On integral representation and algorithmic approaches to the evaluation of combinatorial sums

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Abstract

Integral representation is applied to various problems of algorithmic indefinite and definite summation and for generating combinatorial identities. An integral representation approach to rational summation is compared to known algorithmic approaches. It is shown that the integral representation can be used for practical improvements of known summation algorithms. A new solution to Riordan's problem of combinatorial identities classification is presented.

1 Introduction and basic notions

Integral representation (contour integration) is a classical technique for evaluating infinite sums ([13]), evaluating combinatorial sums ([8]), proving combinatorial identities ([6, 8]), finding inverse combinatorial relations ([9, 28, 30]), etc. In this paper we use ideas of integral representation to improve known algorithms of indefinite and definite rational summation (Section 2), show that a similar technique works for evaluation of single and double Abel-type sums (Section 3) and present an improved integral representation approach to the problem of classification and generation of combinatorial identities (Section 4). Examples obtained by our Maple implementation are given.

1.1 The notion and properties of the res operator

We briefly recall the definition and manipulation rules of the **res** operator, together with an outline of applications of this notion in combinatorial summation problems.

Let G be a set of formal power series in w containing only finitely many terms with negative powers over a field K.

• The *order* of a monomial $c_k w^k$ is k.
• The *order of series* $C(w) = \sum_k c_k w^k$ from G is the minimal order of monomials with nonzero coefficients.
• Denote by G_k a set of series of order k, then $G = \cup_{k=-\infty}^{\infty} G_k$.
• Two series $A(w) = \sum_k a_k w^k$ and $B(w) = \sum_k b_k w^k$ from G are equal if and only if $a_k = b_k$ for all k. We can introduce in G operations of addition, multiplication, substitution, inversion, differentiation and integration (as in the well-known Cauchy algebra). After this G forms a field w.r.t. to addition and multiplication.
• For $C(w)$ from G define the *formal residue* as $\mathbf{res}_w C(w) = c_{-1}$.
• Let $A(w) = \sum_{k=0}^{\infty} a_k w^k$ be a generating function for a sequence $\{a_k\}$. Then

$$a_k = \mathbf{res}_w A(w) w^{-k-1}, \quad k = 0, 1, 2, \ldots$$

For example, one of the possible representations of the binomial coefficient sequence is

$$\binom{n}{k} = \mathbf{res}_w (1+w)^n w^{-k-1}, \quad k = 0, 1, 2, \ldots, n, \tag{1}$$

for natural n. Similarly, for an exponential coefficient

$$\frac{\alpha^k}{k!} = \mathbf{res}_w \exp(\alpha w) w^k, \quad k = 0, 1, 2, \ldots, \tag{2}$$

for an arbitrary complex constant α.

There are several properties (rewriting rules) for the operator **res** which immediately follow from its definition and general properties of Cauchy algebras. We list only a few of them which will be used in this paper. Let $A(w) = \sum_{k=0}^{\infty} a_k w^k$ and $B(w) = \sum_{k=0}^{\infty} b_k w^k$ be generating functions from G.

R1. (**res** removal)

$$\mathbf{res}_w A(w) w^{-k-1} = \mathbf{res}_w B(w) w^{-k-1}$$

for all $k = 0, 1, 2, \ldots$ if and only if $A(w) = B(w)$.

R2. (linearity) for any α, β from K

$$\alpha\, \mathbf{res}_w A(w) w^{-k-1} + \beta\, \mathbf{res}_w B(w) w^{-k-1} = \mathbf{res}_w (\alpha A(w) + \beta B(w)) w^{-k-1}.$$

R3. (substitution) for $f(w)$ from G_k $(k \geq 1)$ and $A(w)$ any element of G or for $A(w)$ polynomial and $f(w)$ any element of G including a constant

$$\sum_{k=0}^{\infty} f(w)^k\, \mathbf{res}_z (A(z) z^{-k-1}) = A(f(w))$$

R4. (inversion) for $f(w)$ from G_0

$$\sum_{k=0}^{\infty} z^k\, \mathbf{res}_w (A(w) f(w)^k w^{-k-1}) = \left. \frac{A(w)}{f(w) \frac{d}{dw} h(w)} \right|_{w=g(z)},$$

where $g(w)$ from G_1 is the inverse in G of the series $h(w) = \frac{w}{f(w)}$ from G_1.

If a formal power series $C(w)$ from G converges in a punctured neighbourhood of zero, then the definition of $\mathbf{res}_w C(w)$ coincides with the usual definition of $\mathbf{res}_{w=0} C(w)$ used in the theory of analytic functions (Grothendieck residue). The formula

$$a_k = \mathbf{res}_w A(w) w^{-k-1}, \quad k = 0, 1, 2, \ldots,$$

where $A(w) = \sum_{k=0}^{\infty} a_k w^k$, is an analog of the integral Cauchy formula

$$a_k = \frac{1}{2\pi i} \oint_{|w|=\rho} A(w) w^{-k-1} dw$$

for coefficients of the Taylor series in a punctured neighbourhood of zero. Similarly it is possible to introduce the definition of formal residue at the point of infinity, logarithmic residue and the residues theorem.

1.2 Simple applications

Most of the time one can think of **res** as a contour or definite integral. Accordingly, we call a representation of an expression in terms of **res** an integral representation. Typically, the application of the integral representation for evaluation e.g. of a definite sum $\sum_{k=a}^{b} f(k)$ uses the following steps:

1. replace $f(k)$ by an integral representation $\mathbf{res}_w F(k, w)$;

2. in the resulting problem

$$\sum_{k=a}^{b} \mathbf{res}_w F(k, w)$$

change the order of the operators **res** and $\sum$ using linearity of **res** obtaining

$$\mathbf{res}_w \sum_{k=a}^{b} F(k, w);$$

solve summation problem under the **res** sign: find $G(w, a, b) = \sum_{k=a}^{b} F(k, w)$ (observe, that in many cases this summation problem is much simpler than initial summation problem, usually it degenerates simply to geometric summation in k);

3. compute closed form $g(a, b) = \mathbf{res}_w G(w, a, b)$ which gives an answer to the original summation problem.

A similar sequence of steps (probably involving multiple residues, and use of more than just linearity rule) is applied to other problems. Steps **1** and **3** above do not look algorithmic for an arbitrary $f(k)$, but as we will show further in many particular cases it is possible to perform them in algorithmic fashion.

We remark also that **res** operator commutes not only with the definite summation operator, but with the shift operator E_k, difference operator $\Delta_k = E_k - 1$ and indefinite summation operator $\sum_k$. If a summation problem under the **res** sign becomes a geometric summation problem, then the usual indefinite summability condition has a straightforward analog in the integral representation. A closed form function $f(k)$ is said to have a closed form sum $g(k)$ (usually written as $g(k) = \sum_k f(k)$) if $(E_k - 1)g(k) = f(k)$. Let $f(k)$ have an integral representation $\mathbf{res}_w F(k,w)$ with kernel $F(k,w)$ being geometric with respect to k, i.e.

$$E_k F(k,w) = q(w)F(k,w).$$

Then

$$(E_k - 1)F(k,w) = (q(w) - 1)F(k,w),$$

and

$$\sum_k F(k,w) = \frac{1}{q(w)-1}F(k,w). \tag{3}$$

In other words, in integral representation with the geometric in k kernel the application of the operator Δ_k corresponds to the multiplication of the kernel by $(q(w) - 1)$, and the application of the operator $\sum_k$ corresponds to the multiplication of the kernel by $\frac{1}{q(w)-1}$ [1]. An operator form of the summability equality

$$(E_k - 1)\sum_k = \sum_k (E_k - 1) = 1$$

corresponds to the trivial "summation unit" cancellation equality

$$(q(w) - 1)\frac{1}{q(w)-1} = 1.$$

If $f(k) = \mathbf{res}_w F(k,w)$ and $g(k) = \mathbf{res}_w G(k,w)$, then the summability condition $(E_k - 1)g(k) = f(k)$ translates into divisibility of $F(k,w)$ by $(q(w) - 1)$. Observe, that $f(k)$, $g(k)$ do not need to be geometric or hypergeometric in order for $F(k,w)$, $G(k,w)$ to be geometric. The reasonable question here is, how often the kernel of an integral representation of the given closed form function $f(k)$ is geometric? In our opinion often enough in order to try to apply the integral representation in algorithmic fashion, as will be shown in later sections.

We first demonstrate an application of the above mentioned scheme (**1-3**) to the problem of indefinite summation with the help of a simple example.

Example 1. (Indefinite Halmos's sum) We need to evaluate an indefinite sum $S(n) = \sum_k (-1)^k \binom{n}{k}$. Using (1) write (step **1**)

$$S(n) = \sum_k (-1)^k \ \mathbf{res}_w (1+w)^n w^{-k-1} = (\text{step } \mathbf{2}) = \ \mathbf{res}_w \frac{(1+w)^n}{w} \sum_k (-w)^{-k} =$$

(we ended up with simple indefinite geometric summation problem)

$$= \ \mathbf{res}_w \frac{(1+w)^n}{w}\left(-\frac{\left(-\frac{1}{w}\right)^k w}{1+w}\right) = \ \mathbf{res}_w \left(-(1+w)^{n-1}\left(-\frac{1}{w}\right)^k\right) =$$

[1] This last value is sometimes referred to as a "summation unit" in integral representation literature.

$$(-1)^{k-1}\,\mathbf{res}_w(1+w)^{n-1}w^{-k} = (\text{using (1) backwards (step } \mathbf{3})) = (-1)^{k-1}\binom{n-1}{k-1}.$$

Observe that the cancellation of $(1+w)$ in the denominator allowed us to use (1) for the second time. As was mentioned above, such a cancellation plays an important role in the decision procedure of summability for different classes of summands.

2 Rational summation revised

Consider

$$y(k+1) - y(k) = F(k), \tag{4}$$

where $F(k)$ is a rational function over a field K of characteristic 0. The *decomposition problem* is to find whether (4) has a rational solution, and if it does not, then to extract an additive rational part $R(k)$ from the solution such that the remaining part satisfies a simpler difference equation, where the denominator of the new right-hand side has the lowest possible degree. This gives an equality

$$\sum_k F(k) = R(k) + \sum_k H(k), \tag{5}$$

where $H(k)$ is a rational function whose denominator has the lowest possible degree. Algorithmic treatment of rational summation and decomposition problems started with the work of S.A.Abramov [2, 3]. There were a number of algorithms and improvements developed over the following years, see e.g. [5, 21, 24, 26, 17] (in particular [26] gives complete overview of these algorithms and improvements to them). Most of these algorithms' description explicitly avoid polynomial factorization in $K[k]$. Before discussing the issues with such an approach we recall a criterion of rational summability as found in [4, 5] (we essentially quote the definitions and the criterion from [5] here).

2.1 Criterion of rational summability

Consider (5), assuming w.l.o.g. that $F(k)$ is a proper rational function. We temporarily replace the coefficient field K by its algebraic closure $\overline{K}$. The partial fraction decomposition of $F(k)$ has the form

$$F(k) = \sum_{i=1}^{m}\sum_{j=1}^{t_i} \frac{\beta_{ij}}{(k-\alpha_i)^j}. \tag{6}$$

Write $\alpha_i \sim \alpha_j$ if $\alpha_i - \alpha_j$ is an integer. Obviously, $\sim$ is an equivalence relation in the set $\{\alpha_1, \ldots, \alpha_m\}$. Each of the corresponding equivalence classes has a largest element in the sense that the other elements of the class are obtained by subtracting positive integers from it. Let $\alpha_1, \ldots, \alpha_v$ be the largest elements of all the classes ($v \leq m$). Then (6) can be rewritten as

$$F(k) = \sum_{i=1}^{v}\sum_{j=1}^{l_i} M_{ij}(E_k)\frac{1}{(k-\alpha_i)^j}. \tag{7}$$

Here $M_{ij}(E_k)$ is a linear difference operator with constant coefficients (a polynomial in E_k over $\overline{K}$). Let $F(k)$ have the form (7) and suppose that (4) possesses a solution $R(k) \in K(k)$. The rational function $R(k)$ can be written in a form analogous to (7):

$$\sum_{i=1}^{v} \sum_{j=1}^{l_i} L_{ij}(E_k) \frac{1}{(k-\alpha_i)^j}. \tag{8}$$

This presentation is unique and therefore

$$L_{ij}(E_k)(E_k - 1) = M_{ij}(E_k). \tag{9}$$

From here we read the **rational summability criterion**:

> A necessary and sufficient condition for existence of a rational solution of (4) is that for all $i = 1, \ldots, v$; $j = 1, \ldots, l_i$ there is an operator $L_{ij}(E_k)$ s.t. (9) holds.

Then, (4) has the solution (8) and all other rational solutions of (4) can be obtained by adding arbitrary constants. If at least one polynomial $M_{ij}(E_k)$ is not divisible by $E_k - 1$ then (4) has no rational solution. We want then to construct (5). Consider one term from (7) writing it for simplicity in the form

$$M(E_k) \frac{1}{(k-\alpha)^j}, \; j \geq 1,$$

compute the quotient $L(E_k)$ and the remainder w:

$$M(E_k) = L(E_k)(E_k - 1) + w, \quad w \in \overline{K}, \tag{10}$$

and write the right-hand side of (5) in the form

$$L(E_k) \frac{1}{(k-\alpha)^j} + \sum_k \frac{w}{(k-\alpha)^j}. \tag{11}$$

This gives a solution to the decomposition problem for this single term, since the denominator of the rational function under the sign of the indefinite sum has obviously the lowest possible degree.

Note that instead of (10) one can consider the reduction modulo $E_k - 1$ of the form

$$M(E_k) = V(E_k)(E_k - 1) + vE_k^c, v \in \overline{K}, \tag{12}$$

where c is some convenient nonnegative integer. It is easy to see that if $c < \deg M(E_k)$ then $\deg V(E_k) \leq \deg L(E_k)$. If $M(E_k) = M_{ij}(E_k)$ then one can take $c = \delta_i$, where δ_i is s.t. $M_{i,l_i}(E_k)$ is divisible by $E_k^{\delta_i}$ and is not divisible by $E_k^{\delta_i+1}$.

Now we will describe an analog of this criterion in integral representation framework. For this we will use the following representation

$$\frac{1}{(k-\alpha)^j} = \frac{1}{(j-1)!} \int_0^\infty e^{-(k-\alpha)u} u^{j-1} du = \frac{1}{(j-1)!} \int_0^\infty e^{-ku} \left(e^{\alpha u} u^{j-1} \right) du. \tag{13}$$

Consider again only one term $M(E_k)\frac{1}{(k-\alpha)^j}$, $(j \geq 1)$, from (7). Let for clarity

$$M(E_k) = \beta_t E_k^t + \ldots + \beta_1 E_k + \beta_0, \quad L(E_k) = \gamma_{t-1} E_k^{t-1} + \ldots + \gamma_1 E_k + \gamma_0.$$

Consider two commutative polynomials with the same coefficients

$$P(X) = \beta_t X^t + \ldots + \beta_1 X + \beta_0, \quad Q(x) = \gamma_{t-1} X^{t-1} + \ldots + \gamma_1 X + \gamma_0.$$

Noting, that

$$\beta_l E_k^l \frac{1}{(k-\alpha)^j} = \frac{1}{(j-1)!}\int_0^\infty \beta_l e^{-lu} e^{-ku}\left(e^{\alpha u} u^{j-1}\right) du, \quad 0 \leq l \leq t,$$

write

$$M(E_k)\frac{1}{(k-\alpha)^j} = \frac{1}{(j-1)!}\int_0^\infty P(e^{-u}) e^{-ku}\left(e^{\alpha u} u^{j-1}\right) du,$$

i.e. the kernel of integral representation of the summand is geometric in k with the base e^{-u}. Now (as in (3))

$$\sum_k M(E_k)\frac{1}{(k-\alpha)^j} = \frac{1}{(j-1)!}\int_0^\infty \frac{P(e^{-u})e^{-ku}}{e^{-u}-1}\left(e^{\alpha u} u^{j-1}\right) du.$$

If $M(E_k) = L(E_k)(E_k - 1)$ then $P(e^{-u}) = (e^{-u} - 1)Q(e^{-u})$,

$$\sum_k M(E_k)\frac{1}{(k-\alpha)^j} = \frac{1}{(j-1)!}\int_0^\infty Q(e^{-u}) e^{-ku}\left(e^{\alpha u} u^{j-1}\right) du,$$

and we can use (13) backwards to get a closed form expression for the result of the rational summation. In other words, divisibility of $M(E_k)$ by $E_k - 1$ condition is equivalent to the divisibility of $P(e^{-u})$ by $(e^{-u} - 1)$ (or $P(X)$ by $(X - 1)$). The last in turn means that $P(1) = 0$ or $\sum_{l=0}^t \beta_l = 0$ and this condition has to hold for all terms (for all i, j) in (7) in order for $F(k)$ to be rational summable.

Note that the property of coefficients of full partial fraction decomposition proven in [23] is immediate corollary of summability criterion (divisibility of $M_{ij}(E_k)$ by $E_k - 1$ in (7)).

If $P(1) = w \neq 0$, then $P(X) - w$ is divisible by $X - 1$. In this case (10) corresponds to $P(X) = Q(X)(X - 1) + w$, and

$$\frac{1}{(j-1)!}\int_0^\infty Q(e^{-u}) e^{-ku}\left(e^{\alpha u} u^{j-1}\right) du,$$

provides the rational part of the decomposition (11) while

$$\frac{1}{(j-1)!}\int_0^\infty \frac{we^{-ku}}{e^{-u}-1}\left(e^{\alpha u} u^{j-1}\right) du$$

provides the nonrational part of the decomposition (11).

It is easy to see, that $P(X) - wX^c$ for $1 \leq c \leq t$ will be also divisible by $(X-1)$, which leaves us with the same amount of flexibility in expressing the rational decomposition result as equation (12) does. Needles to say that very similar observations hold in the case of the definite rational summation.

2.2 Rational summation algorithms

Let $F(k) = \frac{f(k)}{g(k)}$. Define the dispersion of $F(k)$ ($\mathbf{dis}F(k)$) [2] to be the maximal integer distance between roots of the denominator $g(k)$. It can be computed e.g. as the largest nonnegative integer root of the polynomial

$$r(h) = Res_k(g(k), g(k+h)).$$

Denote $\rho = \mathbf{dis}F(k)$. If $\rho = 0$ than we can take in (5) $R(k) = 0$ and $H(k) = F(k)$ (see [2, 5, 26]).

Now, let $\rho > 0$. All algorithms mentioned above carefully avoid factorization in $K[k]$ and fall into one of the two categories.

Iterative (Hermite reduction like) algorithms will start with $R(k) = 0$ and $H(k) = F(k)$ and decrease dispersion of $H(k)$ by 1 at each iteration, reducing nonrational part $H(k)$ and growing rational part $R(k)$. The number of iterations is equal to ρ.

Non-iterative (Ostrogradsky analog) algorithms first build universal denominators $u(k)$ and $v(k)$ such that denominator of $R(k)$ will divide $u(k)$, that denominator of $H(k)$ will divide $v(k)$ and then reduce the problem to linear algebra, solving a system of linear equations with size $\sim \deg u(k)$ (see [24, 5, 26]). In its turn, usually $\deg u(k) = \Theta(\rho)$. The choice of $u(k)$ of the lowest possible degree is obviously crucial here. In [17] an algorithm which a gives sharp bound $u(k)$ in the case when $F(k)$ is rational summable ($H(k) = 0$) is presented.

In both of classes of algorithms if $\rho = \mathbf{dis}F(k) >> \deg g(k)$ the complexity of rational function decomposition is defined by the value of ρ.

Another issue with these algorithms is as follows. If $H(k) \neq 0$ in (5), then the answer to the decomposition problem is not unique. Also summation algorithms ensure that the degree of the denominator of $H(k)$ is the lowest possible, they do not guarantee that the degree of the denominator of $R(k)$ is minimal. For example,

$$\sum_k 1/10\, \frac{125\,k^3 - 153\,k^2 - 20\,k - 270 + k^5 - 21\,k^4}{(k+1)\,(k-9)\,(k-3)\,k\,(k-10)\,(k-4)} =$$

$$= -1/5\, \frac{2\,k^5 - 35\,k^4 + 180\,k^3 - 345\,k^2 + 195\,k + 30}{(k-10)\,(k-4)\,(k-3)\,(k-2)\,k\,(k-1)} + 1/10 \sum_k \frac{1}{k} =$$

$$= 1/10\, \frac{6\,k^7 - 249\,k^6 + 4250\,k^5 - 38235\,k^4 + 191998\,k^3 - 517302\,k^2 + 624180\,k - 151200}{(k-10)\,(k-9)\,(k-8)\,(k-7)\,(k-6)\,(k-5)\,(k-4)\,k} +$$

$$+1/10 \sum_k \frac{1}{k-10} =$$

$$= \frac{1}{k\,(k-10)\,(k-4)} + 1/10 \sum_k \frac{1}{k-4},$$

and first two answers are typical for implementation of rational summation in different CAS's. The problem of choosing the "right" value of c in the remainder in (12) is easy to solve ([26]) in cases when only one shift equivalence class is presented in the partial fraction decomposition of $F(k)$. When there is more than one shift equivalence class, the desire of authors to avoid factorization does not allow an easy solution to the problem of minimization of the denominator of the rational part $R(k)$.

2.3 Do not avoid factorization and follow criterion

It was already observed that while solving the rational decomposition problem (5) factorization in $K[k]$ should not be avoided. It is shown in [22] how the use of factorization improves timings of computing $\mathbf{dis}F(k)$.

After computing $\mathbf{dis}F(k)$ we can use factorization (which is already performed and which is effective [18]) to easily split $F(k)$ into shift equivalence classes and explicitly build (7) using simple observations :
a) if $\deg g_1(k) \neq \deg g_2(k)$ then linear factors over $\overline{K}$ of the denominators $g_1(k)$ and $g_2(k)$ fall into different shift equivalence classes in (7);
b) finding among given polynomials of equal degree those which are shift equivalent is an easy task ([22]);
c) full partial fraction decomposition over $\overline{K}$ is effective [7] at least when $K = Q$;
d) computing a quotient and remainder in (10) is trivial (e.g., remainder is obtained by substitution of 1 instead of E_k into $M(E_k)$ in (10)).

Treating different shift equivalence classes separately will allow one to minimize the degree of the denominator of the rational part in each class as in [26].

The complexity of partial fraction decomposition does not depend on the dispersion of the given rational function, which means that at least in the case $\mathbf{dis}F(k) >> \deg g(k)$ this leads to practical and efficient algorithms. Efficiency of this straightforward solution is confirmed by our prototype implemented in Maple. Compare, for example, the result of computation of

$$\sum_k -\frac{-7350 - 14099\,k^2 - 9198\,k^3 - 14400\,k + k^8 + 404\,k^4 + 4\,k^7 + 8\,k^5 + 8\,k^6}{(k^3 + 3\,k^2 + 4\,k + 3)\,(k-49)\,(k+1)\,(k^3+k+1)\,(k-50)\,k} =$$

$$\frac{k^4 + k^2 + 50}{(k^3 + k + 1)\,(k-50)\,k},$$

which takes standard Maple rational summation procedure (Ostrogradski analog) 4.77 seconds, to the result of the same summation problem

$$\frac{1}{k-50} - \frac{1}{k} + \frac{k^2+1}{k^3+k+1},$$

produced in 0.24 seconds on the same machine using full partial fraction decomposition, and treating the shift equivalence classes separately.

There are even more reasons not to avoid partial fraction decomposition in the case of the definite rational summation. We refer here to the summation problems of the form $\sum_{k=0}^{n} f(k,n)$, where partial fraction decomposition of $f(k,n)$ w.r.t. k has pairs of terms as, e.g. $\frac{1}{nk+1} - \frac{1}{n(n-k)+1}$ or $\frac{1}{k^2+1} - \frac{1}{(n-k)^2+1}$. These kinds of problems are not directly treatable by known summation algorithms, which avoid factorization. For example, the "W-Z" method is not applicable here, because such terms are not proper hypergeometric. The usual answer from computer algebra systems for this type of summation involves a linear combination of values of the Ψ function, which is equivalent to 0 but is not recognized as such. After performing full partial fraction decomposition it takes little effort to find such cases. The use of the integral representation is even more advisable here, because after performing geometric definite summation under integral sign the terms in the kernel of integral representation corresponding to the above terms of the input expression will cancel each other.

3 Integral representation and Abel type sums

Abel-type sums and combinatorial identities with them [1] are widely used in combinatorial enumeration. In [28]–[30] the classical generating functions technique was used to prove different identities involving Abel-type sums. In [20] the "$W-Z$" method was used to check correctness of recurrences related to the Abel-type identities. Evaluation of the Abel-type sums with the help of the integral representation, considered in this section, is interesting for several reasons:

– it is a direct method of evaluation of Abel-type sums (methods of [25] can not be applied to Abel-type sums, because they are not of the hypergeometric type);

– detailed analysis of the condition of summability for Abel-type sums ((22)–(24) and (20)) leads to new formulae involving double sums of the same kind ((25)–(26));

– finally, it is easy to generate multivariate analogs of Abel-type identities based on the same form of integral representation (see [8], §§ 3.3, 5.1 – 5.3), their q-analogs [19], some applications of Abel-type identities in [8] § 3.2, etc).

Proposition 1 *Let i,j=0,1,2,...; x,y — real such that* $x, y \neq 0, -1, -2, \ldots$ *Then*

$$\sum_{k=0}^{n} \frac{(x+k)^{k-i}(y+n-k)^{n-k-j}}{k!(n-k)!} = \sum_{s=0}^{N} c(s)\frac{(x+y+n)^{n-s}}{(n-s)!}, \tag{14}$$

where

$$\begin{cases} N = \min(n, i+j-1), \quad c(s) = \sum_{t=0}^{s} q(t,i,x)p(s-t,j,y), & \textit{for } i>0, j\geq 0, \\ N = n, \quad c(s) = 1, & \textit{for } i=0, j=0, \end{cases} \tag{15}$$

and values p, q are found as

$$p(t,i,x) = \begin{cases} \frac{1}{x^i}, \quad t=0, \\ \frac{1}{x}p(t,i-1,x) - \frac{1}{x}p(t-1,i-1,x+1), \quad 0<t\leq i, \\ 0, \quad \textit{otherwise}, \end{cases} \tag{16}$$

$$q(l,i,x) = \begin{cases} \sum_{t=0}^{l} p(t,i,x), 0 \leq l \leq i-1, \\ 0, \quad \textit{otherwise}. \end{cases} \tag{17}$$

Proof of this proposition uses the method of integral representation and splits into the following lemmas.

Lemma 1 *Let numbers $p(t,i,x)$ be defined as coefficients of expansion (18) in the polynomial basis $\frac{(x+k)^t}{t!}$, $t = 0, 1, \ldots, i$,*

$$\frac{(x+k)^{k-i}}{k!} = \sum_{t=0}^{i} p(t,i,x)\frac{(x+k)^{k-t}}{(k-t)!}. \tag{18}$$

Then coefficients $p(t,i,x)$ satisfy (16), and

$$\sum_{t=0}^{i} p(t,i,x) = 0. \tag{19}$$

Proof follows from representation

$$\begin{aligned}
T(k,i,x) &= (x+k)^{k-i}/k! = \frac{(x+k)^{k-i+1}}{x\cdot k} - \frac{(x+k)^{k-i}}{x\cdot(k-1)!} = \\
&= \frac{1}{x}T(k,i-1,x) - \frac{1}{x}T(k-1,i-1,x+1) = \\
&= \frac{1}{x}\sum_{t=0}^{i-1} p(t,i-1,x)\frac{(x+k)^{k-t}}{(k-t)!} - \frac{1}{x}\sum_{t=0}^{i-1} p(t,i-1,x+1)\frac{(x+k)^{k-t+1}}{(k-t+1)!} = \\
&= \frac{1}{x}\sum_{t=0}^{i-1} p(t,i-1,x)\frac{(x+k)^{k-t}}{(k-t)!} - \frac{1}{x}\sum_{t=1}^{i} p(t-1,i-1,x+1)\frac{(x+k)^{k-t}}{(k-t)!}.
\end{aligned}$$

Comparing coefficients near $\frac{(x+k)^{k-t}}{(k-t)!}$ in the last expansion and in (18) we obtain (16).

The proof of (19) is by induction on i, taking into account that

$$R_i(x) = \sum_{t=0}^{i} p(t,i,x) = \frac{1}{x}\sum_{t=0}^{i-1} p(t,i-1,x) - \frac{1}{x}\sum_{t=0}^{i-1} p(t,i-1,x+1) =$$

$$= \frac{1}{x}R_{i-1}(x) - \frac{1}{x}R_{i-1}(x+1).$$

■

Lemma 2 *Consider polynomial $P_i(u,x) = \sum_{t=0}^{i} p(t,i,x)u^t$ with $p(t,i,x)$ from (16). It can be represented as*

$$P_i(u,x) = (1-u)Q_i(u,x), \quad Q_i(u,x) = (1-u)^{-1}P_i(u,x), \tag{20}$$

where

$$Q_i(u,x) = \sum_{t=0}^{i-1} q(t,i,x)u^t,$$

and $q(t,i,x)$ are from (17).

Proof. Formula (19) is equivalent to $P_i(1,x)=0$, which proves (20). Since

$$Q_i(u,x)=(1-u)^{-1}P_i(u,x)=(1+\sum_{t=1}^{\infty}u^t)P_i(u,x),$$

comparison of coefficients near u^t gives the relation (17). ■

Lemma 3 *The following integral representations hold*

$$\frac{(x+k)^{k-i}}{k!}=\sum_{k=0}^{n}\ \mathbf{res}_v P_i(v,x)\exp((x+k)v)v^{-k-1}, \tag{21}$$

$$\sum_{k=0}^{n}\frac{(x+k)^{k-i}}{k!}\cdot\frac{(y+n-k)^{n-k-j}}{(n-k)!}=$$

$$=\ \mathbf{res}_u\frac{P_i(u,x)}{1-u}\cdot P_j(u,y)\exp((x+y+n)u)u^{-n-1}= \tag{22}$$

$$=\ \mathbf{res}_u Q_i(u,x)P_j(u,y)\exp((x+y+n)u)u^{-n-1}. \tag{23}$$

Proof. From (18), integral representations

$$\begin{aligned} p(t,i,x) &= \mathbf{res}_u P_i(u,x)u^{-t-1},\\ (x+k)^{k-t}/(k-t)! &= \mathbf{res}_v \exp((x+k)v)v^{-k+t-1}, \end{aligned}$$

and linearity and substitution rules for **res** it follows, that

$$\frac{(x+k)^{k-i}}{k!}=\sum_{t=0}^{i}p(t,i,x)\frac{(x+k)^{k-t}}{(k-t)!}=\sum_{t=0}^{i}\ \mathbf{res}_u P_i(u,x)u^{-t-1}\ \mathbf{res}_v\exp((x+k)v)v^{-k+t-1}=$$

(we can extend the summation to ∞ here because $p(t,i,x)=0$ for $t>i$)

$$=\sum_{t=0}^{\infty}\ \mathbf{res}_u P_i(u,x)u^{-t-1}\ \mathbf{res}_v\exp((x+k)v)v^{-k+t-1}=$$

$$=\ \mathbf{res}_v\left(\exp((x+k)v)v^{-k-1}\left\{\sum_{t=0}^{\infty}v^t\ \mathbf{res}_u P_i(u,x)u^{-t-1}\right\}\right)=$$

$$=\ \mathbf{res}_v\left(\exp((x+k)v)v^{-k-1}\left\{P_i(u,x)\right\}\Big|_{u=v}\right).$$

■

From (21)–(22) for each multiplier of summand in (14) using linearity and inversion rules of the operator **res**, we obtain

$$\sum_{k=0}^{n}\frac{(x+k)^{k-i}}{k!}\cdot\frac{(y+n-k)^{n-k-j}}{(n-k)!}=$$

$$= \sum_{k=0}^{n} \mathbf{res}_v \left(P_i(v,x) \exp((x+k)v) v^{-k-1} \, \mathbf{res}_u P_j(u,y) \exp((y+n-k)u) u^{-n+k-1} \right) =$$

(we can extend the summation to ∞ here using standard properties of residues)

$$= \sum_{k=0}^{\infty} \mathbf{res}_v \left(P_i(v,x) \exp((x+k)v) v^{-k-1} \, \mathbf{res}_u P_j(u,y) \exp((y+n-k)u) u^{-n+k-1} \right) =$$

$$= \mathbf{res}_u \left(P_j(u,y) \exp((y+n)u) u^{-n-1} \cdot \right.$$

$$\left. \cdot \left\{ \sum_{k=0}^{\infty} (\exp(-u)u)^k \, \mathbf{res}_v P_i(v,x) \exp(xv) (\exp v)^k v^{-k-1} \right\} \right) =$$

$$= \mathbf{res}_u \left(P_j(u,y) \exp((y+n)u) u^{-n-1} \cdot \left\{ \frac{P_i(v,x) \exp(xv)}{1-v} \right\} |_{v=u} \right),$$

which gives (22).

Formula (23) follows from (22), taking into account (20).

Writing in (23) the coefficient near u^n of the series under the **res** sign, we obtain (14) and first clause of (15). Similarly from (22) for $i = j = 0$, taking into account $P_0(u,x) = P_0(u,y) = 1$, we obtain in the right hand side of (14) the desired expression $\sum_{s=0}^{n} \frac{(x+y+n)^{n-s}}{(n-s)!}$, which completes the proof of proposition. ■

Corollary 1 *With the same conditions as in Proposition 1 and $\alpha = const$*

$$\sum_{r=0}^{n} \sum_{k=0}^{r} \frac{(\alpha - r)^{(n-r)}}{(n-r)!} \cdot \frac{(x+k)^{k-i}}{k!} \cdot \frac{(y+r-k)^{(n-k-j)}}{(r-k)!} =$$

$$= \mathbf{res}_u \exp((\alpha + x + y)u) P_i(u,x) P_j(u,y) (1-u)^{-2} u^{-n-1} = \tag{24}$$

$$= \begin{cases} \sum_{s=0}^{n} (\alpha + x + y)^{n-s} (s+1)/(n-s)!, & \text{if } i = j = 0, \\ \sum_{s=0}^{i-1} (\alpha + x + y)^{n-s}/(n-s)! \left(\sum_{t=0}^{s} \sum_{l=0}^{t} q(l,i,x) \right), & i = 1,2,\ldots, j = 0, \\ \sum_{s=0}^{i+j-2} (\alpha + x + y)^{n-s}/(n-s)! \left(\sum_{t=0}^{s} q(t,i,x) q(s-t,j,y) \right), & i,j = 1,2,\ldots. \end{cases} \tag{25}$$

For small α ($\alpha \neq 1$)

$$\sum_{n=0}^{\infty} \sum_{r=0}^{n} \alpha^n \exp(-\alpha n) \frac{(x+k)^{k-i}}{k!} \cdot \frac{(y+n-k)^{n-k-j}}{(n-k)!} =$$

$$= \begin{cases} \exp((x+y)\alpha)/(1-\alpha)^2, & \text{if } i = j = 0, \\ \exp((x+y)\alpha) Q_i(\alpha,x)/(1-\alpha), & i = 1,2,\ldots, j = 0, \\ \exp((x+y)\alpha) Q_i(\alpha,x) Q_j(\alpha,y), & i,j = 1,2,\ldots. \end{cases} \tag{26}$$

Proof of (24)–(25) uses a scheme similar to the above and integral representation (22) for the sum (14). One can easily obtain the following integral representations for sums (24) and (26):

$$\mathbf{res}_u \exp((\alpha + x + y)u)P_i(u, x)P_j(u, y)(1 - u)^{-2}u^{-n-1},$$

$$\mathbf{res}_u(\exp((x + y)\alpha)P_i(\alpha, x)P_j(\alpha, y)(1 - \alpha)^{-2}u^{-n-1}).$$

These formulae generate (taking into account (20)) the right hand sides of (25) and (26) for different values i and j.

Note that the summability condition in (14) for $i = 1, 2, \ldots$ is equivalent to the representability of the polynomial $P_i(u, x)$ as $(1-u)Q_i(u, x)$, which allows one to remove the factor $\frac{1}{(1-u)}$ under the **res** sign. This effect is analogous to the summability condition of rational function in the previous section. Note also, that the inversion formula of Lagrange and cancellation effect was used twice when obtaining (25) and (26). That is why the result of evaluation of double sum for $i, j = 1, 2, \ldots$ in these formulae are sometimes simpler (more compact) compared to the result of summation in (14) for the same values of i and j.

We have a Maple implementation which tries to recognize Abel-type sums and produces the answer using (14)–(17) as in the following example (with proper assumptions on symbolic parameters a, x):

$$\sum_{k=1}^{n} (ax + k)^{k-2} (n - k)^{n-k} \binom{n}{k} =$$

$$= -\frac{-(ax + n)^n ax - (ax + n)^n + (ax + n)^{n-1} nax + n^n ax + n^n}{a^2x^2(ax + 1)}.$$

4 On Riordan's problem of combinatorial identities classification

J.Riordan ([28], Introduction) posed the problem of characterization of known pairs of inverse identities of the form

$$a_m = \sum_{k=0}^{\infty} c_{mk}b_k, \quad b_m = \sum_{k=0}^{\infty} d_{mk}a_k, \quad m = 0, 1, 2, \ldots, \tag{27}$$

where $C = (c_{mk})$ is an invertible infinite lower triangular matrix whose general term is a linear combination of known combinatorial numbers, and $D = (d_{mk})$ is its inverse. An interpretation of such identities with the help of generating function technique is given in [28, 29, 30]. Each identity of this form generates for $m, n = 0, 1, 2, \ldots$ a combinatorial identity

$$\sum_k c_{nk}d_{km} = \delta(m, n), \quad n, m = 0, 1, 2, \ldots,$$

where $\delta(m,n)$ — Kronecker symbol.

The first complete solution to the Riordan problem was given in [8]-[10] by defining and studying properties of special type of matrices. In this section we extend the results of [8]-[10] and demonstrate how the integral representation can be used in a unified approach for generating new types of combinatorial identities. Proofs of theorems from this section are technical and can be found in [12].

Definition 1 *We say that relation (27) is relation of type R (R^q or $R^q(\alpha_m, \beta_k; \varphi, f, \psi)$), if the general term of matrix (c_{mk}) is defined by the formula*

$$c_{mk} = \frac{\beta_k}{\alpha_m} \mathbf{res}_z(\varphi(z) f(z)^k \psi(z)^m z^{-m+qk-1}),$$

where q is a natural number, $\alpha_m, \beta_k \neq 0$; $\varphi(z), f(z), \psi(z) \in G(0)$.

In particular, for $q = 1$, the matrix (c_{mk}) is infinite lower triangular with the general term

$$c_{mk} = \frac{\beta_k}{\alpha_m} \mathbf{res}_z \left[(\varphi(z) f(z)^k \psi(z)^m z^{-m+k-1}) \right]. \tag{28}$$

A finite $m \times m$ matrix of the type R , $k = 0, 1, \ldots, m$, is called matrix of the type R_m.

Theorem 1 *([8]) Relations of the type R^1 are equivalent to the functional relations between generating functions $A(w) = \sum_{m \geq 0} \alpha_m a_m w^m$, $B(w) = \sum_{k \geq 0} \beta_k b_k w^k$:*

$$A(w\psi^{-1}(w)) z'(w) \psi(w) = \varphi(w) B(wf(w)),$$

where $z = w/\psi(w)$, $\quad z'(w) = \frac{\partial}{\partial w}(w/\psi(w))$.

Matrix (d_{mk}), the inverse of the matrix (c_{mk}) of the type R^1 exists, is of the type R^1, and has the following general term

$$d_{mk} = \frac{\alpha_k}{\beta_m} \mathbf{res}_t \left\{ \varphi^{-1}(w) z'(w) t'(w) \psi^{-k+1} f^{-m-1}(w)/w^{m-k+1} \right\},$$

where $t = wf(w) \in G(1)$.

Theorem 2 *(on splitting) A matrix of the type $R(\alpha_m, \beta_k; \varphi, f, \psi)$ splits into the product of matrices of the types*

$$R(\alpha_m, c_k; \varphi_1, 1, \psi),\ R(c_m, d_k; \varphi_2, 1, 1) \text{ and } R(d_m, \beta_k; \varphi_3, f, 1), \tag{29}$$

where $\varphi_j \in G(0), j = 1, 2, 3$, $\varphi_1\varphi_2\varphi_3 = 1$ and $\varphi_1(0) = \varphi_2(0) = \varphi_3(0) = 1$.

Remark 1. Theorem 2 is stronger than the analogous result in [8] (see also [10, 11]), while the scheme of the proof remains almost the same. The presence of new weighting coefficients and series φ_1, φ_2, φ_3 in matrices of type R in elements of expansion (29):
– allows us to formulate new results on algebraic characterization (Theorem 5);
– gives new classification of known pairs of inverse identities (Theorem 6);
– generates new identities of Riordan type and introduces new objects (methods), as e.g. the Lagrange summation matrix below.

Theorem 3 *(on classification [8]) Inverse identities of the standard type, of Goulde, of Legendre–Tshebyshev, of Abel (standard and exponential), of Legendre ([29], Tables 2.1–2.6, 3.1–3.6) are identities of the type* $R^q(\alpha_m, \beta_k; \varphi, f, \psi)$.

Theorem 4 *(on combinatorial characterization) Binomial coefficients, Stirling numbers (standard and generalized) of the first and second kind and many other combinatorial numbers belong to the type R.*

The proof of this theorem is by comparison of integral representation for combinatorial numbers ([8], pp.269–274) with the matrix of type R. For example, formula (1) for binomial coefficients implies that $\binom{n}{k} = \binom{n}{n-k}$ is of type $R(1,1;1,1,1+w)$.

Remark 2. Results of Theorems 3 and 4 are not surprising, because an integral representation of the type R typically appears in the evaluation of combinatorial sums of different kinds (see [8], main theorem). This allows one to give a combinatorial interpretation to summation formulae, related to matrices of the type R. Weighting coefficients α_m could be interpreted for example as a number of terms or a value of the sum under investigation. Operations of multiplication, substitution and inversion in Cauchy algebra of series, hidden in the construction of matrix of the R type, also have a combinatorial interpretation (see [14, 16]), explaining in every particular case the algebraic structure of the enumeration object under investigation. The result of Theorem 2 plays a similar role (compare to the result of Theorem 7 below). Example 2 of the generation of inverse identities of Legendre-Tshebyshev type can be viewed as an extension of Riordan approach.

Theorem 5 *(algebraic characterization) Let G be the set of all matrices of the type R. Then G forms a subgroup of the unitriangular group* $UTN(\mathbf{R})$, *when*

$$\alpha_m = \beta_m \varphi(0) f^m(0) \psi^m(0).$$

The Riordan group [31] is a subgroup of the group G.

Definition 2 *We refer to the matrix (28) as the summation method of type* $R = R(\alpha_m, \beta_k; \varphi, f, \psi)$, *if all* c_{mk} *are nonnegative and* $\lim_{m\to\infty} c_{mk} = 0$ *for any k.*
We refer to summation methods of types
$R(\alpha_m, c_k; 1, 1, \psi)$, $R(c_m, d_k; \varphi, 1, 1)$ *and* $R(d_m, \beta_k; 1, f, 1)$
as summation methods of Lagrange, of Voronoy and analytic respectively.

Theorem 6 *Let* $A(w)$, $B(w)$, $C(w)$ *and* $D(w)$ *be generating functions for sequences* α_m, β_k, c_k *and* d_k *respectively. For a summation matrix of type R to be regular ([15]), it necessary and sufficient that*

$$A(w/\psi)\psi(w/\psi)' = \varphi B(wf(w)).$$

Similarly, summation matrices of Lagrange, Voronoy and analytic will be regular, if an only if respectively

$$A(w/\psi)\psi(w/\psi)' = C(w),$$

$$C(w) = \varphi D(w),$$
$$D(w) = B(wf(w)).$$

The proof follows immediately from the Toepliz-Shur theorem [15].

Theorem 7 *The well known summation matrices of divergent series due to Vallee-Poussin, Obreshkov, Cezaro, Euler, $P(q,r,s)$, general methods of Lagrange, Voronoy, Gronwall etc. are particular cases of the regular summation method of type R.*
A regular summation method of type R splits into the product of summation methods of Lagrange, of Voronoy and analytic.

Remark 3. Summation method of Lagrange is introduced here for the first time. Gronwall's method is a particular case of the summation method of type R for $\psi(w) = 1$. The second part of the last theorem is an extension of known result in divergent summation theory, that Gronwall matrix splits into the product of matrices of summing divergent series of Voronoy and analytic.

Note, that construction (28) and results of theorems 1–7 can be easily extended in several variants to the multidimensional case with the help of the main theorem in [8].

Example 2. Inverse identities of Legendre–Tshebyshev type ([29], table 2.6, relation 5). Let $p > 0$, $r > 0$, and in (27)

$$C = (c_{mk}) = \left(\binom{rm+p}{m-k} - (r-1)\binom{rm+p}{m-k-1}\right), \quad m,k = 0,1,2,\dots. \tag{30}$$

Then
a) matrix (30) defines the relation of type

$$R^{(1)} = R^{(1)}(1,1;(1+z)^p(1-(r-1)z),1,(1+z)^r);$$

b) inverse identities defined by matrix (30) are equivalent to the following functional identities

$$A(z(1+z)^{-r}) = (1+z)^{p+1}B(z),$$
$$B(z) = (1+z)^{-p-1}A(z(1+z)^{-r}).$$

c) matrix $D = (d_{mk})$ (inverse of the matrix (30)) is defined as

$$D = (d_{mk}) = \left((-1)^{m-k}\binom{m+p+rk-k}{m-k}\right).$$

d) matrices C and D can be expanded as $C = A\,B\,I$, $D = I\,B^{-1}\,A^{-1}$, where I is the identity matrix,

$$A = (a_{mk}) = \left(\binom{mr}{m-k}\right),$$

$$B = (b_{mk}) = \left(\binom{p}{m-k} - (r-1)\binom{p}{m-k-1}\right),$$

$$A^{-1} = (a_{mk}^{(-1)}) = \left((-1)^{m-k} \binom{m+rk-k-1}{m-k} \right),$$

$$B^{-1} = (b_{mk}^{(-1)}) = \left((-1)^{m-k} \binom{p+m-k}{m-k} \right).$$

e) matrix relations

$$C\,D = I, \quad C = A\,B\,I, \quad D = I\,B^{-1}\,A^{-1},$$
$$A\,B\,B^{-1}\,A^{-1} = I, \quad B\,B^{-1}\,A^{-1} = A^{-1}, \quad A\,B\,B^{-1} = A$$

generate the following combinatorial identities

$$\sum_{s=k}^{m} (-1)^{s-k} \left\{ \binom{rm+p}{m-s} - (r-1)\binom{rm+p}{m-s-1} \right\} \binom{s+p+rk-k}{s-k} = \delta(m,k),$$

$m, k = 0, 1, 2, \cdots,$

$$\sum_{s=k}^{m} \binom{mr}{m-s} \left\{ \binom{p}{s-k} - (r-1)\binom{p}{s-k-1} \right\} =$$
$$= \left(\binom{rm+p}{m-k} - (r-1)\binom{rm+p}{m-k-1} \right), \quad m, k = 0, 1, 2, \cdots,$$

$$\sum_{s=k}^{m} \binom{p+m-s}{m-s} \binom{s+rk-k-1}{s-k} = \binom{m+p+rk-k}{m-k}, \quad m, k = 0, 1, 2, \cdots,$$

$$\sum_{n=k}^{m} \sum_{t=n}^{m} \sum_{s=t}^{m} (-1)^{t-k} \binom{mr}{m-s} \left(\binom{p}{s-t} - (r-1)\binom{p}{s-t-1} \right) \cdot$$
$$\cdot \binom{p+t-n}{t-n} \binom{n+rk-k-1}{n-k} = \delta(m,k), \quad m, k = 0, 1, 2, \cdots,$$

$$\sum_{t=k}^{m} \sum_{s=t}^{m} \left(\binom{p}{m-s} - (r-1)\binom{p}{m-s-1} \right) (-1)^{s-k} \binom{p+s-t}{s-t} \cdot$$
$$\cdot \binom{t+rk-k-1}{t-k} = (-1)^{m-k} \binom{m+rk-k-1}{m-k}, \quad m, k = 0, 1, 2, \cdots,$$

$$\sum_{t=k}^{m} \sum_{s=t}^{m} \binom{mr}{m-s} \left(\binom{p}{s-t} - (r-1)\binom{p}{s-t-1} \right) \cdot$$
$$\cdot (-1)^{t-k} \binom{p+t-k}{t-k} = \binom{mr}{m-k}, \quad m, k = 0, 1, 2, \cdots.$$

Proof. From the definition of the general term of matrix C and from the integral representation of binomial coefficient (1) taking into account (29), it follows that

$$c_{mk} = \mathbf{res}_z (1+z)^{rm+p} z^{-m+k-1} - (r-1)\, \mathbf{res}_z (1+z)^{rm+p} z^{-m+k} = \\ = \mathbf{res}_z \left((1+z)^{rm+p} (1-(r-1)z)/z^{m-k+1} \right).$$

Comparison of this expression for c_{mk} with (28) proves the claim a) of this example, if we let

$$\alpha_m = \beta_k = 1,\ \varphi(z) = (1+z)^p (1-(r-1)z),\ f(z) = 1,\ \psi(z) = (1+z)^r,$$

Other claims follow from properties of the operator **res** and of the relations of type R^1. ∎

5 Conclusion

We have shown that ideas of integral representation can be used in an algorithmic fashion and can be implemented in computer algebra systems. Our Maple implementation of improvements to the rational summation algorithms and of Abel-type summation is based on a few simple formulae of the integral representation. The use of wider set of basic formulae of integral for different classes of expressions ([8], pp.269–274, [27]) will allow further extensions. We also plan to continue investigation of the applicability of similar ideas to the Gosper and "W-Z" types of summation problems. Improvements to some classes of hypergeometric summation based on the integral representation ideas are given in [12] and implemented in Maple.

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