

On the Zeros of the Weierstrass \wp -Function

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The Weierstrass \wp -function, defined for $\tau \in \mathfrak{H}$ (upper half-plane) and $z \in \mathbb{C}$ by

$$\wp(z, \tau) = \frac{1}{z^2} + \sum_{\substack{\omega \in \mathbb{Z} + \mathbb{Z}\tau \\ \omega \neq 0}} \left(\frac{1}{(z + \omega)^2} - \frac{1}{\omega^2} \right),$$

is the basic and most famous function of elliptic function theory. As is well known, $\wp(z, \tau)$ is for fixed τ doubly periodic in z and takes on each value in $\mathbb{C} \cup \{\infty\}$ exactly twice (counting multiplicity) as z ranges over $\mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$. In particular, since $\wp(z, \tau)$ is an even function of z , there is for each $\tau \in \mathfrak{H}$ a number $z_0(\tau)$, well-defined up to sign and translation by $\mathbb{Z} + \mathbb{Z}\tau$, such that

$$\wp(z, \tau) = 0 \Leftrightarrow z \equiv \pm z_0(\tau) \pmod{\mathbb{Z} + \mathbb{Z}\tau}.$$

The purpose of this note is to prove the following explicit formula for $z_0(\tau)$ which, despite the long history of the function \wp , seems not to have been noticed earlier; this formula arose out of the authors' investigation of "Jacobi forms" [functions on $\mathbb{C} \times \mathfrak{H}$ satisfying a transformation law of a certain kind under the transformations $(z, \tau) \rightarrow \left(\frac{z + m + n\tau}{c\tau + d}, \frac{a\tau + b}{c\tau + d} \right)$ with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$, $m, n \in \mathbb{Z}$] and is a special case of more general results, but in view of the special interest attaching to the \wp -function it seemed worthwhile to publish it separately.

Theorem. *The zeros of $\wp(z, \tau)$ ($\tau \in \mathfrak{H}$, $z \in \mathbb{C}$) are given by*

$$z = m + \frac{1}{2} + n\tau \pm \left(\frac{\log(5 + 2\sqrt{6})}{2\pi i} + 144\pi i \sqrt{6} \int_{\tau}^{i\infty} (t - \tau) \frac{\Delta(t)}{E_6(t)^{3/2}} dt \right)$$

($m, n \in \mathbb{Z}$), where $E_6(t)$ and $\Delta(t)$ ($t \in \mathfrak{H}$) denote the normalized Eisenstein series of weight 6 and unique normalized cusp form of weight 12 on $\text{SL}_2(\mathbb{Z})$, respectively, and the integral is to be taken over the vertical line $t = \tau + i\mathbb{R}_+$ in \mathfrak{H} .

For $\tau=i$ we know a priori that $\wp(z, \tau)$ vanishes at the point $z = \frac{1+i}{2}$ [because $(\frac{\partial \wp}{\partial z})^2 = 4\wp^3 - g_2(\tau)\wp - g_3(\tau)$ and $g_3(i)=0$, so the zero of $\wp(z, i)$ is a double one and hence a 2-division point; that it is $\frac{1+i}{2}$ rather than $\frac{1}{2}$ or $\frac{i}{2}$ is easily verified].

Comparing this with the result of the theorem yields the integral identity

$$\int_1^\infty \frac{A(it)}{E_6(it)^{3/2}} (t-1) dt = \frac{\pi - \log(5 + 2\sqrt{6})}{288\pi^2 \sqrt{6}}.$$

Expanding $A/E_6^{3/2}$ in a Fourier series and integrating term by term, we obtain the following amusing corollary, in which all mention of the \wp -function (or, for that matter, of modular forms) has been suppressed:

Corollary. Define integers A_n ($n \geq 1$) by the formal power series expansion

$$\sum_{n=1}^\infty A_n q^n = \frac{q \prod_{n=1}^\infty (1 - q^n)^{24}}{\left(1 - 504 \sum_{n=1}^\infty \sum_{d|n} d^5 q^n\right)^{3/2}}$$

($A_1 = 1, A_2 = 732, A_3 = 483336, \dots$). Then

$$\sum_{n=1}^\infty \frac{A_n}{n^2} e^{-2\pi n} = \frac{\pi - \log(5 + 2\sqrt{6})}{72\sqrt{6}}.$$

We remark that the series in the corollary converges very slowly since (as is not hard to show) A_n satisfies the asymptotic formula

$$A_n \sim C n^{1/2} e^{2\pi n} \quad (C = 1/216 \sqrt{2\pi}).$$

First Proof (Modular Forms)

We begin by observing that the function \wp satisfies the transformation law

$$\wp\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 \wp(z, \tau) \quad \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})\right),$$

as one easily checks from the definition. It follows that the function $z_0(\tau)$ satisfies the transformation equation

$$\frac{z_0(\tau)}{c\tau+d} \equiv \pm z_0\left(\frac{a\tau+b}{c\tau+d}\right) \pmod{\mathbb{Z} + \mathbb{Z} \frac{a\tau+b}{c\tau+d}},$$

i.e. $z_0(\tau)$ is a (many-valued) modular form of weight -1 on $\text{SL}_2(\mathbb{Z})$. Now the function $z_0(\tau)$ has infinitely many branches, but (except for sign) these differ from each other by the linear functions $\tau \rightarrow m + n\tau$ ($m, n \in \mathbb{Z}$), so the second derivative $z_0''(\tau)$

is well-defined up to sign and the function $z_0''(\tau)^2$ is single-valued. Writing the transformation law of z_0 as

$$z_0(\tau) = m + n\tau \pm (c\tau + d)z_0\left(\frac{a\tau + b}{c\tau + d}\right)$$

and differentiating twice we obtain

$$z_0''(\tau) = \pm (c\tau + d)^{-3} z_0''\left(\frac{a\tau + b}{c\tau + d}\right),$$

i.e. the function $z_0''(\tau)^2$ transforms under the action of $SL_2(\mathbb{Z})$ like a modular form of weight 6.

At first sight it appears that z_0 , and hence z_0'' , is a holomorphic function of τ , since it (or each branch of it) is locally bounded and is defined implicitly by the vanishing of a meromorphic function in two variables. However, since z_0 is many-valued we also have to worry about ramification or coalescing of different branches. The branches $z_0(\tau) + m + n\tau$ ($m, n \in \mathbb{Z}$) can never meet, since $m + n\tau \neq 0$ for $(m, n) \neq (0, 0)$, but the branches $z_0(\tau)$ and $-z_0(\tau) + m + n\tau$ can meet. This happens when $\wp(\tau, z)$ has a zero at one of the three 2-division points

$$z \equiv \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2} \pmod{\mathbb{Z} + \mathbb{Z}\tau}$$

[the point $z=0$ also satisfies $z \equiv -z \pmod{\mathbb{Z} + \mathbb{Z}\tau}$ but cannot be a zero of \wp since it is always a pole], i.e. whenever the function $\wp\left(\frac{1}{2}, \tau\right)\wp\left(\frac{\tau}{2}, \tau\right)\wp\left(\frac{1+\tau}{2}, \tau\right)$ vanishes at τ . But this function is easily checked to be a holomorphic modular form of weight 6 on $SL_2(\mathbb{Z})$ and hence a multiple (in fact $-1/864$) of $E_6(\tau)$. Thus

$$\begin{aligned} \text{two branches of } z_0(\tau) \text{ meet} &\Leftrightarrow \wp(z, \tau) \text{ has a double zero} \\ &\Leftrightarrow E_6(\tau) = 0. \end{aligned}$$

[This can also be seen from the equation $\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$, since g_3 is a multiple of E_6 . Also, as is well-known, $E_6(\tau)$ vanishes precisely when τ is $SL_2(\mathbb{Z})$ -equivalent to i , but we shall not need this fact.] At such a point τ_0 , exactly two branches of $z_0(\tau)$ coincide [since the function $\wp(\cdot, \tau) : \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau \rightarrow \mathbb{C} \cup \{\infty\}$ is exactly 2-to-1], so each branch of z_0 has the local form

$$z_0(\tau) = \pm \sum_{r=0}^{\infty} c_r(\tau - \tau_0)^{r + \frac{1}{2}} + \frac{m}{2} + \frac{n}{2}\tau \quad (\tau \rightarrow \tau_0)$$

with $c_0 \neq 0$; differentiating twice and squaring we get

$$z_0''(\tau)^2 = \frac{c_0^2}{16}(\tau - \tau_0)^{-3} - \frac{3}{8}c_0c_1(\tau - \tau_0)^{-2} + \dots$$

Thus $z_0''(\tau)^2$ is meromorphic with a triple pole at each τ with $E_6(\tau) = 0$, so the function $E_6(\tau)^3 z_0''(\tau)^2$, which transforms like a modular form of weight $3 \times 6 + 2 \times 3 = 24$, is holomorphic in the whole upper half-plane. In a moment we shall

see that it is also holomorphic at the cusp of $\mathfrak{H}/\text{SL}_2(\mathbb{Z})$ and in fact has a zero of order 2 there, so it must be a constant multiple of $\Delta(\tau)^2$, i.e.

$$\begin{aligned} z_0''(\tau) &= \pm C_0 \frac{\Delta(\tau)}{E_6(\tau)^{3/2}} \quad (\tau \in \mathfrak{H}) \\ &= \pm C_0(e^{2\pi i\tau} + 732e^{4\pi i\tau} + \dots) \quad (\text{Im}(\tau) > 1) \end{aligned}$$

for some $C_0 \neq 0$; integrating twice, we obtain the final formula

$$\begin{aligned} z_0(\tau) &= C_1 + C_2\tau \pm C_0 \int_{\tau}^{i\infty} \frac{\Delta(t)}{E_6(t)^{3/2}}(t - \tau)dt \quad (\tau \in \mathfrak{H}) \\ &= C_1 + C_2\tau \mp \frac{C_0}{4\pi^2}(e^{2\pi i\tau} + 183e^{4\pi i\tau} + \dots) \quad (\text{Im}(\tau) > 1) \end{aligned}$$

for (each branch of) $z_0(\tau)$.

To prove the last assertion and determine the values of the constants of integration C_0 , C_1 , and C_2 , we investigate the asymptotic behaviour of $z_0(\tau)$ as $\tau \rightarrow i\infty$. Computing the Fourier development of $\wp(z, \tau)$ in the same way as one calculates the Fourier development of Eisenstein series, one finds

$$\begin{aligned} (2\pi i)^{-2} \wp(z, \tau) &= \frac{1}{\zeta - 2 + \zeta^{-1}} + \frac{1}{12} + (\zeta - 2 + \zeta^{-1})q \\ &\quad + (2\zeta^2 + \zeta - 6 + \zeta^{-1} + 2\zeta^{-2})q^2 + \dots, \end{aligned}$$

where we have set $q = e^{2\pi i\tau}$ and $\zeta = e^{2\pi iz}$ [the coefficient of q^n for $n > 0$ is $\sum_{d|n} d(\zeta^d - 2 + \zeta^{-d})$, but we shall not need more terms than those given]. The ambiguity $z \rightarrow \pm z + m + n\tau$ ($m, n \in \mathbb{Z}$) in z corresponds to the ambiguity $\zeta \rightarrow q^n \zeta^{\pm 1}$ ($n \in \mathbb{Z}$) in ζ . To ask whether some branch of $z_0(\tau)$ has a finite limit as $\tau \rightarrow i\infty$ is equivalent to asking whether there is some finite value of ζ making the above expansion vanish for $q = 0$. Clearly, such a value is given by $\zeta - 2 + \zeta^{-1} = -12$ or $\zeta = -\varepsilon^{\pm 1}$, where $\varepsilon = 5 + 2\sqrt{6}$ is the fundamental unit of $\mathbb{Q}(\sqrt{6})$. In terms of $z = \frac{1}{2\pi i} \log \zeta$ this corresponds to

$$z = m + \frac{1}{2} \pm \frac{1}{2\pi i} \log \varepsilon \quad (m \in \mathbb{Z}).$$

Thus $z_0(\tau)$ has a branch tending to each of these values of z as $\tau \rightarrow i\infty$. To find the Fourier expansion of this branch, we write

$$z_0(\tau) = m + \frac{1}{2} \pm \frac{1}{2\pi i} (\log \varepsilon + Ae^{2\pi i\tau} + Be^{4\pi i\tau} + \dots)$$

with as yet unknown coefficients A, B, \dots , substitute the corresponding value

$$\zeta = e^{2\pi iz_0(\tau)} = -\varepsilon^{\pm 1} \left(1 + Aq + \left(\frac{A^2}{2} + B \right) q^2 + \dots \right)^{\pm 1}$$

into the above expansion for $\wp(z, \tau)$, and successively equate the coefficients of each power of q to 0. This gives the values

$$A = \pm 72\sqrt{6}, \quad B = \pm 13176\sqrt{6} = 183A, \dots;$$

comparing this expansion of $z_0(\tau)$ with the one given above we obtain

$$C_0 = \pm 144\pi i \sqrt{6}, \quad C_1 = m + \frac{1}{2} \pm \frac{1}{2\pi i} \log \varepsilon, \quad C_2 = n \quad (m, n \in \mathbb{Z})$$

and hence the assertion of the theorem.

We observe that the same method can be applied to find the solutions of any equation of the form

$$\wp(z, \tau) = \phi(\tau)$$

where ϕ is a modular form of weight 2 [of course, since there are no holomorphic modular forms of weight 2 on $SL_2(\mathbb{Z})$ we must take $\phi(\tau)$ to be either a meromorphic modular form or else modular on a subgroup Γ of $SL_2(\mathbb{Z})$]: Again the solutions are of the form

$$z \equiv \pm z_\phi(\tau) \pmod{\mathbb{Z} + \mathbb{Z}\tau},$$

where z_ϕ transforms up to sign and translation by $\mathbb{Z} + \mathbb{Z}\tau$ like a modular form of weight -1 and hence $z_\phi''(\tau)^2$ like a modular form of weight 6 on Γ . By considering the ramification points of $z_\phi(\tau)$, i.e. the points where $z_\phi(\tau)$ is a 2-division point on $\mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$, one sees that

$$z_\phi''(\tau) = \pm \psi(\tau) / (4\phi(\tau)^3 - g_2(\tau)\phi(\tau) - g_3(\tau))^{3/2},$$

where $\psi(\tau)$ is a modular form of weight 12 on Γ which is holomorphic wherever ϕ is. By substituting the Fourier expansions of \wp and ϕ into the defining equation for z_ϕ we can compute as many terms as desired of the Fourier development of ψ and hence identify this function entirely, after which z_ϕ is obtained by two-fold integration.

Second Proof (Elliptic Integrals)

Following a suggestion of Hirzebruch, we can also prove the formulas for z_0 and z_ϕ by using elliptic integrals rather than modular forms. Indeed, from the formula for the derivative of $\wp(z)$ we obtain

$$dz = \frac{d\wp(z)}{\wp'(z)} = \frac{d\wp(z)}{\sqrt{4\wp(z)^3 - g_2\wp(z) - g_3}}$$

and hence

$$z_0(\tau) = \int_{\wp=\infty}^{\wp=0} dz = \pm \int_0^\infty \frac{dX}{\sqrt{4X^3 - g_2X - g_3}}.$$

Here

$$g_2 = 60 \times 2\zeta(4)E_4 = \frac{4\pi^4}{3} E_4, \quad g_3 = 140 \times 2\zeta(6)E_6 = \frac{8\pi^6}{27} E_6,$$

where

$$E_4 = 1 + 240 \sum_{n \geq 1} \sigma_3(n)q^n, \quad E_6 = 1 - 504 \sum_{n \geq 1} \sigma_5(n)q^n$$

($\sigma_v(n) = \sum_{d|n} d^v, q = e^{2\pi i\tau}$) are the normalized Eisenstein series of weight 4 and 6; the

change of variables $X = \frac{\pi^2}{3} t$ gives

$$z_0(\tau) = \pm \frac{\sqrt{3}}{2\pi} \int_0^\infty \frac{dt}{\sqrt{t^3 - 3E_4t - 2E_6}}.$$

We now compute the second derivative of this by differentiating with respect to τ under the integral sign. The derivatives of E_4 and E_6 are given by

$$\frac{1}{2\pi i} \frac{d}{d\tau} E_4 = \frac{1}{3}(E_2E_4 - E_6), \quad \frac{1}{2\pi i} \frac{d}{d\tau} E_6 = \frac{1}{2}(E_2E_6 - E_4^2),$$

where $E_2 = 1 - 24 \sum_{n \geq 1} \sigma_1(n)q^n$ (which is not a modular form). Hence

$$\frac{1}{2\pi i} \frac{\partial}{\partial \tau} \frac{1}{(t^3 - 3E_4t - 2E_6)^{1/2}} = \frac{1}{2} \frac{(E_2E_4 - E_6)t + E_2E_6 - E_4^2}{(t^3 - 3E_4t - 2E_6)^{3/2}}.$$

To differentiate again we also need the derivative of E_2 , which is given by

$$\frac{1}{2\pi i} \frac{d}{d\tau} E_2 = \frac{1}{12}(E_2^2 - E_4);$$

we then find after some computation

$$\begin{aligned} & \frac{1}{(2\pi i)^2} \frac{\partial^2}{\partial \tau^2} \frac{1}{(t^3 - 3E_4t - 2E_6)^{1/2}} \\ &= (t^3 - 3E_4t - 2E_6)^{-5/2} \left\{ \frac{5}{24}(E_2^2E_4 + E_4^2 - 2E_2E_6)t^4 \right. \\ & \quad + \frac{7}{24}(E_2^2E_6 + E_4E_6 - 2E_2E_4^2)t^3 + \frac{1}{8}(E_2^2E_4^2 - 2E_2E_4E_6 + 6E_6^2 - 5E_4^3)t^2 \\ & \quad + \frac{1}{24}(5E_2^2E_4E_6 + 6E_2E_4^3 - 16E_2E_6^2 + 5E_4^2E_6)t \\ & \quad \left. + \frac{1}{12}(2E_2^2E_6^2 - 4E_2E_4^2E_6 + 9E_4^4 - 7E_4E_6^2) \right\} \\ &= \frac{\partial}{\partial t} \frac{At^2 + Bt + C}{(t^3 - 3E_4t - 2E_6)^{3/2}}, \end{aligned}$$

where

$$\begin{aligned}
 A &= -\frac{1}{12}(E_2^2 E_4 + E_4^2 - 2E_2 E_6) = -\frac{3}{5} \cdot \frac{1}{(2\pi i)^2} \frac{\partial^2 E_4}{\partial \tau^2} = -144 \sum_{n \geq 1} n^2 \sigma_3(n) q^n, \\
 B &= -\frac{1}{12}(E_2^2 E_6 + E_4 E_6 - 2E_2 E_4^2) = -\frac{2}{7} \cdot \frac{1}{(2\pi i)^2} \frac{\partial^2 E_6}{\partial \tau^2} = 144 \sum_{n \geq 1} n^2 \sigma_5(n) q^n, \\
 C &= +\frac{1}{6}(E_4^3 - E_6^2) = 288\Delta = 288q \prod_{n \geq 1} (1 - q^n)^{24}.
 \end{aligned}$$

Integrating from $t=0$ to $t=\infty$ gives

$$\frac{-1}{4\pi^2} z_0''(\tau) = \frac{\sqrt{3}}{2\pi} \frac{At^2 + Bt + C}{(t^3 - 3E_4 t - 2E_6)^{3/2}} \Big|_0^\infty = \pm \frac{36i\sqrt{6}}{\pi} \frac{\Delta(\tau)}{E_6(\tau)^{3/2}},$$

in accordance with the formula obtained earlier. To complete the proof we must still compute $z_0(\tau)$ for $\tau \rightarrow i\infty$. But at infinity the discriminant Δ of the cubic $t^3 - 3E_4 t - 2E_6$ tends to 0, so the cubic degenerates into the product of a linear factor and the square of a linear factor and the elliptic integral defining z_0 becomes elementary. More precisely, for $\tau \rightarrow i\infty$ we have $E_4 \rightarrow 1, E_6 \rightarrow 1$ and hence

$$\begin{aligned}
 z_0(i\infty) &= \pm \frac{\sqrt{3}}{2\pi} \int_0^\infty \frac{dt}{\sqrt{t^3 - 3t - 2}} \\
 &= \pm \frac{\sqrt{3}}{2\pi} \left(\int_0^2 + \int_2^\infty \right) \frac{dt}{(t+1)\sqrt{t-2}};
 \end{aligned}$$

making the substitution $t = 2 - 3x^2$ in the first integral and $t = 2 + 3x^2$ in the second we find

$$\begin{aligned}
 z_0(i\infty) &= \pm \frac{1}{\pi i} \int_0^{\sqrt{2/3}} \frac{dx}{1-x^2} \pm \frac{1}{\pi} \int_0^\infty \frac{dx}{1+x^2} \\
 &= \pm \frac{1}{2\pi i} \log \frac{1+x}{1-x} \Big|_0^{\sqrt{2/3}} \pm \frac{1}{\pi} \arctan x \Big|_0^\infty \\
 &= \pm \frac{1}{2\pi i} \log(5 + 2\sqrt{6}) \pm \frac{1}{2},
 \end{aligned}$$

and combining this with the result already obtained for $z_0''(\tau)$ we recover the formula of the theorem.

The same method applies to the solution $z = z_\phi(\tau)$ of the equation $\wp(z, \tau) = \phi(\tau)$, where $\phi(\tau)$ is any meromorphic function of τ : we have

$$z_\phi(\tau) = \frac{\sqrt{3}}{2\pi} \int_{\frac{3}{\pi^2} \phi(\tau)}^\infty \frac{dt}{\sqrt{t^3 - 3E_4 t - 2E_6}}$$

(with the indeterminacy coming from the choice of square root and of path of integration) and from this we can determine $z_\phi(i\infty)$ and $z_\phi''(\tau)$ in the same way as in

the special case $\phi = 0$. Namely, if $\phi(\tau)$ has a limit λ as $\text{Im}(\tau) \rightarrow \infty$, then

$$\begin{aligned} z_\phi(i\infty) &= \frac{\sqrt{3}}{2\pi} \int_{\frac{3\lambda}{\pi^2}}^\infty \frac{dt}{(t+1)\sqrt{t-2}} = \frac{1}{2} - \frac{1}{\pi} \arctan\left(\sqrt{\frac{\lambda}{\pi^2} - 2}\right) \\ &= \frac{1}{2} \pm \frac{1}{2\pi i} \log \frac{1 + \sqrt{\frac{2}{3} - \frac{\lambda}{\pi^2}}}{1 - \sqrt{\frac{2}{3} - \frac{\lambda}{\pi^2}}}. \end{aligned}$$

As to the second derivative, we find

$$\begin{aligned} z'_\phi(\tau) &= -\frac{\sqrt{3}}{2\pi} \cdot \frac{3}{\pi^2} \phi'(\tau) \left(\frac{27}{\pi^6} \phi(\tau)^3 - \frac{9}{\pi^2} E_4 \phi(\tau) - 2E_6 \right)^{-1/2} \\ &\quad + \frac{\sqrt{3}}{2\pi} \int_{\frac{3}{\pi^2} \phi(\tau)}^\infty \frac{\partial}{\partial \tau} [(t^3 - 3E_4 t - 2E_6)^{-1/2}] dt, \\ z''_\phi(\tau) &= -\frac{3\sqrt{3}}{2\pi^3} \frac{d}{d\tau} \left(\frac{\phi'(\tau)}{\sqrt{\frac{27}{\pi^6} \phi^3 - \frac{9}{\pi^2} E_4 \phi - 2E_6}} \right) \\ &\quad - \frac{3\sqrt{3}}{2\pi^3} \phi'(\tau) \frac{\partial}{\partial \tau} [(t^3 - 3E_4 t - 2E_6)^{-1/2}] \Big|_{t=\frac{3}{\pi^2} \phi(\tau)} \\ &\quad + \frac{\sqrt{3}}{2\pi} \int_{\frac{3}{\pi^2} \phi(\tau)}^\infty \frac{\partial^2}{\partial \tau^2} [(t^3 - 3E_4 t - 2E_6)^{-1/2}] dt. \end{aligned}$$

Using the formulas given earlier for $\frac{d}{d\tau} E_4$, $\frac{d}{d\tau} E_6$, and $\frac{\partial^2}{\partial \tau^2} [(t^3 - 3E_4 t - 2E_6)^{-1/2}]$ we find after some computation the result

$$\begin{aligned} z''_\phi(\tau) &= \pm \frac{1}{72\pi^2} (4\phi^3 - g_2\phi - g_3)^{-3/2} \{ -2(4\phi^3 - g_2\phi - g_3)\phi^{**} \\ &\quad + (12\phi^2 - g_2)\phi^{*2} + (36g_3\phi + 2g_2^2)\phi^* \\ &\quad + 12g_2\phi^4 + 3g_2^2\phi^2 + 6g_2g_3\phi - g_2^3 + 27g_3^2 \}, \end{aligned}$$

where we have set

$$\phi^* = 12\pi^2 \left(\frac{1}{2\pi i} \frac{\partial}{\partial \tau} - \frac{1}{6} E_2 \right) \phi, \quad \phi^{**} = 12\pi^2 \left(\frac{1}{2\pi i} \frac{\partial}{\partial \tau} - \frac{1}{3} E_2 \right) \phi^*.$$

This formula holds for any meromorphic function $\phi(\tau)$; if ϕ is a (meromorphic) modular form of weight 2 on a subgroup Γ of $\text{SL}_2(\mathbb{Z})$ then, as is well-known, ϕ^* and ϕ^{**} are modular forms on Γ of weight 4 and weight 6, respectively, so we have obtained an explicit formula of the form

$$z''_\phi(\tau) = \frac{\psi(\tau)}{(4\phi(\tau)^3 - g_2(\tau)\phi(\tau) - g_3(\tau))^{3/2}},$$

where ψ is a (meromorphic) modular form of weight 12 on Γ and is a cusp form if ϕ is a holomorphic modular form. As a further corollary we observe that the equation $\wp(z, \tau) = \phi(\tau)$ for the special function

$$\phi(\tau) = \wp(a\tau + b, \tau) \quad (a, b \in \mathbb{C}, \text{ not both } \in \mathbb{Z})$$

[which is a modular form of weight 2 on $\Gamma(N)$ if $a, b \in \frac{1}{N}\mathbb{Z}$] has the special solution $z = a\tau + b$ with $z''(\tau) = 0$. Hence the form ψ must vanish identically in this case and we have obtained a non-linear second order differential equation satisfied by all the functions $\wp(a\tau + b, \tau)$.