ALGEBRAIC RELATIONS FOR RECIPROCAL SUMS OF FIBONACCI NUMBERS

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Let $\{F_n\}_{n\geq 0}$ and $\{L_n\}_{n\geq 0}$ be Fibonacci numbers and Lucas numbers. In this paper, we prove the algebraic independence of the numbers

$$\sum_{n=1}^{\infty}\frac{1}{F_n^2}\,,\quad \sum_{n=1}^{\infty}\frac{1}{F_n^4}\,,\quad \sum_{n=1}^{\infty}\frac{1}{F_n^6}\qquad \left(\text{respectively,}\quad \sum_{n=1}^{\infty}\frac{1}{L_n^2}\,,\quad \sum_{n=1}^{\infty}\frac{1}{L_n^4}\,,\quad \sum_{n=1}^{\infty}\frac{1}{L_n^6}\right)\,,$$

and write each

$$\sum_{n=1}^{\infty}\frac{1}{F_n^{2s}}\qquad \left(\text{respectively,}\quad \sum_{n=1}^{\infty}\frac{1}{L_n^{2s}}\right)\qquad (s=4,5,6,\dots)\,,$$

as a rational (respectively, algebraic) function of these three numbers over $\mathbb{Q}$. Similar results are obtained for the alternating sums

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{F_n^{2s}}\qquad \left(\text{respectively,}\quad \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{L_n^{2s}}\right)\qquad (s=1,2,\dots)$$

Our theorems cover more general binary recurrences including such as Pell numbers.