\gamma_i}^* \zeta^{n-\gamma_i} \right) (\alpha_2 \zeta^n)^j,$$

where $\gamma_i = \pi_{s_i}$ for $i = 1, 2, \ldots, k-j$. What is left is to combine the above with (3.7), and prove that

$$\sum_{\substack{1 \le \pi_1 < \cdots < \pi_k \le n \\ 1 \le s_1 < \cdots < s_{k-j} \le k}} \prod_{i=1}^{k-j} d^*_{\gamma_i} \zeta^{n-\gamma_i} = \binom{n-k+j}{j} \sum_{1 \le \nu_1 < \cdots < \nu_{k-j} \le n} \prod_{i=1}^{k-j} d^*_{\nu_i} \zeta^{n-\nu_i}.$$

Indeed, $k-j$ variables $\gamma_i = \pi_{s_i}$ take values over the set $\{1, \ldots, n\}$, and the remaining j variables π_l of $\pi_1, \ldots, \pi_k$ we can choose from the set $[n] \setminus \{\gamma_1, \ldots, \gamma_{k-j}\}$ in $\binom{n-(k-j)}{j}$ ways. By definition, the right-hand side of the above is the value of the $\mathfrak{H}^*$-Delannoy number of the first kind.

(b) Substituting $v_i = v^*_i + \alpha_1 \zeta^i$ to (3.4b) we obtain

$$\begin{aligned}
\left\{ {n \atop k} \right\}_{\mathfrak{L}} &= \sum_{0 \le \sigma_1 \le \cdots \le \sigma_k \le n} \prod_{i=1}^{k} (v^*_{\sigma_i} \zeta^{n-\sigma_i} + \alpha_1 \zeta^n) \\
&= \sum_{0 \le \sigma_1 \le \cdots \le \sigma_k \le n} \sum_{j=0}^{k} \left(\sum_{1 \le s_1 < \cdots < s_{j-k} \le k} \prod_{i=1}^{k-j} v^*_{\sigma_{s_i}} \zeta^{n-\sigma_{s_i}} \right) (a_2 \zeta^n)^j \\
&= \sum_{j=0}^{k} \left(\sum_{\substack{0 \le \sigma_1 \le \cdots \le \sigma_k \le n \\ 1 \le s_1 < \cdots < s_{j-k} \le k}} \prod_{i=1}^{k-j} v^*_{\sigma_{s_i}} \zeta^{n-\sigma_{s_i}} \right) (a_2 \zeta^n)^j.
\end{aligned}$$

What is left is to show

$$\sum_{\substack{0 \le \sigma_1 \le \cdots \le \sigma_k \le n \\ 1 \le s_1 < \cdots < s_{k-j} \le k}} \prod_{i=1}^{k-j} v^*_{\sigma_{s_i}} \zeta^{n-\sigma_{s_i}} = \binom{n+k}{j} \sum_{0 \le \nu_1 \le \cdots \le \nu_{k-j} \le n} \prod_{i=1}^{k-j} v^*_{\nu_i} \zeta^{n-\nu_i}.$$

First, observe that $(k-j)$ variables $\nu_1 = \sigma_{s_1}, \ldots, \nu_{k-j} = \sigma_{s_{k-j}}$ take values over the set $\{0, 1, \ldots, n\}$ with repetition allowed. The remaining j variables σ_l variables lie between these ν_i and depend on values $\nu_1, \nu_2 - \nu_1, \ldots, \nu_{k-j} - \nu_{k-j-1}, n - \nu_{k-j}$. Denote by $A = A(n, k, j, \nu_1, \ldots, \nu_{k-j})$ the number of ways to select these remaining ν_l, where $l \ne s_i$ for $i = 1, 2, \ldots, k-j$. We have

$$A = \sum_{\substack{i_1 + \cdots + i_{k-j+1} = j \\ i_1, \ldots, i_{k-j+1} \ge 0}} \binom{\nu_1 + i_1}{i_1} \binom{\nu_2 - \nu_1 + i_2}{i_2} \cdots \binom{n - \nu_{k-j} + i_{k-j+1}}{i_{k-j+1}}.$$

Using standard methods of generating functions [29] we obtain

$$A = [z^j] \frac{1}{(1-z)^{\nu_1+1}} \frac{1}{(1-z)^{\nu_2-\nu_1+1}} \cdots \frac{1}{(1-z)^{\nu_{k-j}-\nu_{k-j-1}+1}} \frac{1}{(1-z)^{n-\nu_{k-j}+1}}.$$

Since there are $k-j+1$ terms we have

$$A = [z^j]\frac{1}{(1-z)^{n+k-j+1}} = \binom{n+k}{j}.$$

which completes the proof. $\square$

Note that the generalization of Carlitz formulae for the generalized Stirling numbers delivered by Médicis and Leroux [22, Th.2.2] is a special case of the above with $\zeta = 1$ and $\alpha_2 = w_0$ (see Section 8.6 for more details).

Corollary 11. *Let $\mathfrak{I} = \langle \mathbf{v}, \mathbf{1}, \mathbf{v}^*, \zeta\rangle$, where $\mathbf{v} = (v_0, v_1, \ldots)$ and $\mathbf{v}^* = (0, v_0, v_1, \ldots)$. Then for all $n, k \in \mathbb{N}$, we have*

$$\sum_{i=0}^{k} (-1)^i \begin{bmatrix} n \\ i \end{bmatrix}_{\mathfrak{I}} \begin{Bmatrix} n-1 \\ k-i \end{Bmatrix}_{\mathfrak{I}} = 0, \tag{3.8a}$$

$$\sum_{i=0}^{k} (-1)^{k-i} \begin{bmatrix} n \\ k-i \end{bmatrix}_{\mathfrak{I}} \begin{Bmatrix} n-1 \\ i \end{Bmatrix}_{\mathfrak{I}} = 0. \tag{3.8b}$$

Proof. We need to observe that for every integer $n \geq 1$, the generating functions $C_n(z)$ and $S_n(z)$ of the $\mathfrak{I}$-Delannoy numbers of the first and second kind satisfy $C_n(-z)S_{n-1}(z) \equiv 1$. $\square$

Proposition 12. *Let $n, k \geq 0$. If for every $0 \leq i, j \leq n$ such that $i \neq j$ we have $v_i \neq v_j \zeta^{i-j}$, then*

$$\left\langle {n \atop k} \right\rangle_{\mathfrak{L}} = \sum_{i=0}^{n} \frac{\prod_{j=1}^{n}(h_j v_i + d_j \zeta^{i-j})}{\prod_{\substack{j=0 \\ j\neq i}}^{n}(v_i - v_j\zeta^{i-j})} \left(v_i \zeta^{n-i}\right)^k. \tag{3.9}$$

Proof. Using the partial fraction decomposition of the generating function (3.2) we obtain $F_n(z) = \sum_{i=0}^{n} W_i/(1 - v_i\zeta^{n-i}z)$. To find the coefficients $W_0, W_1, \ldots, W_n$ we multiply both sides by $\prod_{i=0}^{n}(1 - v_i\zeta^{n-i}z)$. For $s = 0, 1, \ldots, n$, we evaluate it with $z = (v_s\zeta^{n-s})^{-1}$ to obtain

$$\prod_{j=1}^{n}(h_j + \frac{d_j}{v_s}\zeta^{s-j}) = W_s \prod_{\substack{j=0 \\ j\neq s}}^{n}(1 - \frac{v_j}{v_s}\zeta^{s-j}).$$

The rest of the proof is straightforward. $\square$

Theorem 13. *Let $n, k \geq 0$, if $h_i \neq 0$ for $i \geq 1$, then*

$$\left\langle {n \atop k+1} \right\rangle_{\mathfrak{L}} = \frac{1}{k+1} \sum_{i=0}^{k} \left\langle {n \atop i} \right\rangle_{\mathfrak{L}} \sigma_n(k-i), \tag{3.10}$$

where

$$\sigma_n(i) = \sum_{j=0}^{n} \left(v_j \zeta^{n-j}\right)^{i+1} + (-1)^i \sum_{j=1}^{n} \left(\frac{d_j}{h_j}\zeta^{n-j}\right)^{i+1}. \tag{3.11}$$

Proof. Let us consider the following equality $\partial/\partial z \log(F_n(z)) = 1/F_n(z)\partial/\partial z F_n(z)$. We insert the power series expansion of $F_n(z)$ and $\partial/\partial z F_n(z)$ to get

$$\frac{\partial}{\partial z} \log\left\{\frac{1}{1-v_0\zeta^n z}\prod_{i=1}^{n}\frac{h_i + d_i\zeta^{n-i}z}{1-v_i\zeta^{n-i}z}\right\} F_n(z) = \sum_{k\geq 0}(k+1)\left\langle {n \atop k+1} \right\rangle_{\mathfrak{L}} z^k.$$

Using the Cauchy product of the series in the left-hand side and equating coefficients of like powers of z we obtain the formula. □

Applying the above to the $\mathfrak{B}$-Delannoy numbers we obtain a recurrence relation for $D(n,k)$, that is,

$$D(n,k) = \frac{1}{k}\sum_{i=0}^{k-1} D(n,i)\left(1+n+(-1)^{k-i-1}n\right).$$

4. Weighted Central Delannoy Numbers

Let $\mathfrak{W} = \langle \mathbf{v}^c, \mathbf{h}^c, \mathbf{d}^c, 1\rangle$, where $\mathbf{v}^c = (v, v, \ldots), \mathbf{h}^c = (h, h, \ldots), \mathbf{d}^c = (d, d, \ldots)$ are constant sequences of $v, h, d \in \mathbb{C}$. The generalized Delannoy numbers reduce to the so-called unrestricted weighted lattice paths studied, among others, by Fray and Roselle [13]. They showed, for instance, that

$$\sum_{n\geq 0}\left\langle {n \atop k} \right\rangle_{\mathfrak{W}} z^n = \frac{(v+dz)^k}{(1-hz)^{k+1}} = \sum_{n\geq 0}\sum_{r=0}^{n} h^{n-r}v^{k-r}d^r\binom{k}{r}\binom{n+k-r}{k}z^n.$$

The central weighted Delannoy numbers were studied by Hetyei [17] who showed that the total weight of all Delannoy paths may be expressed by substitution into a shifted Jacobi polynomial with the appropriate parameters. Namely,

$$\left\langle {n+\beta \atop n} \right\rangle_{\mathfrak{W}} = h^\beta(-d)^n \widetilde{P}_n^{(0,\beta)}\left(-\frac{hv}{d}\right),$$

where $\widetilde{P}_n^{(0,\beta)}(x)$ is a shifted Legendre polynomial defined with the help of nth Jacobi polynomial $P_n^{(0,\beta)}$ of type $(0,\beta)$, i.e., $\widetilde{P}_n^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(2x-1)$. We refer the reader also to Sulanke [26].

Let us define auxiliary numbers. We denote by $S(n)$ the sum of weights of all Delannoy paths from the origin to (n,n) that do not go above the line $y = x$. For

$v = h = d = 1$ we obtain the sequence of the so-called *large Schroeder numbers* [9]. The sequence

$$\{S(n)\}_{n\geq 0} = (1, 2, 6, 21, 77, 286, 1066, 3977, 14841, \ldots)$$

is denoted by A006318 in OEIS [23].

Proposition 14. *$S(0) = 1$, and for $n = 1, 2, \ldots$, we have*

$$S(n) = d\, S(n-1) + h\, v \sum_{i=0}^{n-1} S(i) S(n-1-i). \tag{4.1}$$

Proof. We partition the family of paths from $(0, 0)$ to (n, n) that do not go above the line $y = x$ into two classes A and B, such that the last step of paths from A is diagonal, and the last one from B is vertical. The weighted sum of paths in A is $S(n-1)$. To calculate the size of B we partition B into pairwise disjoint classes $B_0, B_1, \ldots, B_{n-1}$ such that B_i contains these paths of B such that the last horizontal step $a_l = ((l, l), (l + 1, l)$ of π is a_i, for $0 \leq l \leq n-1$. It is clear that the size of B_i is $S(i)S(n-1-i)$. □

Let $G(z) = \sum_{n\geq 0} S(n)z^n$. From (4.1), the function $G(z)$ satisfies the following functional equation

$$G(z) = 1 + d\, zG(z) + h\, v\, zG(z)^2, \tag{4.2}$$

which yields

$$G(z) = \frac{1 - d\, z - \sqrt{d^2\, z^2 - z(2\, d + 4\, h\, v) + 1}}{2\, h\, v\, z}. \tag{4.3}$$

We take the negative square root due to the initial condition $[z^0]G(z) = 1$.

Theorem 15. *For $t \geq 1$ we have $[z^0]G(z)^t = 1$, and for $n = 1, 2, \ldots$, we have*

$$[z^n]G(z)^t = \sum_{l=0}^{t} \sum_{k=0}^{n} \sum_{i=0}^{n-l-2k} \sum_{j=0}^{k+l} \frac{l}{n} \binom{t}{l} \binom{n}{k} \binom{n-k}{l+k} \binom{n-l-2k}{i} \binom{k+l}{j} \cdot d^{i+j} 2^{n-l-2k-i} (hv)^{n-i-j}. \tag{4.4}$$

Proof. The function $G(z)$ satisfies the functional equation (4.2). Letting $u = G(z) - 1$, we obtain $u = z\phi(u)$, where $\phi(u) = d + h\, v + u(d + 2\, h\, v) + h\, v\, u^2$. From the Lagrange Inversion Formula [24, Th. 5.4.2] we obtain

$$\begin{aligned}
[z^n]u(z)^t &= \frac{1}{n}[u^{n-1}] \left\{ t u^{t-1} \phi(u)^n \right\} \\
&= \frac{t}{n}[u^{n-t}] \left\{ (d + h\, v + u(d + 2\, h\, v) + h\, v\, u^2)^n \right\} \\
&= \frac{t}{n} \sum_{k=0}^{n} \sum_{i=0}^{n-t-2k} \sum_{j=0}^{k+t} \binom{n}{k} \binom{n-k}{n-t-2k} \binom{n-t-2k}{i} \binom{k+t}{j} \\
&\quad \cdot d^{i+j} 2^{n-t-2k-i} (hv)^{n-i-j}.
\end{aligned}$$

Finally, to get the series expansion of $G(z)^t = (u+1)^t$ we use the binomial theorem,

$$[z^n]G(z)^t = \sum_{l=0}^{t} \binom{t}{l} [z^n]u(z)^l$$

and the formula follows. $\square$

Proposition 16. *For $n \geq 1$, we have*

$$\left\langle \begin{matrix} n \\ n \end{matrix} \right\rangle_{\mathfrak{W}} = \sum_{k=1}^{n} \sum_{\substack{i_1+\cdots+i_k=n \\ i_1,\ldots,i_k \geq 1}} \prod_{j=1}^{k} \left(2\,h\,v\,S(i_j-1) + d\,\delta_{i_j,1}\right), \tag{4.5}$$

where $\delta_{a,b} = 1$ when $a = b$, and zero otherwise.

Proof. Let us partition $L(n,n)$ into n subclasses $A_1, A_2, \ldots, A_n$ such that paths from A_k contains exactly k lattice points from the set $\{(1,1), \ldots, (n,n)\}$. We call (j,j) the jth *diagonal point*. Let us consider the subclass A_k. These k diagonal points (j,j) can be determined by the solution of the equation $i_1 + \cdots + i_k = n$, where $i_1, \ldots, i_k \geq 1$. Indeed, the first diagonal point is (i_1, i_1), the second one is $(i_1 + i_2, i_1 + i_2)$, and so on. For the solution $i_1, \ldots, i_k$, the number of paths that contains corresponding k diagonal points is the product of the numbers $g_1, \ldots, g_k$ of weighted Delannoy paths π from $(j-1)$th diagonal point to the jth diagonal point, for $j = 1, 2, \ldots, k$, and π does not contain any other diagonal point between these fixed ones. For the convenience, we assume that the 0th diagonal point is $(0,0)$. To calculate g_j we observe that for every such path π we have three cases: (a) π does not go above the line $y = x$, (b) π does not go below the line $y = x$, and (c) π is a single diagonal step. Simple verification shows that $g_j = 2\,v\,h\,S(i_j - 1) + d\,\delta_{i_j,1}$. $\square$

Theorem 17. *We have*

$$F(z) = \sum_{n \geq 0} \left\langle \begin{matrix} n \\ n \end{matrix} \right\rangle_{\mathfrak{W}} z^n = \frac{1}{\sqrt{d^2\,z^2 - z(2\,d + 4\,h\,v) + 1}}. \tag{4.6}$$

Proof. Let us denote by $g(z) = \sum_{n\geq 1}(2\,h\,v\,S(n-1) + d\,\delta_{n,1})z^n$ which implies $g(z) = h\,v\,2G(z)z + d\,z$. Using the generating function (4.3) of the numbers $S(n)$ we get explicit form of $g(z) = 1 - \sqrt{d^2z^2 - z(2\,d + 4\,h\,v) + 1}$. On the other hand, by (4.5) we obtain $[z^n]F(z) = [z^n]\sum_{k=1}^{n} g(z)^k = [z^n]\sum_{k\geq 1} g(z)^k$. The last equality is due to $g(0) = 0$. The value of the central $\mathfrak{W}$-Delannoy number is one for $n = 0$, thus we can change the summation range from $k \geq 1$ to $k \geq 0$. Therefore, $F(z) = 1/(1 - g(z))$ which simplifying gives the formula. $\square$

Corollary 18. *For every $n \geq 0$, we have*

$$\left\langle \begin{matrix} n \\ n \end{matrix} \right\rangle_{\mathfrak{W}} = \sum_{k=0}^{n} \sum_{s=0}^{k} \binom{k}{s} h^s v^s 2^s d^{k-s} [z^{n-k}]G(z)^s.$$

Proof. By Theorem 17 we see that

$$\left\langle \begin{matrix} n \\ n \end{matrix} \right\rangle_{\mathfrak{W}} = \sum_{k=0}^{n} [z^n](h\,v\,2\,G(z)z + d\,z)^k = \sum_{k=0}^{n}\sum_{s=0}^{k}\binom{k}{s} h^s v^s 2^s d^{k-s}[z^{n-k}]G(z)^s.$$

We use the binomial theorem and the explicit formula for $[z^n]G(z)^s$ from Theorem 15. □

5. Counting Lattice Paths with Restrictions

Let us define three classes of segments: vertical $S^v(k)$, horizontal $S^h(k)$ and diagonal $S^d(k)$, as follows

$$\begin{aligned} S^v(k) &= \{((k,i),(k,i+1)) : i = 0,1,\dots\}, && \text{for } k \geq 0,\\ S^h(k) &= \{((k-1,i),(k,i)) : i = 0,1,\dots\}, && \text{for } k \geq 1,\\ S^d(k) &= \{((k-1,i),(k,i+1)) : i = 0,1,\dots\}, && \text{for } k \geq 1. \end{aligned}$$

Let $S^v = \bigcup_{k\geq 0} S^v(k)$, we denote S^h and S^d analogously. With this notation, the family $\mathcal{S}$ of all lattice segments (vertical, horizontal, and diagonal), is $S^v \cup S^h \cup S^d$. Let $H \subseteq \mathcal{S}$, we denote by $L(n,k;H)$ the family of all lattice paths from $(0,0)$ to (n,k) consisting of steps from H. Let $D(n,k;H) = |L(n,k;H)|$, and let

$$G_n(z;H) = \sum_{k\geq 0} D(n,k;H)z^k. \tag{5.1}$$

For the case $H = \mathcal{S}$ we have $L(n,k) = L(n,k;\mathcal{S})$ and $D(n,k) = D(n,k;\mathcal{S})$, and thus $G_n(z;\mathcal{S}) = (1+z)^n/(1-z)^{n+1}$. We are interested in finding the numbers $D(n,k;H)$ for sets H which can be obtained from the whole $\mathcal{S}$ by removing certain classes of segments S^v, S^h, and S^d.

Proposition 19. *If $r \geq 1$, then for every $k \geq 0$ and $n \geq r$, we have*

$$S^h(r) \cap H = \emptyset \quad \text{and} \quad S^d(r) \cap H = \emptyset \quad \text{implies} \quad L(n,k;H) = \emptyset. \tag{5.2}$$

Fix $n \geq 1$. Let H' be the set of lattice segments obtained from $\mathcal{S}$ by removing r horizontal classes of segments $S^h(i_1), \dots, S^h(i_r)$, where $0 \leq r \leq n$. It is clear that we can remove at most $(n-r)$ diagonal classes of segments $S^d(j_1), \dots, S^d(j_{n-r})$ from the set H' to obtain a set H'' such that $L(n,k;H'') \neq \emptyset$. Indeed, the sets of indices $I = \{i_1, \dots, i_r\}$ and $J = \{j_1, \dots, j_{n-r}\}$ can not share a common element, i.e., $I \cap J = \emptyset$.

Let $[n] = \{1,2,\dots,n\}$ and $[n]_0 = \{0,1,\dots,n-1\}$. Let A be a set of non-negative integers. From now on we denote by $\mathcal{P}(A)$ the set of all subsets of A. Let $\sigma \in \mathcal{P}([n])$, we denote by $S^v[\sigma]$ the union

$$S^v[\sigma] = \bigcup_{\sigma_i \in \sigma} S^v(\sigma_i).$$

We denote $S^h[\sigma]$ and $S^d[\sigma]$ similarly.

Theorem 20. *For a fixed $n \geq 0$, let $H = \mathcal{S} \setminus (S^v[\eta] \cup S^h[\sigma] \cup S^d[\tau])$, where $\tau \in \mathcal{P}([n])$, $\sigma \in \mathcal{P}([n])$, and $\eta \in \mathcal{P}([n+1]_0)$ such that $\tau \cap \sigma = \emptyset$. Then*

$$G_n(z; H) = \frac{z^{|\sigma|}(1+z)^{n-|\sigma|-|\tau|}}{(1-z)^{n+1-|\eta|}}, \tag{5.3}$$

$$D(n, k; H) = \sum_{j=0}^{k} \binom{n-|\sigma|-|\tau|}{j-|\sigma|} \binom{n-|\eta|+k-j}{n-|\eta|}. \tag{5.4}$$

Proof. Let $\tau = \{\tau_1, \ldots, \tau_a\}$, $\sigma = \{\sigma_1, \ldots, \sigma_b\}$, and $\eta = \{\eta_1, \ldots, \eta_c\}$. Let us consider the generating function (3.2) for the $\mathfrak{R}$-Delannoy numbers, where the 4-tuple $\mathfrak{R} = \langle \mathbf{v}, \mathbf{h}, \mathbf{d}, \zeta \rangle$ is defined as follows. We set $\zeta = 1$, $h_0 = 0$, $d_0 = 0$, and

(a) for $i \in [n+1]_0$, if $i \in \eta$ then $v_i = 0$, and $v_i = 1$ otherwise, (vertical);
(b) for $i \in [n]$, if $i \in \sigma$ then $h_i = 0$, and $h_i = 1$ otherwise, (horizontal);
(c) for $i \in [n]$, if $i \in \tau$ then $d_i = 0$, and $d_i = 1$ otherwise, (diagonal).

On one hand, $v_i = 0$ ($d_i = 0$ and $h_i = 0$, respectively) means that the weight $w^{\mathfrak{R}}(\pi)$ of $\pi \in L(n, k)$ consisting of a segment from $S^v(i)$ ($S^d(i)$ and $S^h(i)$, respectively) is equal to zero. On the other hand, for every π consisting of segments from H the weight function $w^{\mathfrak{R}}(\pi)$ is equal to one. Thus the value of the corresponding $\mathfrak{R}$-Delannoy number is equal to the number of paths from $(0, 0)$ to (n, k) whose segments lie in H, i.e.,

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{\mathfrak{R}} = \sum_{\pi \in L(n,k)} w^{\mathfrak{R}}(\pi) = \sum_{\pi \in L(n,k;H)} 1 = D(n, k; H). \tag{5.5}$$

$\square$

Example 21. Let $H = \mathcal{S} \setminus (S^d(1) \cup S^d(n))$. Then

$$G_n(z; H) = \frac{(1+z)^{n-2}}{(1-z)^{n+1}}, \quad D(n, k; H) = \sum_{j=0}^{k} \binom{n-2}{j} \binom{n+k-j}{n}.$$

See Fig. 5 (a) for $n = 4$.

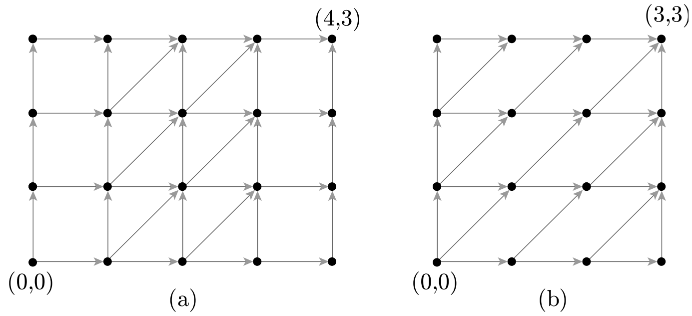

Figure 5: Lattices with removed segments.

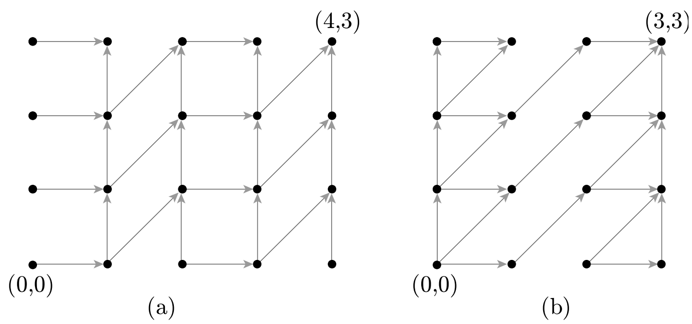

Figure 6: Lattices with removed segments.

Example 22. Let $H = \mathcal{S} \setminus (S^v(1) \cup S^v(2) \cup \cdots \cup S^v(n-1))$. Then

$$G_n(z; H) = \frac{(1+z)^n}{(1-z)^2}, \quad D(n, k; H) = \sum_{j=0}^{k} \binom{n}{j}(k-j+1).$$

See Fig. 5 (b) for $n = 3$.

Example 23. Let $H = (S^v \setminus S^v(0)) \cup B$, where $B = S^h(1) \cup S^d(2) \cup S^h(3) \cup S^d(4) \cup \cdots$. Then

$$G_n(z; H) = \frac{z^{\lfloor n/2 \rfloor}}{(1-z)^n}, \quad D(n, k; H) = \sum_{j=0}^{k} \binom{0}{j - \lfloor n/2 \rfloor} \binom{n-1+k-j}{n-1}.$$

See Fig. 6 (a) for $n = 4$.

Example 24. Let $H = S^d \cup S^v(0) \cup S^v(n) \cup B$, where $B = S^h(1) \cup S^h(3) \cup S^h(5) \cup \cdots$. Then

$$G_n(z; H) = \frac{z^{\lfloor n/2 \rfloor}(1+z)^{n-\lfloor n/2 \rfloor}}{(1-z)^2}, \quad D(n, k; H) = \sum_{j=0}^{k} \binom{n - \lfloor n/2 \rfloor}{n-j}(k-j+1).$$

See Fig. 6 (b) for $n = 3$.

6. A General Case of Generalized Delannoy Numbers

In previous sections we have been working under the assumption that lattice paths begin at $(0,0)$. To study the general case let x, y be any two lattice points, and let us denote by $L[x,y]$ the set of all Delannoy paths from x to y. Let

$$\left\langle {n \atop k} \right\rangle_{\mathfrak{L}}^{(m,l)} = \sum_{\pi \in L[x,y]} w^{\mathfrak{L}}(\pi), \tag{6.1}$$

where $x = (m,l)$ and $y = (m+n, l+k)$. Denote by $D^{(m,l)}(n,k)$ the cardinality of the family $L[(m,l),(m+n,l+k)]$. It is clear that $D^{(m,l)}(n,k) = D(n,k)$. For all $n, k, m, l \geq 0$, we have

$$\begin{aligned} \left\langle {n \atop k} \right\rangle_{\mathfrak{L}}^{(m,l)} &= v_{m+n} \left\langle {n \atop k-1} \right\rangle_{\mathfrak{L}}^{(m,l)} + h_{m+n}\, \zeta^{l+k} \left\langle {n-1 \atop k} \right\rangle_{\mathfrak{L}}^{(m,l)} \\ &\quad + d_{m+n}\, \zeta^{l+k-1} \left\langle {n-1 \atop k-1} \right\rangle_{\mathfrak{L}}^{(m,l)}. \end{aligned} \tag{6.2}$$

Likewise,

$$\begin{aligned} \left\langle {n \atop k} \right\rangle_{\mathfrak{L}}^{(m,l)} &= v_{m} \left\langle {n \atop k-1} \right\rangle_{\mathfrak{L}}^{(m,l+1)} + h_{m+1}\, \zeta^{l} \left\langle {n-1 \atop k} \right\rangle_{\mathfrak{L}}^{(m+1,l)} \\ &\quad + d_{m+1}\, \zeta^{l} \left\langle {n-1 \atop k-1} \right\rangle_{\mathfrak{L}}^{(m+1,l+1)} \end{aligned} \tag{6.3}$$

with $\left\langle {0 \atop 0} \right\rangle_{\mathfrak{L}}^{(m,l)} = 1$. For $n < 0$ or $k < 0$, we assume that $\left\langle {n \atop k} \right\rangle_{\mathfrak{L}}^{(m,l)} = 0$. See Fig. 7.

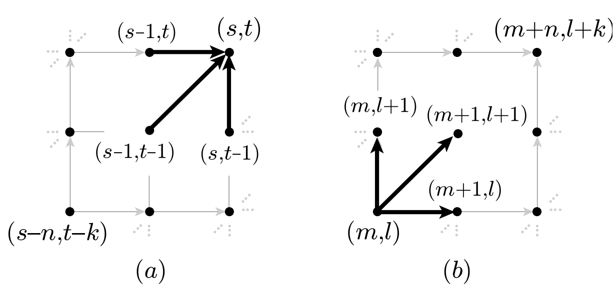

Figure 7: Recurrence relations for the general case.

Theorem 25. *For every $n \geq 0$, let*

$$F_n^{(m,l)}(z) = \sum_{k\geq 0} \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{\mathfrak{L}}^{(m,l)} z^k. \tag{6.4}$$

Then for every $n \geq 1$, we have

$$F_n^{(m,l)}(z) = \frac{\zeta^{nl}}{(1 - v_m \zeta^n z)} \prod_{i=1}^{n} \frac{h_{m+i} + d_{m+i}\zeta^{n-i} z}{1 - v_{m+i}\zeta^{n-i} z}, \tag{6.5}$$

and $F_0^{(m,l)}(z) = 1/(1 - v_m z)$.

Proof. Applying (6.2) to (6.4) we obtain

$$F_n^{(m,l)}(z) = \frac{(h_{m+n}\zeta^l + d_{m+n}\zeta^l z)}{1 - v_{m+n} z} F_{n-1}^{(m,l)}(\zeta z) \tag{6.6}$$

with $F_0^{(m,l)}(z) = 1/(1 - v_m z)$. Solving this recurrence relation yields (6.5). □

In a similar way we define generalized $\mathfrak{L}$-Delannoy numbers of the first and the second kind, respectively. For a given 4-tuple $\mathfrak{L} = \langle \mathbf{v}, \mathbf{h}, \mathbf{d}, \zeta \rangle$ we set $\mathfrak{L}_1 = \langle \mathbf{0}, \mathbf{h}, \mathbf{d}, \zeta \rangle$ and $\mathfrak{L}_2 = \langle \mathbf{v}, \mathbf{1}, \mathbf{0}, \zeta \rangle$. We have

$$\left[\begin{matrix} n \\ k \end{matrix} \right]_{\mathfrak{L}}^{(m,l)} = \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{\mathfrak{L}_1}^{(m,l)}, \qquad \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{\mathfrak{L}}^{(m,l)} = \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{\mathfrak{L}_2}^{(m,l)}.$$

Let

$$C_n^{(m,l)}(z) = \sum_{k\geq 0} \left[\begin{matrix} n \\ k \end{matrix} \right]_{\mathfrak{L}}^{(m,l)} z^k, \qquad S_n^{(m,l)}(z) = \sum_{k\geq 0} \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{\mathfrak{L}}^{(m,l)} z^k, \tag{6.7}$$

then $C_0(z) = 1$, and

$$C_n^{(m,l)}(z) = \zeta^{nl} \prod_{i=1}^{n} \left(h_{m+i} + d_{m+i}\zeta^{n-i} z \right), \qquad \text{for } n \geq 1, \tag{6.8a}$$

$$S_n^{(m,l)}(z) = \zeta^{nl} \prod_{i=0}^{n} \frac{1}{(1 - v_{m+i}\zeta^{n-i} z)}, \qquad \text{for } n \geq 0. \tag{6.8b}$$

Proposition 26 (Symmetric-like functions). *For all $n, k, m, l \geq 0$, we have*

$$\left[\begin{matrix} n \\ k \end{matrix} \right]_{\mathfrak{L}}^{(m,l)} = \zeta^{nl} \sum_{\pi \in \mathcal{P}_k(n)} \prod_{i=1}^{k} d_{m+\pi_i} \zeta^{n-\pi_i} \prod_{i=1}^{n-k} h_{m+\overline{\pi}_i}, \tag{6.9a}$$

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{\mathfrak{L}}^{(m,l)} = \zeta^{nl} \sum_{0\leq j_1 \leq j_2 \leq \cdots \leq j_k \leq n} \prod_{s=1}^{k} v_{m+j_s} \zeta^{n-j_s}, \tag{6.9b}$$

where $\pi = \{\pi_1, \ldots, \pi_k\}$ and $\overline{\pi} = \{\overline{\pi}_1, \ldots, \overline{\pi}_{n-k}\}$.

Proposition 27 (Convolution-like formulae). *For all $n, m, k \geq 1$, we have*

$$\begin{bmatrix} n+m \\ k \end{bmatrix}_{\mathfrak{L}} = \sum_{j=0}^{k} \begin{bmatrix} n \\ j \end{bmatrix}_{\mathfrak{L}} \begin{bmatrix} m \\ k-j \end{bmatrix}_{\mathfrak{L}}^{(n,j)}, \tag{6.10a}$$

$$\begin{Bmatrix} n+m \\ k+m \end{Bmatrix}_{\mathfrak{L}} = \sum_{j=0}^{m} \begin{Bmatrix} j \\ m-j \end{Bmatrix}_{\mathfrak{L}} \begin{Bmatrix} n+m-j \\ k+j \end{Bmatrix}_{\mathfrak{L}}^{(j,m-j)}. \tag{6.10b}$$

Proof. (a) To show the first equation let us consider the set $L_1(n+m,k)$ of lattice paths with diagonal and horizontal segments, and observe that every such path π can be "cut" at the point (n,j) into two paths π_1 and π_2, where $0 \leq j \leq k$. The weight of π is the product of weights of π_1 and π_2. See Fig. 8 (a).

(b) In the second equality we consider the set $L_2(n{+}m, k{+}m)$ of lattice paths with horizontal and vertical segments. As in (a) we can show that $\pi \in L_2(n+m, k+m)$ can be "cut" at the point $(j, m-j)$, where $0 \leq j \leq m$. See Fig. 8 (b). □

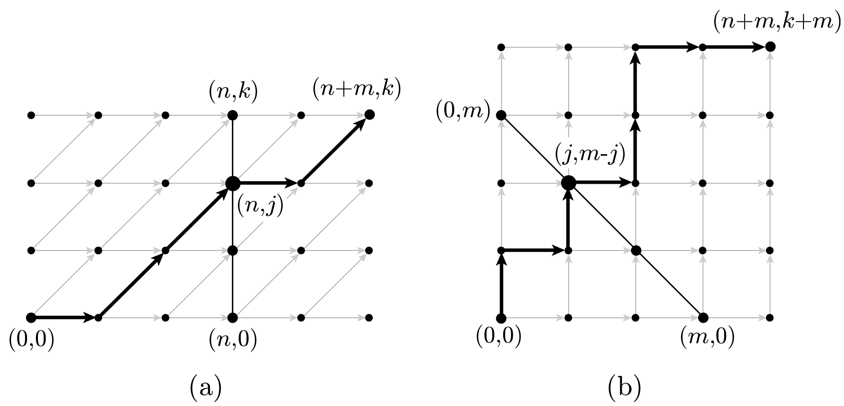

Figure 8: Illustration of the convolution-like formulae.

Let $\pi \in L[x,y]$, and let α be a segment (vertical, horizontal or diagonal) of the lattice. We write $\alpha \in \pi$ if the path π contains the segment α. A set of segments $\lambda = \{\alpha_1, \ldots, \alpha_s\}$ is called the *layer* of the set $L[x,y]$ if for every $\pi \in L[x,y]$ there is exactly one segment $\alpha_i \in \lambda$ such that $\alpha_i \in \pi$. Denote by $\Lambda[x,y]$ the family of all layers of the set $L[x,y]$. If $x = (0,0)$, then we write $\Lambda[y]$ for short.

Example 28. Let $\lambda = \{\alpha^v, \alpha^h, \alpha^d\}$, where $\alpha^v = ((0,0),(0,1))$, $\alpha^h = ((0,0),(1,0))$, and $\alpha^d = ((0,0),(1,1))$. The set λ is a layer of $L(n,k)$. Indeed, every path from $L(n,k)$ contains exactly one segment from λ.

Lemma 29. *For $n, k \geq 0$ and $\lambda \in \Lambda[(n,k)]$, we have*

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{\mathfrak{L}} = \sum_{\alpha \in \lambda} \left\langle \begin{matrix} m \\ l \end{matrix} \right\rangle_{\mathfrak{L}} w^{\mathfrak{L}}(\alpha) \left\langle \begin{matrix} n-m' \\ k-l' \end{matrix} \right\rangle_{\mathfrak{L}}^{(m',l')}, \tag{6.11}$$

where $\alpha = ((m,l),(m',l'))$.

Proof. Let $\lambda = \{\alpha_1, \alpha_2, \ldots, \alpha_s\}$. By definition, the set $\lambda \in \Lambda[(n,k)]$ determines a partition of the family $L(n,k)$ into pairwise-disjoint classes of paths $A_1, \ldots, A_s$ such that $\pi \in A_i$ if and only if $\alpha_i \in \pi$, for $i = 1, 2, \ldots, s$. Therefore, we have

$$\sum_{\pi \in L[(n,k)]} w^{\mathfrak{L}}(\pi) = \sum_{\alpha \in \lambda} \sum_{\pi \in L[(m,l)]} w^{\mathfrak{L}}(\pi) \cdot w^{\mathfrak{L}}(\alpha) \cdot \sum_{\eta \in L[(m',l'),(n,k)]} w^{\mathfrak{L}}(\eta),$$

where $\alpha = ((m,l),(m',l'))$. □

Corollary 30. *For all* $n, k \geq 0$ *and* $\lambda \in \Lambda[(n,k)]$, *we have*

$$D(n,k) = \sum_{((m,l),(m',l')) \in \lambda} D(m,l) D(n-m', k-l'). \tag{6.12}$$

Proposition 31 (Vertical summation). *For* $m, n, k \geq 1$, *we have*

$$\begin{aligned} \left\langle {m+n \atop k} \right\rangle_{\mathfrak{L}} &= h_{m+1} \sum_{i=0}^{k} \zeta^i \left\langle {m \atop i} \right\rangle_{\mathfrak{L}} \left\langle {n-1 \atop k-i} \right\rangle_{\mathfrak{L}}^{(m+1,i)} \\ &\quad + d_{m+1} \sum_{i=0}^{k-1} \zeta^i \left\langle {m \atop i} \right\rangle_{\mathfrak{L}} \left\langle {n-1 \atop k-i-1} \right\rangle_{\mathfrak{L}}^{(m+1,i+1)}. \end{aligned} \tag{6.13}$$

Proof. Let us define two classes of segments:

- $H = \{(x,y) : x = (m,i), y = (m+1,i), i = 0,1,\ldots,k\}$ (horizontal);
- $S = \{(x,y) : x = (m,i), y = (m+1,i+1), i = 0,1,\ldots,k-1\}$ (diagonal).

The union $H \cup S$ is a layer of the set $L(n,k)$. Applying Lemma 29 and simplifying the formula we obtain our claim. See Fig. 9. □

The convolution-like formula for the $\mathfrak{L}$-Delannoy numbers of the second kind takes a form

$$\left\{ {m+n \atop k} \right\}_{\mathfrak{L}} = \sum_{i=0}^{k} \left\{ {m \atop i} \right\}_{\mathfrak{L}} \zeta^i \left\{ {n-1 \atop k-i} \right\}_{\mathfrak{L}}^{(m+1,i)}.$$

Proposition 32 (Horizontal summation). *For all* $n, k, m \geq 1$, *we have*

$$\begin{aligned} \left\langle {n \atop k+m} \right\rangle_{\mathfrak{L}} &= \sum_{i=0}^{n} v_i \left\langle {i \atop m} \right\rangle_{\mathfrak{L}} \left\langle {n-i \atop k-1} \right\rangle_{\mathfrak{L}}^{(i,m+1)} \\ &\quad + \zeta^m \sum_{i=0}^{n-1} d_{i+1} \left\langle {i \atop m} \right\rangle_{\mathfrak{L}} \left\langle {n-i-1 \atop k-1} \right\rangle_{\mathfrak{L}}^{(i+1,m+1)}. \end{aligned} \tag{6.14}$$

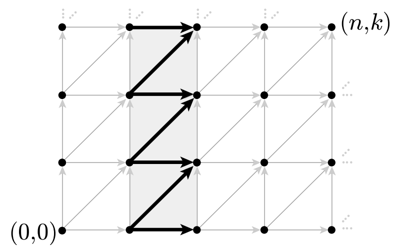

Figure 9: A layer of $L[(n,k)]$ used in vertical summation.

Proof. Let us define two classes of segments:

- $V = \{(x,y) : x = (i,m), y = (i,m+1), i = 0,1,\dots,n\}$,
- $D = \{(x,y) : x = (i,m), y = (i+1,m+1), i = 0,1,\dots,n-1\}$.

The union $V \cup D$ is a layer of the set $L(n,k)$. See Fig. 10. □

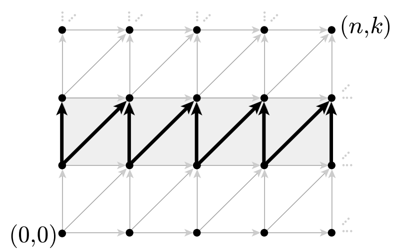

Figure 10: A layer of $L[(n,k)]$ used in horizontal summation.

Proposition 33 (Diagonal summation). *For all $n,k,m \geq 1$, we have*

$$\left\langle {n+m \atop k+m} \right\rangle_{\mathfrak{L}} = \sum_{i=0}^{m} r(i) \left\langle {i \atop m-i} \right\rangle_{\mathfrak{L}} + \sum_{i=0}^{m-1} p(i) \left\langle {i \atop m-1-i} \right\rangle_{\mathfrak{L}}, \tag{6.15}$$

where

$$\begin{aligned} r(i) &= v_i \left\langle {n+m-i \atop k+i-1} \right\rangle_{\mathfrak{L}}^{(i,m-i+1)} \\ &\quad + h_{i+1} \zeta^{m-i} \left\langle {n+m-i-1 \atop k+i} \right\rangle_{\mathfrak{L}}^{(i+1,m-i)} \\ &\quad + d_{i+1} \zeta^{m-i} \left\langle {n+m-i-1 \atop k+i-1} \right\rangle_{\mathfrak{L}}^{(i+1,m-i+1)}, \end{aligned} \tag{6.16}$$

$$p(i) = d_{i+1} \zeta^{m-1-i} \left\langle {n+m-i-1 \atop k+i} \right\rangle_{\mathfrak{L}}^{(i+1,m-i)}. \tag{6.17}$$

Proof. In this case we have four classes of segments that together form a layer of the set $L(n,k)$ (see Fig. 11). Namely,

- $V = \{(x,y) : x = (i, m-i), y = (i, m-i+1), i = 0, 1, \dots, m\}$,
- $H = \{(x,y) : x = (i, m-i), y = (i+1, m-i), i = 0, 1, \dots, m\}$,
- $D = \{(x,y) : x = (i, m-i), y = (i+1, m-i+1), i = 0, 1 \dots, m\}$,
- $D' = \{(x,y) : x = (i, m-1-i), y = (i+1, m-i), i = 0, 1 \dots, m-1\}$.

$\square$

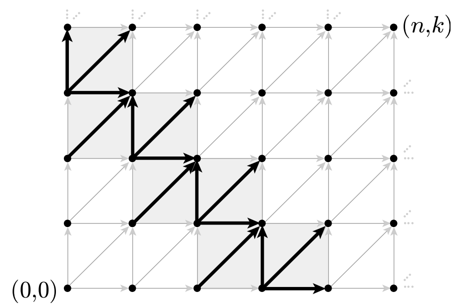

Figure 11: A layer of $L[(n,k)]$ used in diagonal summation.

The above three propositions and Corollary 30 applied to the $\mathfrak{B}$-numbers give the following recurrence relations for the standard Delannoy numbers $D(n,k)$.

Corollary 34. *For $n, k, m \geq 1$, we have*

$$D(m+n,k) = D(m,k) + \sum_{i=0}^{k-1} D(m,i)(D(n-1,k-i) + D(n-1,k-i-1)),$$

$$D(n,k+m) = D(n,m) + \sum_{i=0}^{n-1} D(i,m)(D(n-i,k-1) + D(n-i-1,k-1)),$$

$$D(n+m,k+m) = \sum_{i=0}^{m} D(i,m-i)r(i) + \sum_{i=0}^{m-1} D(i,m-1-i)D(n+m-i-1,k+i),$$

where

$$r(i) = (D(n+m-i,k+i-1) + D(n+m-i-1,k+i) + D(n+m-i-1,k+i-1)).$$

7. Connection with Partitions of Numbers

In this section we deal with the 4-tuple $\mathfrak{P} = \langle \mathbf{v}, \mathbf{1}, \mathbf{d}, \zeta \rangle$. We follow the notation of Andrews [1]. By the *partition* λ of a non-negative integer n we mean a finite non-increasing sequence of non-negative integers $(\lambda_1, \lambda_2, \ldots, \lambda_r)$ such that $\lambda_1 + \cdots + \lambda_r = n$. Each $\lambda_i > 0$ is called a *part* of λ.

We define two functions on partitions. Namely, let

$$w^d(\lambda, m) = \prod_{i=1}^{r} d_{m-\lambda_i}, \qquad w^v(\lambda, m) = \prod_{i=1}^{r} v_{m-\lambda_i}.$$

For all $n, k \geq 1$, we denote by $\mathcal{D}(m, n, k)$ the set of all partitions of the number n into k distinct parts from the set $\{0, 1, \ldots, m\}$. We denote by $\mathcal{R}(m, n, k)$ the set of such partitions with repetition of their parts allowed. Then we set

$$\alpha(m, n, k) = \sum_{\lambda \in \mathcal{D}(m,n,k)} w^d(\lambda, m), \tag{7.1}$$

$$\beta(m, n, k) = \sum_{\lambda \in \mathcal{R}(m,n,k)} w^v(\lambda, m). \tag{7.2}$$

Let us consider the generating function of the $\mathfrak{L}$-Delannoy numbers of the first kind $C_m(z)$ and the second kind $S_m(z)$, respectively. Observe that

$$\prod_{i=0}^{m-1} \left(1 + d_{m-i}\zeta^i z\right) = \sum_{k=0}^{m-1} \sum_{n \geq 0} \alpha(m-1, n, k)\zeta^n z^k,$$

$$\prod_{i=0}^{m} \frac{1}{(1 - v_{m-i}\zeta^i z)} = \sum_{k \geq 0} \sum_{n \geq 0} \beta(m, n, k)\zeta^n z^k.$$

Therefore, the $\mathfrak{P}$-Delannoy numbers of the first and second kind, respectively, are generating functions of α and β, in the sense that

$$\begin{bmatrix} m \\ k \end{bmatrix}_{\mathfrak{P}} = \sum_{n \geq 0} \alpha(m-1, n, k)\zeta^n, \qquad \begin{Bmatrix} m \\ k \end{Bmatrix}_{\mathfrak{P}} = \sum_{n \geq 0} \beta(m, n, k)\zeta^n.$$

Thus

$$\left\langle \begin{matrix} m \\ k \end{matrix} \right\rangle_{\mathfrak{P}} = \sum_{j=0}^{k} \begin{bmatrix} m \\ j \end{bmatrix}_{\mathfrak{P}} \begin{Bmatrix} m \\ k-j \end{Bmatrix}_{\mathfrak{P}} = \sum_{n \geq 0} \gamma(m, n, k)\zeta^n,$$

where

$$\gamma(m, n, k) = \sum_{j=0}^{k} \sum_{l=0}^{n} \alpha(m-1, l, j)\beta(m, n-l, k-j).$$

But what do the numbers $\gamma(m,n,k)$ really count in terms of partitions? From Corollary 9 we have

$$\left\langle \begin{matrix} m \\ k \end{matrix} \right\rangle_{\mathfrak{P}} = \sum_{j=0}^{k} \sum_{\substack{1\le\pi_1<\cdots<\pi_j\le m \\ 0\le\sigma_1\le\cdots\le\sigma_{k-j}\le m}} d_{\pi_1}\cdots d_{\pi_j} v_{\sigma_1}\cdots v_{\sigma_{k-j}} \zeta^{km-S} \tag{7.3}$$

$$= \sum_{n\ge 0} \gamma(m,n,k)\zeta^n, \tag{7.4}$$

where $S = \pi_1 + \cdots + \pi_j + \sigma_1 + \cdots + \sigma_{k-j}$, and

$$\gamma(m,n,k) = \sum_I \prod_{i=1}^{j} d_{m-\pi_i} \prod_{i=1}^{k-j} v_{m-\sigma_i},$$

where the summation range I is given by

$$I = \begin{cases} 0 \le \pi_1 < \pi_2 < \cdots < \pi_j \le m-1; \\ 0 \le \sigma_1 \le \sigma_2 \le \cdots \le \sigma_{k-j} \le m; \\ \pi_1 + \cdots + \pi_j + \sigma_1 + \cdots + \sigma_{k-j} = n. \end{cases}$$

Let H be a finite set of non-negative integers. We denote by $\mathcal{D}(H;n,k)$ the family of partitions of n into k distinct parts from the set H. We define $\mathcal{R}(H;n,k)$ similarly (repetition of parts is allowed). It is well-known that

$$\sum_{n,k\ge 0} |\mathcal{D}(H;n,k)| z^k y^n = \prod_{i\in H} (1+y^i z), \tag{7.5a}$$

$$\sum_{n,k\ge 0} |\mathcal{R}(H;n,k)| z^k y^n = \prod_{i\in H} \frac{1}{(1-y^i z)}. \tag{7.5b}$$

Fix $m \ge 1$, and for a given subset $H \subseteq \{0,1,\ldots,m-1\}$, let $\mathfrak{P}(H)$ be the 4-tuple $\langle \mathbf{a}, \mathbf{1}, \mathbf{a}, \zeta \rangle$ such that $\mathbf{a} = \{0, a_1, \ldots, a_m, 0, 0\ldots\}$ is defined as follows. For $i = 0,1,\ldots,m-1$, if $i \in H$ then $a_{m-i} = 1$, and $a_{m-i} = 0$ otherwise.

Theorem 35. *We have*

$$\begin{bmatrix} m \\ k \end{bmatrix}_{\mathfrak{P}(H)} = \sum_{n\ge 0} |\mathcal{D}(H;n,k)|\zeta^n, \tag{7.6a}$$

$$\left\{ \begin{matrix} m \\ k \end{matrix} \right\}_{\mathfrak{P}(H)} = \sum_{n\ge 0} |\mathcal{R}(H;n,k)|\zeta^n. \tag{7.6b}$$

Proof. (a) Due to the definition of the sequence $\mathbf{a}$ of the 4-tuple $\mathfrak{P}(H) = \langle \mathbf{a}, \mathbf{1}, \mathbf{a}, \zeta \rangle$, we have

$$\sum_{k\ge 0} \begin{bmatrix} m \\ k \end{bmatrix}_{\mathfrak{P}(H)} z^k = \prod_{i=1}^{m} \left(1 + a_i \zeta^{m-i} z\right)$$

$$= \prod_{i\in H} \left(1 + \zeta^i z\right) = \sum_{k\ge 0} \sum_{\pi} \zeta^{\pi_1+\cdots+\pi_k}\ z^k,$$

where the summation range π is over all subsets $A \subseteq H$ such that $|A| = k$. Comparing coefficients of z^k we obtain (7.5a). (b) In a similar way we show the second equality, i.e.,

$$\sum_{k\geq 0} \left\{ \begin{matrix} m \\ k \end{matrix} \right\}_{\mathfrak{P}(H)} z^k = \prod_{i=1}^{m} \frac{1}{(1 - a_i \zeta^{m-i} z)} = \prod_{i\in H} \frac{1}{(1-\zeta^i z)}.$$

$\square$

8. Examples and Remarks

In this section we present some numbers of general interest which are special cases of $\mathfrak{L}$-Delannoy numbers.

8.1. The Binomial Coefficients

Setting $\mathfrak{B} = \langle \mathbf{1}, \mathbf{1}, \mathbf{1}, 1 \rangle$ gives

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\mathfrak{B}} = \binom{n}{k}, \qquad \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{\mathfrak{B}} = \binom{n+k}{k}, \qquad \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{\mathfrak{B}} = D(n,k).$$

8.2. The p, q-Binomial Coefficients

The p,q-binomial coefficient [4, 7, 11] is defined as

$$\binom{n}{k}_{p,q} = \frac{(q^n - p^n)(q^{n-1} - p^{n-1}) \cdots (q^{n-k+1} - p^{n-k+1})}{(q^k - p^k)(q^{k-1} - p^{k-1}) \cdots (q-p)}.$$

One shows that

$$\prod_{i=1}^{n} \left(1 + q^{i-1} p^{n-i}\, z\right) = \sum_{k\geq 0} q^{\binom{k}{2}} p^{\binom{k}{2}} \binom{n}{k}_{p,q} z^k,$$

$$\prod_{i=0}^{n} \frac{1}{(1 - q^i p^{n-i}\, z)} = \sum_{k\geq 0} \binom{n+k}{k}_{p,q} z^k.$$

Setting $\mathfrak{P} = \langle \mathbf{q}, \mathbf{1}, \mathbf{q}^*, p \rangle$, where $\mathbf{q} = (1, q, q^2, \ldots)$ and $\mathbf{q}^* = (1, 1, q, q^2, \ldots)$, we obtain

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\mathfrak{P}} = p^{\binom{k}{2}} q^{\binom{k}{2}} \binom{n}{k}_{p,q}, \qquad \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{\mathfrak{P}} = \binom{n+k}{k}_{p,q}.$$

Setting $p = (1+\sqrt{5})/2$ and $q = (1-\sqrt{5})/2$ we obtain the Fibonomial coefficients [10, 15, 16, 18, 19, 20] defined as

$$\binom{n}{k}_{Fib} = \frac{F_n F_{n-1} \cdots F_{n-k+1}}{F_k F_{k-1} \cdots F_1},$$

where F_n is the n-th Fibonacci number. Thus we have

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\mathfrak{P}} = (-1)^{\binom{k}{2}} \binom{n}{k}_{Fib}, \qquad \begin{Bmatrix} n \\ k \end{Bmatrix}_{\mathfrak{P}} = \binom{n+k}{k}_{Fib}.$$

For the general case of the $\mathfrak{P}$-Delannoy numbers we have

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\mathfrak{P}}^{(m,l)} = p^{n\,l} q^{k\,m} \begin{bmatrix} n \\ k \end{bmatrix}_{\mathfrak{P}}, \quad \begin{Bmatrix} n \\ k \end{Bmatrix}_{\mathfrak{P}}^{(m,l)} = p^{n\,l} q^{k\,m} \begin{Bmatrix} n \\ k \end{Bmatrix}_{\mathfrak{P}}.$$

Due to the above correspondence, all the results from Section 6 take simple forms. For instance, Proposition 27 yields

$$\begin{bmatrix} n+m \\ k \end{bmatrix}_{\mathfrak{P}} = \sum_{j=0}^{k} \begin{bmatrix} n \\ j \end{bmatrix}_{\mathfrak{P}} p^{m\,j} q^{(k-j)n} \begin{bmatrix} m \\ k-j \end{bmatrix}_{\mathfrak{P}},$$

$$\begin{Bmatrix} n+m \\ k+m \end{Bmatrix}_{\mathfrak{P}} = \sum_{j=0}^{m} \begin{Bmatrix} j \\ m-j \end{Bmatrix}_{\mathfrak{P}} p^{(n+m-j)(m-j)} q^{(k+j)j} \begin{Bmatrix} n+m-j \\ k+j \end{Bmatrix}_{\mathfrak{P}}.$$

Corcino [7] asked about the generalization of q-Vandermonde identity for the p,q-binomials. From the above we answer to his question.

Corollary 36 (p,q-Vandermonde like identity). *For $n, m, k \geq 1$, we have*

$$\binom{n+m}{k}_{p,q} = \sum_{j=0}^{k} p^{(m-k+j)\,j} q^{(k-j)(n-j)} \binom{n}{j}_{p,q} \binom{m}{k-j}_{p,q}, \tag{8.1}$$

$$\binom{n+2m+k}{k+m}_{p,q} = \sum_{j=0}^{m} \binom{m}{m-j}_{p,q} p^{(n+m-j)(m-j)} q^{(k+j)j} \binom{n+m+k}{k+j}_{p,q}.$$

8.3. The q-Binomial Coefficients

For $p = 1$, we obtain the q-binomial coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\mathfrak{Q}} = q^{\binom{k}{2}} \binom{n}{k}_q, \qquad \begin{Bmatrix} n \\ k \end{Bmatrix}_{\mathfrak{Q}} = \binom{n+k}{k}_q.$$

Recall that $\binom{n}{k}_q$ is the number of k-dimensional subspaces of the n-dimensional vector space over the finite field of order q. From Theorem 1 we have

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{\mathfrak{Q}} = \sum_{i=0}^{k} q^{\binom{i}{2}} \binom{n}{i}_q \binom{n+k-i}{k-i}_q.$$

There is natural question about an analogous combinatorial interpretation of the above in terms of counting certain subspaces. The convolution (8.1) reduces to the q-Vandermonde identity [14].

8.4. The Stirling Numbers

Let us define the ordinary Stirling numbers of the first and second kind as follows:

$$\prod_{i=1}^{n}(1+iz) = \sum_{k=0}^{n} \begin{bmatrix} n+1 \\ n+1-k \end{bmatrix} z^k, \quad \prod_{i=1}^{n} \frac{1}{(1-iz)} = \sum_{k=0}^{n} \begin{Bmatrix} n+k \\ n \end{Bmatrix} z^k.$$

Setting $\mathfrak{S} = \langle \mathbf{n}, \mathbf{1}, \mathbf{n}, 1\rangle$, where $\mathbf{n} = (0, 1, 2, 3, \ldots)$ we obtain

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\mathfrak{S}} = \begin{bmatrix} n+1 \\ n+1-k \end{bmatrix}, \qquad \begin{Bmatrix} n \\ k \end{Bmatrix}_{\mathfrak{S}} = \begin{Bmatrix} n+k \\ n \end{Bmatrix}.$$

From Theorem 1 we have

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{\mathfrak{S}} = \sum_{i=0}^{k} \begin{bmatrix} n+1 \\ n+1-i \end{bmatrix} \begin{Bmatrix} n+k-i \\ n \end{Bmatrix}$$

and as in previous example, we may ask about its interpretation.

8.5. The p, q-Stirling Numbers

Wachs and White [27] define the p, q-Stirling numbers by the following recursion

$$S_{p,q}(n, k) = [k]_{p,q} S_{p,q}(n-1, k) + p^{k-1} S_{p,q}(n-1, k-1)$$

with $S_{p,q}(0, 0) = 1$, where $[k]_{p,q} = \sum_{i=1}^{k} p^{k-i} q^{i-1}$ and $[0]_{p,q} = 0$. The generating function is

$$\sum_{k\geq 0} S_{p,q}(n+k, n) z^k = p^{\binom{n}{2}} \prod_{i=0}^{n} \frac{1}{(1 - z[i]_{p,q})}.$$

Setting $\mathfrak{S} = \langle \mathbf{i}, \mathbf{1}, \mathbf{i}, 1\rangle$, where $\mathbf{i} = ([0]_{p,q}, [1]_{p,q},\ [2]_{p,q} \ldots)$ we obtain

$$p^{\binom{n}{2}} \begin{Bmatrix} n \\ k \end{Bmatrix}_{\mathfrak{S}} = S_{p,q}(n+k, n).$$

8.6. The $\mathfrak{A}$-Stirling Numbers

Let $\mathbf{w} = (w_0, w_1, \ldots)$. In the notation of Médicis and Leroux [22] we define $\mathfrak{A}$-Stirling numbers of the first kind $c^{\mathfrak{A}}$ and the second kind $S^{\mathfrak{A}}$ as follows

$$\sum_{k\geq 0} c^{\mathfrak{A}}(n, k) z^k = (1 + w_0 z)(1 + w_1 z) \cdots (1 + w_{n-1} z),$$

$$\sum_{k\geq 0} S^{\mathfrak{A}}(n+k, n) z^k = \frac{1}{(1 + w_0 z)} \frac{1}{(1 + w_1 z)} \cdots \frac{1}{(1 + w_n z)}.$$

Setting $\mathfrak{A} = \langle \mathbf{w}, \mathbf{1}, \mathbf{w}^*, 1\rangle$, where $\mathbf{w}^* = (1, w_0, w_1, \ldots)$ we obtain

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\mathfrak{A}} = c^{\mathfrak{A}}(n,k), \qquad \begin{Bmatrix} n \\ k \end{Bmatrix}_{\mathfrak{A}} = S^{\mathfrak{A}}(n+k,n).$$

The generalization of Carlitz formulae for the $\mathfrak{A}$-Stirling numbers delivered by Médicis and Leroux [22, Th.2.2] is a special case of Theorem 10 for $\zeta = 1$ and $\alpha_2 = w_0$, that is,

$$S^{\mathfrak{A}}(n,k) = \begin{Bmatrix} k \\ n-k \end{Bmatrix}_{\mathfrak{H}} = \sum_{j=k}^{n} \binom{n}{j} w_0^{n-j} \begin{Bmatrix} k \\ j-k \end{Bmatrix}_{\mathfrak{H}^*}. \tag{8.2}$$

8.7. The ζ-Analogues of the Stirling Numbers

Let $w_1, w_2, \ldots$ be a sequence of complex numbers. For $n \geq 1$, let $\mathbf{w}_n(\zeta) = (w_1\zeta^{n-1}, w_2\zeta^{n-2}, \ldots, w_n)$, and recall that ζ-analogues [12] of the Stirling numbers are defined as

$$\hat{C}_k^n(\hat{\mathbf{w}}_n) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \hat{w}_{i_1}\hat{w}_{i_2}\cdots\hat{w}_{i_k},$$
$$\hat{S}_k^n(\hat{\mathbf{w}}_n) = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n} \hat{w}_{i_1}\hat{w}_{i_2}\cdots\hat{w}_{i_k},$$

where $\hat{w}_i = w_i\zeta^{n-i}$. If we set $\mathfrak{Z} = \langle \mathbf{w}, \mathbf{1}, \mathbf{w}, \zeta\rangle$, where $\mathbf{w}$ is the sequence $(0, w_1, w_2, \ldots)$, then

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\mathfrak{Z}} = \hat{C}_k^n(\mathbf{w}_n(\zeta)), \qquad \begin{Bmatrix} n \\ k \end{Bmatrix}_{\mathfrak{Z}} = \hat{S}_k^n(\mathbf{w}_n(\zeta)).$$

Acknowledgements I would like to thank the referee for very careful reading of the manuscript and for bringing some papers about Delannoy numbers to my attention.

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