A NEW GENERALIZATION OF FIBONACCI SEQUENCE AND EXTENDED BINET'S FORMULA

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Abstract

Consider the Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$ with initial conditions $F_0 = 0$, $F_1 = 1$ and recurrence relation $F_n = F_{n-1} + F_{n-2}$ $(n \geq 2)$. The Fibonacci sequence has been generalized in many ways, some by preserving the initial conditions, and others by preserving the recurrence relation. In this article, we study a new generalization $\{q_n\}$, with initial conditions $q_0 = 0$ and $q_1 = 1$, which is generated by the recurrence relation $q_n = aq_{n-1} + q_{n-2}$ (when n is even) or $q_n = bq_{n-1} + q_{n-2}$ (when n is odd), where a and b are nonzero real numbers. Some well-known sequences are special cases of this generalization. The Fibonacci sequence is a special case of $\{q_n\}$ with $a = b = 1$. Pell's sequence is $\{q_n\}$ with $a = b = 2$ while k-Fibonacci sequence has $a = b = k$. We produce an extended Binet's formula for $\{q_n\}$ and, thereby, identities such as Cassini's, Catalan's, d'Ocagne's, etc.

1. Introduction

The Fibonacci sequence, $\{F_n\}_{n=0}^{\infty}$, is a series of numbers, starting with the integer pair 0 and 1, where the value of each element is calculated as the sum of the two preceding it. That is, $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. The first few terms of the Fibonacci sequence are: $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597,$ $2584, \ldots$. The Fibonacci numbers are perhaps most famous for appearing in the rabbit-breeding problem, introduced by Leonardo de Pisa in 1202 in his book called *Liber Abaci*. However, they also occur in Pascal's triangle [18], in Pythagorean triples [18], computer algorithms [1, 9, 33], some areas of algebra [5, 8, 31], graph theory [2, 3], quasicrystals [34, 41], and many other areas of mathematics. They occur in a variety of other fields such as finance, art, architecture, music, etc. (See [10] for extensive resources on Fibonacci numbers.)

However, in this paper, we are most interested in the generalizations of the Fibonacci sequence. Some authors [13, 15, 17, 27, 37] have generalized the Fibonacci sequence by preserving the recurrence relation and altering the first two terms of the sequence, while others [7, 20, 21, 22, 26, 30, 40] have generalized the Fibonacci sequence by preserving the first two terms of the sequence but altering the recurrence

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relation slightly. One example of this latter generalization, called the k-Fibonacci sequence, $\{F_{k,n}\}_{n=0}^{\infty}$, is defined using a linear recurrence relation depending on one real parameter (k) given by

$$F_{k,n} = kF_{k,n-1} + F_{k,n-2} \quad (n \geq 2)$$

where $F_{k,0} = 0$ and $F_{k,1} = 1$. When $k = 1$, the classical Fibonacci sequence is obtained. These generalizations satisfy identities that are analogous to the identities satisfied by the classical Fibonacci sequence [18].

We now introduce a further generalization of the Fibonacci sequence; we shall call it the generalized Fibonacci sequence. Unlike other variations, this new generalization depends on two real parameters used in a non-linear recurrence relation.

Definition 1. For any two nonzero real numbers a and b, the generalized Fibonacci sequence, say $\left\{F_n^{(a,b)}\right\}_{n=0}^{\infty}$, is defined recursively by

$$F_0^{(a,b)} = 0, \;\; F_1^{(a,b)} = 1, \;\; F_n^{(a,b)} = \begin{cases} aF_{n-1}^{(a,b)} + F_{n-2}^{(a,b)}, & \text{if n is even} \\ bF_{n-1}^{(a,b)} + F_{n-2}^{(a,b)}, & \text{if n is odd} \end{cases} \quad (n \geq 2).$$

To avoid cumbersome notation, let us denote $F_n^{(a,b)}$ by q_n. Thus, the sequence $\{q_n\}$ satisfies

$$q_0 = 0, \;\; q_1 = 1, \;\; q_n = \begin{cases} aq_{n-1} + q_{n-2}, & \text{if n is even} \\ bq_{n-1} + q_{n-2}, & \text{if n is odd} \end{cases} \quad (n \geq 2).$$

We now note that this new generalization is in fact a family of sequences where each new choice of a and b produces a distinct sequence. When $a = b = 1$, we have the classical Fibonacci sequence and when $a = b = 2$, we get the Pell numbers. Even further, if we set $a = b = k$, for some positive integer k, we get the k-Fibonacci numbers, the generalization of the Fibonacci numbers mentioned above.

We will describe the terms of the sequence $\{q_n\}$ explicitly by using a generalization of Binet's formula. Therefore, we will start the main content of the paper by deriving a generalization of Binet's formula (via generating functions) and then will present extensions of well-known Fibonacci identities such as Catalan's, Cassini's, and d'Ocagne's. Later, we alter $\{q_n\}$ by allowing arbitrary initial conditions and also consider the convergence of the ratios of successive terms of the sequence. It is well-known that the ratios of successive Fibonacci numbers approach the golden mean, Φ, so it is natural to ask if analogous results exist for the variations and extensions of the Fibonacci sequence. Even for random Fibonacci sequences, there are results related to growth and decay rates [6, 16, 29, 36, 39]. We now give a brief word-combinatorial interpretation of the generalized Fibonacci sequence as this is the context in which we first studied this family of sequences.

Let $0 < \alpha < 1$ be an irrational number. Associate with α a sequence, called the characteristic sequence of α (see [23]), which is denoted by $\omega = \omega(\alpha)$, and given by

$$\omega = \omega_1 \omega_2 \omega_3 \cdots \omega_n \cdots,$$

where

$$\omega_n = \lfloor (n+1)\alpha \rfloor - \lfloor n\alpha \rfloor \quad (n \geq 1).$$

Note that $\omega_n \in \{0, 1\}$, so that $\omega(\alpha)$ is an infinite word consisting of 0's and 1's.

We now outline the relation between the characteristic sequence of α and the continued fraction expansion of α. This connection leads us, via word combinatorics, to the definition of the generalized Fibonacci sequences. Suppose that the continued fraction expansion of $\alpha = [0; 1 + d_1, d_2, d_3, \ldots]$, and define a sequence $\{s_n\}_{n \geq 0}$ of words by

$$s_0 = 1, \ s_1 = 0, \ \text{and} \ s_n = s_{n-1}^{d_{n-1}} s_{n-2}, \ (n \geq 2).$$

Then for $n \geq 1$, each s_n is a prefix of $\omega(\alpha)$ and $\omega(\alpha) = \lim\limits_{n \to \infty} s_n$ (see [23]).

Example 2. (The infinite Fibonacci word) Let $\alpha = [0; 2, 1, 1, 1, \ldots] = \frac{1}{\phi^2}$, where ϕ is the Golden Mean. Then the $\{s_n\}$ are

$$s_0 = 1, s_1 = 0, s_2 = 01, s_3 = 010, s_4 = 01001, s_5 = 01001010, \ldots.$$

The limit of this sequence is the infinite word,

$$\omega = 0100101001001010010100100101010010010\ldots.$$

Since the lengths of s_0 and s_1 are both 1 and s_n is obtained by concatenating s_{n-1} and s_{n-2}, the length of the word s_n, denoted by $|s_n|$, is F_{n+1}, the $n + 1^{st}$ Fibonacci number. Since the lengths of these subwords are Fibonacci numbers, the infinite word ω is called the Fibonacci word.

Now we associate with the generalized Fibonacci sequence, $\left\{ F_n^{(a,b)} \right\} = \{q_n\}$, a unique quadratic irrational number α in the interval $(0, 1)$, whose continued fraction expansion has the form $\alpha = [0; a, b, a, b, \ldots] = [0; a, b, \overline{a, b}]$. Then $\omega(\alpha) = \lim\limits_{n \to \infty} s_n$ where

$$s_0 = 1, \ s_1 = 0, \ s_2 = 0^{a-1}1 = 000\cdots01,$$

and

$$s_n = \begin{cases} s_{n-1}^a s_{n-2}, & \text{if n is even} \\ s_{n-1}^b s_{n-2}, & \text{if n is odd} \end{cases} \quad (n \geq 3).$$

We next define a number sequence $\{r_n\}$ as follows. Let $r_0 = 0$, $r_n = |s_n|$ $(n \geq 1)$. Since $\{r_n\}$ and $\{q_n\}$ satisfy the same initial conditions and have the same recursive definitions, clearly $\{r_n\} = \{q_n\}$.

In conclusion, every generalized Fibonacci sequence with a and b nonnegative integers has a one-to-one correspondence with a quadratic irrational α in the interval $(0,1)$ having the form $\alpha = [0; a, b, a, b, \ldots]$. Moreover, every generalized Fibonacci sequence is intimately connected to an infinite word called the characteristic sequence of α.

Example 3. Let $\alpha = [0; 1, 1, 1, 1, \ldots] = \frac{1}{\phi}$, where ϕ is the Golden Mean. Then the terms of the sequence $\{s_n\}$ are

$$s_0 = 1, s_1 = 0, s_2 = 1, s_3 = 10, s_4 = 101, \ldots.$$

Observe that $\{r_n\} = \{F_n\}$, the Fibonacci sequence and that $\omega(\alpha)$ can be obtained from the infinite Fibonacci word by exchanging 0's and 1's.

2. Generating Function for the Generalized Fibonacci Sequence

Generating functions provide a powerful technique for solving linear homogeneous recurrence relations. Even though generating functions are typically used in conjunction with linear recurrence relations with constant coefficients, we will systematically make use of them for linear recurrence relations with nonconstant coefficients. In this section, we consider the generating functions for the generalized Fibonacci sequences and derive some of the most fascinating identities satisfied by these sequences. As Wilf indicated in [38], "a generating function is a clothesline on which we hang up a sequence of numbers for display."

Theorem 4. *The generating function for the generalized Fibonacci sequence $\{q_n\}$ is*

$$F(x) = \frac{x\left(1 + ax - x^2\right)}{1 - (ab + 2)x^2 + x^4}.$$

Proof. We begin with the formal power series representation of the generating function for $\{q_n\}$,

$$F(x) = q_0 + q_1 x + q_2 x^2 + \cdots + q_k x^k + \cdots = \sum_{m=0}^{\infty} q_m x^m.$$

Note that,

$$bxF(x) = bq_0 x + bq_1 x^2 + bq_2 x^3 + \cdots + bq_k x^{k+1} + \cdots = \sum_{m=0}^{\infty} bq_m x^{m+1} = \sum_{m=1}^{\infty} bq_{m-1} x^m$$

and

$$x^2 F(x) = q_0 x^2 + q_1 x^3 + q_2 x^4 + \cdots + q_k x^{k+2} + \cdots = \sum_{m=0}^{\infty} q_m x^{m+2} = \sum_{m=2}^{\infty} q_{m-2} x^m.$$

Since $q_{2k+1} = b q_{2k} + q_{2k-1}$ and $q_0 = 0, q_1 = 1$, we get

$$\left(1 - bx - x^2\right) F(x) = x + \sum_{m=1}^{\infty} \left(q_{2m} - b q_{2m-1} - q_{2m-2}\right) x^{2m}.$$

Since $q_{2k} = a q_{2k-1} + q_{2k-2}$, we get

$$\left(1 - bx - x^2\right) F(x) = x + \sum_{m=1}^{\infty} \left(a - b\right) q_{2m-1} x^{2m}$$

$$\left(1 - bx - x^2\right) F(x) = x + (a - b) x \sum_{m=1}^{\infty} q_{2m-1} x^{2m-1}.$$

Now let

$$f(x) = \sum_{m=1}^{\infty} q_{2m-1} x^{2m-1}.$$

Since

$$
\begin{aligned}
q_{2k+1} &= b q_{2k} + q_{2k-1} = b\left(a q_{2k-1} + q_{2k-2}\right) + q_{2k-1} \\
&= (ab + 1) q_{2k-1} + b q_{2k-2} = (ab + 1) q_{2k-1} + q_{2k-1} - q_{2k-3} \\
&= (ab + 2) q_{2k-1} - q_{2k-3},
\end{aligned}
$$

we have

$$\left(1 - (ab + 2)x^2 + x^4\right) f(x)$$

$$= x - x^3 + \sum_{m=3}^{\infty} \left(q_{2m-1} - (ab + 2)q_{2m-3} + q_{2m-5}\right) x^{2m-1} = x - x^3.$$

Therefore,

$$f(x) = \frac{x - x^3}{1 - (ab + 2)x^2 + x^4}$$

and as a result, we get

$$\left(1 - bx - x^2\right) F(x) = x + (a - b)x \cdot \frac{x - x^3}{1 - (ab + 2)x^2 + x^4}.$$

After simplifying the above expression we get the desired result

$$F(x) = \frac{x\left(1 + ax - x^2\right)}{1 - (ab + 2)x^2 + x^4}. \qquad \square$$

3. Binet's Formula for the Generalized Fibonacci Sequence and Identities

Koshy refers to the Fibonacci numbers as one of the "two shining stars in the vast array of integer sequences" [18]. We may guess that one reason for this reference is the sheer quantity of interesting properties this sequence possesses. Further still, almost all of these properties can be derived from Binet's formula. A main objective of this paper is to demonstrate that many of the properties of the Fibonacci sequence can be stated and proven for a much larger class of sequences, namely the generalized Fibonacci sequence. Therefore, we will state and prove an extension of Binet's formula for the generalized Fibonacci sequences and so derive a number of mathematical properties including generalizations of Cassini's, Catalan's, and d'Ocagne's identities for the ordinary Fibonacci sequence.

Theorem 5. (Generalized Binet's formula) *The terms of the generalized Fibonacci sequence $\{q_m\}$ are given by*

$$q_m = \left(\frac{a^{1-\xi(m)}}{(ab)^{\lfloor \frac{m}{2} \rfloor}}\right) \frac{\alpha^m - \beta^m}{\alpha - \beta}$$

where $\alpha = \frac{ab + \sqrt{a^2 b^2 + 4ab}}{2}$, $\beta = \frac{ab - \sqrt{a^2 b^2 + 4ab}}{2}$, and $\xi(m) := m - 2\lfloor \frac{m}{2} \rfloor$.

Proof. First, note that α and β are roots of the quadratic equation

$$x^2 - abx - ab = 0$$

and

$$\xi(m) = \begin{cases} 0 & \text{if } m \text{ is even} \\ 1 & \text{if } m \text{ is odd} \end{cases}$$

is the parity function. We have seen that the generating function for the sequence $\{q_m\}$ is given by (see Theorem 1)

$$F(x) = \frac{x\left(1 + ax - x^2\right)}{1 - (ab + 2)x^2 + x^4}.$$

Using the partial fraction decomposition, we rewrite $F(x)$ as

$$F(x) = \frac{1}{\alpha - \beta} \left[\frac{a(\alpha + 1) - \alpha x}{x^2 - (\alpha + 1)} - \frac{a(\beta + 1) - \beta x}{x^2 - (\beta + 1)} \right]$$

where α and β are as above. Since the Maclaurin series expansion of the function $\frac{A - Bz}{z^2 - C}$ is given by

$$\frac{A - Bz}{z^2 - C} = \sum_{n=0}^{\infty} BC^{-n-1} z^{2n+1} - \sum_{n=0}^{\infty} AC^{-n-1} z^{2n},$$

the generating function $F(x)$ can be expressed as

$$\frac{1}{\alpha - \beta} \left[\frac{a(\alpha + 1) - \alpha x}{x^2 - (\alpha + 1)} - \frac{a(\beta + 1) - \beta x}{x^2 - (\beta + 1)} \right]$$

$$= \frac{1}{\alpha - \beta} \left[\sum_{m=0}^{\infty} \frac{-\beta(\alpha + 1)^{m+1} + \alpha(\beta + 1)^{m+1}}{(\alpha + 1)^{m+1}(\beta + 1)^{m+1}} x^{2m+1} \right]$$

$$+ \frac{a}{\alpha - \beta} \left[\sum_{m=0}^{\infty} \frac{(\beta + 1)(\alpha + 1)^{m+1} - (\alpha + 1)(\beta + 1)^{m+1}}{(\alpha + 1)^{m+1}(\beta + 1)^{m+1}} x^{2m} \right].$$

We now simplify using the following properties of α and β.

(i) $(\alpha + 1)(\beta + 1) = 1$, (ii) $\alpha + \beta = ab$, (iii) $\alpha \cdot \beta = -ab$,

(iv) $\alpha + 1 = \frac{\alpha^2}{ab}$, (v) $\beta + 1 = \frac{\beta^2}{ab}$, (vi) $-\beta(\alpha + 1) = \alpha$,

(vii) $-\alpha(\beta + 1) = \beta$.

Using the above identities, we get

$$F(x) = \sum_{m=0}^{\infty} \left(\frac{1}{ab} \right)^{m+1} \frac{-\beta\alpha^{2m+2} + \alpha\beta^{2m+2}}{\alpha - \beta} x^{2m+1}$$

$$+ \sum_{m=0}^{\infty} a \left(\frac{1}{ab} \right)^{m+1} \frac{(\beta + 1)\alpha^{2m+2} - (\alpha + 1)\beta^{2m+2}}{\alpha - \beta} x^{2m}$$

$$= \sum_{m=0}^{\infty} \left(\frac{1}{ab} \right)^{m} \frac{\alpha^{2m+1} - \beta^{2m+1}}{\alpha - \beta} x^{2m+1} + \sum_{m=0}^{\infty} a \left(\frac{1}{ab} \right)^{m} \frac{\alpha^{2m} - \beta^{2m}}{\alpha - \beta} x^{2m}.$$

Combining the two sums, we get

$$F(x) = \sum_{m=0}^{\infty} a^{1-\xi(m)} \left(\frac{1}{ab}\right)^{\lfloor \frac{m}{2} \rfloor} \frac{\alpha^m - \beta^m}{\alpha - \beta} x^m = \sum_{m=0}^{\infty} q_m x^m.$$

Therefore, for all $m \geq 0$, we have

$$q_m = \left(\frac{a^{1-\xi(m)}}{(ab)^{\lfloor \frac{m}{2} \rfloor}}\right) \frac{\alpha^m - \beta^m}{\alpha - \beta}.$$

$\square$

Note that when $a = b = 1$, $q_m = \frac{\alpha^m - \beta^m}{\alpha - \beta}$, which is the original Binet formula for the Fibonacci numbers.

Theorem 6. (Cassini's identity) *For any nonnegative integer n, we have*

$$a^{1-\xi(n)}b^{\xi(n)}q_{n-1}q_{n+1} - a^{\xi(n)}b^{1-\xi(n)}q_n^2 = a(-1)^n.$$

Since Cassini's identity is a special case of Catalan's identity, which is stated below, it is enough to prove Catalan's identity.

Theorem 7. (Catalan's identity) *For any two nonnegative integers n and r, with $n \geq r$, we have*

$$a^{\xi(n-r)}b^{1-\xi(n-r)}q_{n-r}q_{n+r} - a^{\xi(n)}b^{1-\xi(n)}q_n^2 = a^{\xi(r)}b^{1-\xi(r)}(-1)^{n+1-r}q_r^2.$$

Proof. Using the extended Binet's formula, we get

$$a^{\xi(n-r)}b^{1-\xi(n-r)}q_{n-r}q_{n+r}$$

$$= a^{\xi(n-r)}b^{1-\xi(n-r)} \left(\frac{a^{1-\xi(n-r)}}{(ab)^{\lfloor \frac{n-r}{2} \rfloor}}\right) \left(\frac{a^{1-\xi(n+r)}}{(ab)^{\lfloor \frac{n+r}{2} \rfloor}}\right) \frac{\alpha^{n-r} - \beta^{n-r}}{\alpha - \beta} \frac{\alpha^{n+r} - \beta^{n+r}}{\alpha - \beta}$$

$$= \left(\frac{a^{2-\xi(n-r)}b^{1-\xi(n-r)}}{(ab)^{n-\xi(n-r)}}\right) \frac{\alpha^{n-r} - \beta^{n-r}}{\alpha - \beta} \frac{\alpha^{n+r} - \beta^{n+r}}{\alpha - \beta}$$

$$= \left(\frac{a}{(ab)^{n-1}}\right) \frac{\alpha^{2n} - (\alpha\beta)^{n-r}(\alpha^{2r} + \beta^{2r}) + \beta^{2n}}{(\alpha - \beta)^2}$$

and

$$a^{\xi(n)}b^{1-\xi(n)}q_n^2 = a^{\xi(n)}b^{1-\xi(n)} \left(\frac{a^{2-2\xi(n)}}{(ab)^{2\lfloor \frac{n}{2} \rfloor}}\right) \frac{\alpha^{2n} - 2(\alpha\beta)^n + \beta^{2n}}{(\alpha - \beta)^2}$$

$$= \left(\frac{a}{(ab)^{2\lfloor \frac{n}{2} \rfloor + \xi(n) - 1}}\right) \frac{\alpha^{2n} - 2(\alpha\beta)^n + \beta^{2n}}{(\alpha - \beta)^2}$$

$$= \left(\frac{a}{(ab)^{n-1}}\right) \frac{\alpha^{2n} - 2(\alpha\beta)^n + \beta^{2n}}{(\alpha - \beta)^2}.$$

Therefore,

$$a^{\xi(n-r)}b^{1-\xi(n-r)}q_{n-r}q_{n+r} - a^{\xi(n)}b^{1-\xi(n)}q_n^2$$

$$= \left(\frac{a}{(ab)^{n-1}}\right)\frac{2(\alpha\beta)^n - (\alpha\beta)^{n-r}(\alpha^{2r} + \beta^{2r})}{(\alpha - \beta)^2}$$

$$= \left(\frac{-a}{(ab)^{n-1}}\right)(\alpha\beta)^{n-r}\frac{\alpha^{2r} - 2\alpha^r\beta^r + \beta^{2r}}{(\alpha - \beta)^2}$$

$$= \left(\frac{-a}{(ab)^{n-1}}\right)(-ab)^{n-r}\left(\frac{\alpha^r - \beta^r}{\alpha - \beta}\right)^2$$

$$= (-1)^{n+1-r}\frac{a}{(ab)^{r-1}} \cdot \frac{(ab)^{2\lfloor\frac{r}{2}\rfloor}}{a^{2-2\xi(r)}}q_r^2$$

$$= (-1)^{n+1-r}a^{2\xi(r)-1}(ab)^{1-\xi(r)}q_r^2$$

$$= (-1)^{n+1-r}a^{\xi(r)}b^{1-\xi(r)}q_r^2.$$

$\square$

Theorem 8. (d'Ocagne's identity) *For any two nonnegative integers m and n with $m \geq n$, we have*

$$a^{\xi(mn+m)}b^{\xi(mn+n)}q_m q_{n+1} - a^{\xi(mn+n)}b^{\xi(mn+m)}q_{m+1}q_n = (-1)^n a^{\xi(m-n)}q_{m-n}.$$

Proof. First note that

$$\xi(m+1) + \xi(n) - 2\xi(mn+n) = \xi(m) + \xi(n+1) - 2\xi(mn+m) = 1 - \xi(m-n)$$

and

$$\xi(m-n) = \xi(mn+m) + \xi(mn+n).$$

Using the extended Binet's formula and the above identities, we obtain:

$$a^{\xi(mn+m)}b^{\xi(mn+n)}q_m q_{n+1}$$

$$= \left(\frac{a(ab)^{-n}}{(ab)^{\frac{m-n-\xi(m-n)}{2}}}\right)\frac{\alpha^{m+n+1} + \beta^{m+n+1} - (\alpha\beta)^n(\beta\alpha^{m-n} + \alpha\beta^{m-n})}{(\alpha - \beta)^2}$$

and

$$a^{\xi(mn+n)}b^{\xi(mn+m)}q_{m+1}q_n$$

$$= \left(\frac{a(ab)^{-n}}{(ab)^{\frac{m-n-\xi(m-n)}{2}}}\right)\frac{\alpha^{m+n+1} + \beta^{m+n+1} - (\alpha\beta)^n(\alpha^{m-n+1} + \beta^{m-n+1})}{(\alpha - \beta)^2}.$$

Therefore,

$$a^{\xi(mn+m)}b^{1-\xi(mn+m)}q_m q_{n+1} - a^{\xi(mn-n)}b^{1-\xi(mn-n)}q_{m+1}q_n$$

$$= \left(\frac{(-1)^n a}{(ab)^{\frac{m-n-\xi(m-n)}{2}}}\right)\left(\frac{\alpha^{m-n} - \beta^{m-n}}{\alpha - \beta}\right)$$

$$= (-1)^n a a^{\xi(m-n)-1}q_{m-n}$$

$$= (-1)^n a^{\xi(m-n)}q_{m-n}.$$

$\square$

Theorem 9. (Additional identities) (i) *For any two nonnegative integers m and n,*

$$a^{\xi(mn+m)}b^{\xi(mn+n)}q_m q_{n+1} + a^{\xi(mn+n)}b^{\xi(mn+m)}q_{m-1}q_n = a^{\xi(m+n)}q_{m+n}.$$

This identity is equivalent to the Convolution Property given by

$$a^{\xi(km)}b^{\xi(km+k)}q_m q_{k-m+1} + a^{\xi(km+k)}b^{\xi(km)}q_{m-1}q_{k-m} = a^{\xi(k)}q_k.$$

(ii) *For any two nonnegative integers n and k with $n \geq k$,*

$$\left(a^{1-\xi(n+k)}b^{\xi(n+k)}\right)q_{n+k+1}^2 + \left(a^{\xi(n-k)}b^{1-\xi(n-k)}\right)q_{n-k}^2 = aq_{2n+1}q_{2k+1}.$$

(iii) *For any natural number n,*

$$\sum_{k=1}^{2n} aq_{k-1}q_k = q_{2n}^2$$

and

$$\sum_{k=1}^{2n+1} aq_{k-1}q_k = \left(\frac{a}{b}\right)\left[q_{2n+1}^2 - 1\right].$$

(iv) *If $m \mid n$, then $q_m \mid q_n$.*

(v) *For any two natural numbers n and m, we have $\gcd(q_m, q_n) = q_{\gcd(m,n)}$.*

(vi) *For any nonnegative integer n,*

$$q_{n+2}^2 - q_n^2 = a^{1-\xi(n)}b^{\xi(n)}q_{2n+2} \quad and \quad q_{n+2}^2 + q_n^2 = a^{1-\xi(n)}b^{\xi(n)}q_{2n+2} + 2q_n^2.$$

Consequently,

$$\left(a^{1-\xi(n)}b^{\xi(n)}q_{2n+2}\right)^2 + (2q_n q_{n+2})^2 = \left(a^{1-\xi(n)}b^{\xi(n)}q_{2n+2} + 2q_n^2\right)^2.$$

This identity produces Pythagorean triples involving generalized Fibonacci numbers.

For any three nonnegative integers n, k, j with $k \geq j$,

$$a^{(-1)^n \xi(j) \xi(n+k)} b^{-(-1)^n \xi(j) \xi(n+k)} q_k q_{n+j}$$

$$-a^{(-1)^n \xi(k) \xi(n+j)} b^{-(-1)^n \xi(k) \xi(n+j)} q_j q_{n+k} = (-1)^j q_n q_{k-j}.$$

Proof. We leave the proofs to the reader, since they are similar to the proof of the previous theorem.

Theorem 10. (Sums Involving Binomial Coefficients). *For any nonnegative integer n we have*

$$\sum_{k=0}^{n} \binom{n}{k} a^{\xi(k)} (ab)^{\lfloor \frac{k}{2} \rfloor} q_k = q_{2n}$$

and

$$\sum_{k=0}^{n} \binom{n}{k} a^{\xi(k+1)} (ab)^{\lfloor \frac{k+1}{2} \rfloor} q_{k+1} = a q_{2n+1}.$$

Proof. First note that

$$a \frac{\alpha^k - \beta^k}{\alpha - \beta} = (ab)^{\lfloor \frac{k}{2} \rfloor} a^{\xi(k)} q_k$$

for any nonnegative integer k. Therefore,

$$
\begin{aligned}
\sum_{k=0}^{n} \binom{n}{k} a^{\xi(k)} (ab)^{\lfloor \frac{k}{2} \rfloor} q_k &= \sum_{k=0}^{n} \binom{n}{k} a \frac{\alpha^k - \beta^k}{\alpha - \beta} \\
&= \frac{a}{\alpha - \beta} \left[\sum_{k=0}^{n} \binom{n}{k} \alpha^k - \sum_{k=0}^{n} \binom{n}{k} \beta^k \right] \\
&= \frac{a}{\alpha - \beta} \left[(\alpha + 1)^n - (\beta + 1)^n \right] \\
&= \frac{a}{\alpha - \beta} \left[\left(\frac{\alpha^2}{ab} \right)^n - \left(\frac{\beta^2}{ab} \right)^n \right] \\
&= \frac{a}{(ab)^n} \left(\frac{\alpha^{2n} - \beta^{2n}}{\alpha - \beta} \right) \\
&= q_{2n}.
\end{aligned}
$$

Also,

$$\sum_{k=0}^{n} \binom{n}{k} a^{\xi(k+1)} (ab)^{\lfloor \frac{k+1}{2} \rfloor} q_{k+1} = \sum_{k=0}^{n} \binom{n}{k} a \frac{\alpha^{k+1} - \beta^{k+1}}{\alpha - \beta}$$

$$= \frac{a}{\alpha - \beta} \left[\alpha \sum_{k=0}^{n} \binom{n}{k} \alpha^k - \beta \sum_{k=0}^{n} \binom{n}{k} \beta^k \right]$$

$$= \frac{a}{\alpha - \beta} \left[\alpha (\alpha + 1)^n - \beta (\beta + 1)^n \right]$$

$$= \frac{a}{\alpha - \beta} \left[\alpha \left(\frac{\alpha^2}{ab} \right)^n - \beta \left(\frac{\beta^2}{ab} \right)^n \right]$$

$$= \frac{a}{(ab)^n} \left(\frac{\alpha^{2n+1} - \beta^{2n+1}}{\alpha - \beta} \right)$$

$$= a q_{2n+1}.$$

$\square$

Remark 11 The last two identities can be combined and generalized as follows. For any nonnegative integer r, one can easily verify (either using the generating function together with the differential operator or the extended Binet Formula) that

$$\sum_{k=0}^{n} \binom{n}{k} a^{\xi(k+r)} (ab)^{\lfloor \frac{k}{2} \rfloor + \xi(k)\xi(r)} q_{k+r} = a^{\xi(r)} q_{2n+r}.$$

4. A Further Generalization, Convergence Properties, and an Open Problem

Now, we may take the generalized Fibonacci sequence a bit further by allowing arbitrary initial conditions. So, consider the sequence $\{Q_n\}$, where $Q_0 = C, Q_1 = D$, and

$$Q_n = \begin{cases} aQ_{n-1} + Q_{n-2}, & \text{if n is even} \\ bQ_{n-1} + Q_{n-2}, & \text{if n is odd} \end{cases} \quad (n \geq 2).$$

The following theorem states a relationship between the terms of $\{Q_n\}$ and the terms of $\{q_n\}$, the generalized Fibonacci sequence. In addition, the generating function for the $\{Q_n\}$ is given. First part of this theorem can be proven by induction.

Theorem 12. *Let the sequence $\{Q_n\}$ satisfy the above initial conditions and recurrence relation. Then*

$$Q_n = Dq_n + C \left(\frac{b}{a} \right)^{\xi(n)} q_{n-1}$$

for all $n \geq 1$. Moreover, its generating function is given by

$$G(x) = \sum_{n=0}^{\infty} Q_n x^n = \frac{C + Dx + (aD - abC - C)x^2 + (bC - D)x^3}{1 - (ab + 2)x^2 + x^4}.$$

The $\{q_n\}$-identities we studied in this article can be extended to the sequence $\{Q_n\}$ with minor modifications.

Remark 13 (On Convergence Properties). For the classical Fibonacci sequence $\{F_n\}$, (which is $\{q_m\}$ with $a = b = 1$), it is well-known that the ratios of successive terms $\left\{\frac{F_{n+1}}{F_n}\right\}$ converge to the golden ratio, or golden mean, $\phi = \frac{1+\sqrt{5}}{2}$. Consider the generalization obtained when $a = b$. From Theorem 2, we get

$$\frac{q_{m+1}}{q_m} = \frac{1}{a} \cdot \frac{\alpha^{m+1} - \beta^{m+1}}{\alpha^m - \beta^m} = \frac{\alpha}{a} \cdot \frac{1 - \left(\frac{\beta}{\alpha}\right)^{m+1}}{1 - \left(\frac{\beta}{\alpha}\right)^m}.$$

As a result, $\frac{q_{m+1}}{q_m}$ converges to $\frac{\alpha}{a} = \frac{a+\sqrt{a^2+4}}{2}$. Of course, when $a = 1$, the quadratic irrational $\frac{\alpha}{a}$ is called the golden mean. V. W. de Spinadel in [4] gives the names silver mean and bronze mean to the cases when $a = 2$ and $a = 3$, respectively.

Now, if $a \neq b$, the ratios of successive terms do not converge since

$$\frac{q_{m+1}}{q_m} = a^{\xi(m)} b^{1-\xi(m)} + \frac{q_{m-1}}{q_m}$$

and $\left\{a^{\xi(m)} b^{1-\xi(m)}\right\}$ oscillates between a and b. Therefore, for most sequences in the family of generalized Fibonacci sequences, the ratios of successive terms do not converge. However, it is not hard to see that

$$\frac{q_{2m}}{q_{2m-1}} \longrightarrow \frac{\alpha}{b}, \qquad \frac{q_{2m+1}}{q_{2m}} \longrightarrow \frac{\alpha}{a}, \qquad \text{and} \qquad \frac{q_{m+2}}{q_m} \longrightarrow \alpha + 1.$$

Remark 14 (An Open Problem). Let $a_1, a_2, \ldots, a_k$ be positive integers and define a sequence $\{\overline{q_m}\}$ as follows. Set

$$\overline{q_0} = 0, \quad \overline{q_1} = 1,$$

and for all $m \geq 2$,

$$\overline{q_m} = a_t \overline{q_{m-1}} + \overline{q_{m-2}} \text{ where } m \equiv t + 1 \pmod{k} \text{ for some } t \in \{1, \ldots, k\}.$$

When $k = 2$, $\{\overline{q_m}\}$ is the family of generalized Fibonacci sequences we studied in this paper. It remains open to find a closed form of the generating function and a Binet-like formula for $\{\overline{q_m}\}$, provided they exist.

References

[1] J. Atkins, R. Geist, *Fibonacci numbers and computer algorithms*, College Math. J. **18** (1987), 328-336.

[2] Z. R. Bogdonowicz, *Formulas for the number of spanning trees in a fan*, Appl. Math. Sci. **2**, (2008), 781-786.

[3] P. Chebotarev, *Spanning forests and the golden ratio*, Discrete Appl. Math. **156**, (2008), 813-821.

[4] V. W. de Spinadel, *The metallic mean family and multifractal spectra*, Nonlinear Anal. **36** (1999), 721-745.

[5] J. de Souza, E. M. F. Curado, M. A. Rego-Monteiro, *Generalized Heisenberg algebras and Fibonacci series*, J. Phys. A: Math. Gen. **39** 10415-10425.

[6] M. Embree, L.N. Trefethen, *Growth and decay of random Fibonacci sequences*, R. Soc.London. Proc. Ser. A Math. Phys. Eng. Sci. 455 (1999), no.1987, 2471-2485.

[7] S. Falcon, A. Plaza, *The k-Fibonacci sequence and the Pascal 2-Triangle*, Chaos, Solitons and Fractals **33**(2007), 38-49.

[8] A. J. Feingold, *A hyperbolic GCM Lie algebra and the Fibonacci numbers*, Proc. Amer. Math. Soc. **80** (1980) 379-385.

[9] M. L. Fredman, R. E. Tarjan, *Fibonacci heaps and their uses in improved network optimization algorithms*, J. ACM **34**, (1987), 596-615.

[10] Fibonacci numbers and the Golden section,
`http://www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/`

[11] A. H. George, *Some formulae for the Fibonacci sequence with generalizations*, Fibonacci Quart. **7**(1969), 113-130

[12] V. E. Hoggatt, *Fibonacci and Lucas Numbers*, Fibonacci Association, Santa Clara, CA, 1969.

[13] A. F. Horadam, *A generalized Fibonacci sequence*, Amer. Math. Monthly **68**(1961), 455-459

[14] A. F. Horadam, *Fibonacci number triples*, Amer. Math. Monthly **68**(1961), 751-753.

[15] D. V. Jaiswal, *On a generalized Fibonacci sequence*, LabdevJ. Sci. Tech. Part A **7**(1969), 67-71.

[16] E. Janvresse, B. Rittaud, *How do random Fibonacci sequences grow?*, Probab. Theory Related Fields 142 (2008), no.3-4, 619-648.

[17] S. T. Klein, *Combinatorial representation of generalized Fibonacci numbers*, Fibonacci Quart. **29** (1991) 124-131.

[18] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Wiley, New York, 2001.

[19] A. T. Krassimir, *On a second new generalization of the Fibonacci sequence*, Fibonacci Quart. **24**(1986), no. 4, 362-365.

[20] A. T. Krassimir, A. C. Liliya, S. D. Dimitar, *A new perspective to the generalization of the Fibonacci sequence*, Fibonacci Quart. **23**(1985), no. 1, 21-28.

[21] G. Y. Lee, S. G. Lee, H. G. Shin, *On the k-generalized Fibonacci matrix Q_k*, Linear Algebra Appl. **251**(1997), 73-88.

[22] G. Y. Lee, S. G. Lee, J. S. Kim, H. K. Shin, *The Binet formula and representations of k-generalized Fibonacci numbers*, Fibonacci Quart. **39**(2001), no. 2, 158-164.

[23] M. Lothaire, *Algebraic Combinatorics on Words*, Cambridge University Press, Cambridge, 2005.

[24] G. Michael, *A new proof for an old property*, Fibonacci Quart. **2**(1964), 57-58.

[25] I. Niven, H. Zuckerman, H. Montgomery, *An Introduction to the Theory of Numbers*, Fifth Ed., John Wiley and Sons, 1991.

[26] S. P. Pethe, C. N. Phadte, *A generalization of the Fibonacci sequence.* Applications of Fibonacci numbers, Vol. 5 (St. Andrews, 1992), 465-472.

[27] J. C. Pond, *Generalized Fibonacci Summations*, Fibonacci Quart. **6** (1968), 97-108.

[28] J. Riordan, *Generating function for powers of Fibonacci numbers*, Duke Math. J. **29**(1962), 5-12.

[29] B. Rittaud, *On the average growth of random Fibonacci sequences*, J. Integer Seq. 10 (2007), no.2, Article 07.2.4, 32 pp. (electronic).

[30] G. Sburlati, *Generalized Fibonacci sequences and linear congruences*, Fibonacci Quart. **40** (2002), 446-452.

[31] M. Schork, *Generalized Heisenberg algebras and k-generalized Fibonacci numbers*, J. Phys. A: Math. Theor. **40** (2007), 4207-4214.

[32] A. G. Shannon, A. F. Horadam, *Generalized Fibonacci number triples*, Amer. Math. Monthly **80**(1973), 187-190.

[33] I. Stojmenovic, *Recursive algorithms in computer science courses: Fibonacci numbers and binomial coefficients*, IEEE Trans. Educ. **43** (2000), 273-276.

[34] J.-B. Suck, M. Schreiber, P.Häussler, *Quasicrystals: An Introduction to Structure, Physical Properties, and Applications*, Springer-Verlag, Berlin Heidelberg, 2002.

[35] S. Vajda, *Fibonacci & Lucas Numbers, and the Golden Section, Theory and Applications*, Ellis Horwood Ltd., Chichester, 1989.

[36] D. Viswanath, *Random Fibonacci Sequences and the number 1.13198824...*, Math. Comp. 69(2000), 1131-1155.

[37] J. E. Walton, A. F. Horadam, *Some further identities for the generalized Fibonacci sequence $\{H_n\}$*, Fibonacci Quart. 12(1974), 272-280.

[38] H. S. Wilf, *Generating functionology*, Academic Press, 1990

[39] T. G. Wright, L.N. Trefethen, *Computing Lyapunov constants for random occurences with smooth coefficients*, J. Comput. Appl. Math. 132(2001),no.2, 331-340.

[40] L. J. Zai, L.J. Sheng, *Some properties of the generalization of the Fibonacci sequence*, Fibonacci Quart. **25**(1987), no. 2, 111-117.

[41] V. I. Zubov, J. N. Teixeira Rabelo, *Anharmonic effects in Fibonacci-chain quasicrystals*, Phys. Rev. B (3) **49** (1994), 8671-8678.