# The Weierstrass Theory For Elliptic Functions

## Including The Generalisation To Higher Genus

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# Outline

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# What are elliptic functions?

They are complex functions with two independent periods.

## Definition

An elliptic function is a meromorphic function $f$ defined on $\mathbb{C}$ for which there exist two non-zero complex numbers $\omega_1, \omega_2$ such that

$$f(u+\omega_1) = f(u+\omega_2) = f(u) \qquad \text{for all } u \in \mathbb{C}$$

where $\omega_1/\omega_2 \notin \mathbb{R}$.

- The field of elliptic functions with respect to given periods is generated by a Weierstrass $\wp$-function and its derivative $\wp'$.

# The Weierstrass $\wp$-function

## Definition

We define the Weierstrass $\wp$-function with a complex variable $u$ and a pair of complex periods $\omega_1, \omega_2$.

$$\wp(u;\omega_1,\omega_2) = \frac{1}{u^2} + {\sum_{m,n}}' \left\{ \frac{1}{(u - m\omega_1 - n\omega_2)^2} - \frac{1}{(m\omega_1 + n\omega_2)^2} \right\}.$$

where $'$ implies that terms with zero denominators are omitted.

Define the period lattice, $\Lambda$ with points $\Lambda_{m,n} = m\omega_1 + n\omega_2$. Then

$$\wp(u;\omega_1,\omega_2) = \wp(u;\Lambda) = u^{-2} + {\sum_{m,n}}' [(u - \Lambda_{m,n})^{-2} - \Lambda_{m,n}^{-2}]$$

# How the $\wp$-function parameterises an elliptic curve

An elliptic curve is a non-singular algebraic curve with equation

$$y^2 = x^3 + ax + b$$

- Let $g_2$ and $g_3$ be the elliptic invariants defined as below.

$$g_2 = 60 \sum\nolimits'_{m,n} \Lambda_{m,n}^{-4} \qquad g_3 = 140 \sum\nolimits'_{m,n} \Lambda_{m,n}^{-6}. \qquad (*)$$

**The Differential Equation**

Then $\qquad [\wp'(u)]^2 = 4\wp(u)^3 - g_2\wp(u) - g_3$ ▸ g=2

- So the solution to $[y']^2 = 4y^3 - g_2 y - g_3$ is $y = \wp(u+\alpha)$, providing that there are numbers $\omega_1, \omega_2$ which satisfy $(*)$.

$\Longrightarrow$ The $\wp$-function is said to parameterise an elliptic curve

# Properties of the $\wp$-function

## The Second Derivative

Differentiating gives

g=2

$$\wp''(u) = 6\wp(u)^2 - \frac{1}{2}g_2$$

- We see that $\wp(u)$ can be defined by

$$u = \int_{-\infty}^{\wp(u)} \frac{dx}{\sqrt{4x^3 - g_2x - g_3}} = \int_{-\infty}^{\wp} \frac{dx}{y}$$

## Addition Formula

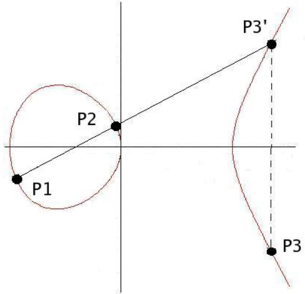
$$\wp(u+v) = \frac{1}{4}\left[\frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)}\right]^2 - \wp(u) - \wp(v)$$

elliptic curve addition
sigma function addition

# Elliptic curve addition

This relates to the addition law for points on an elliptic curve.

▸ The ℘-function addition formula




Given two points $P1$ and $P2$:

1. Find the straight line connecting them.
2. Calculate the third point of intersection $P3'$.
3. Reflect to find $P3$.

Define the addition law as

$$P1 + P2 = P3$$

Points on an elliptic curve (along with an extra point, $\infty$) form an abelian group.

# The Weierstrass $\sigma$-function

We can also associate a $\sigma$-function to the lattice $\Lambda$. It satisfies

$$\wp(u) = -\frac{d^2}{du^2}\ln[\sigma(u)], \qquad \sigma(u) = \sigma(u,\Lambda),$$

higher genus

- The $\sigma$-function has a power series expansion

g=3

$$\sigma(u) = u - \frac{1}{240}g_2u^5 - \frac{1}{840}g_3u^7 - \frac{1}{161280}g_2^2u^9 - \dots$$

## The addition formula for $\sigma(u)$

$$-\frac{\sigma(u+v)\sigma(u-v)}{\sigma(u)^2\sigma(v)^2} = \wp(u) - \wp(v)$$

p-addition g=2 g=3

# (n,s)-curves and the genus

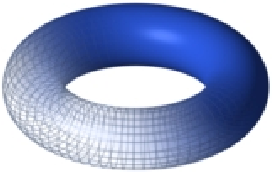
- Define an $(n, s)$-curve as an algebraic curve with equation

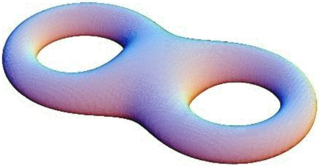
$$y^n = x^s + \lambda_{s-1}x^{s-1} + \ldots + \lambda_1 x + \lambda_0$$

where $n < s$ and $n, s$ coprime.

general algebraic curve

- This will define a surface with genus $g = \frac{1}{2}(n-1)(s-1)$





The genus is roughly thought of as the number of 'holes' in a surface.

# Hyperelliptic curves

## Definition

A hyperelliptic curve is of the form $y^2 = f(x)$ where $f(x)$ is a polynomial of degree $s > 4$, with $s$ distinct roots.

The simplest example is the (2,5)-curve, with $g = 2$

$$C : y^2 = x^5 + \lambda_4 x^4 + \lambda_3 x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0$$

Now, $\sigma$ & $\wp$ are functions of two variables & a period matrix $M$:

$$\sigma = \sigma(\mathbf{u}; M), \qquad \mathbf{u} = (u_1, u_2)$$

where

$$u_1 = \int^{(x_1,y_1)} \frac{dx}{y}, \qquad u_2 = \int^{(x_2,y_2)} \frac{xdx}{y}$$

for two variable points $(x_i, y_i)$ on $C$.

# Hyperelliptic $\wp$-functions

There are now three possibilities for the $\wp$-function

$$\wp_{ij} = -\frac{\partial^2}{\partial u_i \partial u_j} \ln \sigma(\mathbf{u}), \quad i \leq j \in \{1, 2\}$$

elliptic case

$\wp \equiv \wp_{11}$

Baker found a hyperelliptic addition formula: elliptic case g=3

$$\frac{\sigma(\mathbf{u}+\mathbf{v})\sigma(\mathbf{u}-\mathbf{v})}{\sigma(\mathbf{u})^2\sigma(\mathbf{v})^2} = \wp_{22}(\mathbf{u})\wp_{21}(\mathbf{v}) - \wp_{21}(\mathbf{u})\wp_{22}(\mathbf{v}) - \wp_{11}(\mathbf{u}) + \wp_{11}(\mathbf{v})$$

We now extend the new notation to consider higher derivatives

$$\wp_{ijk} = -\frac{\partial^3}{\partial u_i \partial u_j \partial u_k} \ln \sigma(\mathbf{u}), \qquad \wp_{ijkl} = -\frac{\partial^4}{\partial u_i \partial u_j \partial u_k \partial u_l} \ln \sigma(\mathbf{u})$$

$$i \leq j \leq k \leq l \in \{1, 2\}$$

$\wp' \equiv \wp_{111}$ $\qquad \wp'' \equiv \wp_{1111}$

# PDEs for the hyperelliptic case

Baker found other generalisations of the elliptic results:

- Equations for the 10 possible $\wp_{ijk} \cdot \wp_{lmn}$ in terms of $\wp_{qr}$ starting with

$$\begin{aligned}\wp_{222}^2 &= 4\wp_{22}^3 + 4\wp_{12}\wp_{22} + 4\wp_{11} + \lambda_4\wp_{22}^2 + \lambda_2 \\ \wp_{122}\wp_{222} &= 4\wp_{22}^2\wp_{12} + \lambda_4\wp_{22}\wp_{12} + 2\wp_{12}^2 \\ &\quad - 2\wp_{11}\wp_{22} + \tfrac{1}{2}\lambda_3\wp_{22} + \tfrac{1}{2}\lambda_1\end{aligned}$$

elliptic case

- Equations for the five possible $\wp_{ijkl}$ in terms of the $\wp_{lm}$ starting with

$$\wp_{2222} = 6\wp_{22}^2 + \tfrac{1}{2}\lambda_3 + \lambda_4\wp_{22} + 4\wp_{12}$$

elliptic case

# Trigonal curves

- Next consider the trigonal curves. The simplest example is the (3,4)-curve which has genus 3.

$$C : y^3 = x^4 + \lambda_3 x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0$$

- We define the $\wp$-functions as in the hyperelliptic case, but now $\mathbf{u} = (u_1, u_2, u_3)$, so there are six possible $\wp$-functions.
- In the 1990s the first 4-index PDE was found

$$\wp_{3333} = 6\wp_{33}^2 - 3\wp_{22}$$

- Later, an expansion of the $\sigma$-function was calculated, which helped find the other PDEs and addition formula.

# Sato Weights

For every $(n, s)$-curve we can define a set of weights that render all equations homogeneous. These are defined using the Weierstrass Sequence for (n,s) and are labelled the Sato Weights. For the $(3, 4)$ curve they are given by

| **Variable** | $x$ | $y$ | $u_1$ | $u_2$ | $u_3$ | $\lambda_3$ | $\lambda_2$ | $\lambda_1$ | $\lambda_0$ |
|---|---|---|---|---|---|---|---|---|---|
| **Weight** | $-3$ | $-4$ | $5$ | $2$ | $1$ | $-3$ | $-6$ | $-9$ | $-12$ |

**e.g.** The equation defining the curve has weight -12

$$\begin{array}{ccccccccccc} y^3 & = & x^4 & + & \lambda_3 x^3 & + & \lambda_2 x^2 & + & \lambda_1 x & + & \lambda_0 \\ -12 & & -12 & & -3,-9 & & -6,-6 & & -9,-3 & & -12 \end{array}$$

# Sigma expansion

Consider the value of $\sigma(\mathbf{u};\lambda_i)$ when all $\lambda_i = 0$. This is shown to be the Schur-Weierstrass polynomial generated by $(n,s)$.

$$\mathrm{SW}_{3,4} = u_1 - u_3 u_2^2 + \frac{1}{20}u_3^5$$

So the sigma expansion will have weight 5. Write it in the form

$$\sigma(u_1,u_2,u_3) = C_5 + C_8 + C_{11} + C_{14} + C_{17}$$

where $C_{5+3n}$ has weight $(5+3n)$ in the $u_i$ and $-3n$ in the $\lambda_i$.
To find the $C_i$ we

1. Identify the possible terms — those with correct weight.
2. Form the sigma function with unidentified coefficients.
3. Determine coefficients by satisfying known properties.

We are able to find the sigma expansions starting with

elliptic case

$$C_8 = \left(\frac{1}{40}u_3^6 u_2 - \frac{1}{2}u_3^2 u_2^3\right)\lambda_3$$

# Addition formula

Using the method of undetermined coefficients and the $\sigma$-expansion we find the addition formula for the (3,4)-curve

$$\frac{\sigma(\mathbf{u}+\mathbf{v})\sigma(\mathbf{u}-\mathbf{v})}{\sigma(\mathbf{u})^2\sigma(\mathbf{v})^2} = \wp_{11}(\mathbf{v}) - \wp_{11}(\mathbf{u}) + \wp_{12}(\mathbf{v})\wp_{23}(\mathbf{u}) - \wp_{12}(\mathbf{u})\wp_{23}(\mathbf{v}) + \wp_{13}(\mathbf{v})\wp_{22}(\mathbf{u}) - \wp_{13}(\mathbf{u})\wp_{22}(\mathbf{v}) + \frac{1}{3}\left[Q_{1333}(\mathbf{u})\wp_{33}(\mathbf{v}) - Q_{1333}(\mathbf{v})\wp_{33}(\mathbf{u})\right]$$

elliptic case

where $$Q_{ijkl} = \wp_{ijkl} - 2(\wp_{ij}\wp_{kl} + \wp_{ik}\wp_{jl} + \wp_{il}\wp_{jk})$$

A second addition formula was discovered, which has no ananlogue in the elliptic case.

$$\frac{\sigma(\mathbf{u}+\mathbf{v})\sigma(\mathbf{u}+[\xi]\mathbf{v})\sigma(\mathbf{u}+[\xi^2]\mathbf{v})}{\sigma(\mathbf{u})^3\sigma(\mathbf{v})^3} = R(\mathbf{u},\mathbf{v}) + R(\mathbf{v},\mathbf{u})$$

where $\xi^3 = 1$

equianharmonic elliptic case

# Higher Genus Curves and Future Work

- A similar approach worked on the (3,5)-curve ($g = 4$).
- A new result has been found in the Equianharmonic Elliptic Case (when $g_2 = 0$) Trigonal-case

$$\frac{\sigma(u+v)\sigma(u+\xi v)\sigma(u+\xi^2 v)}{\sigma(u)^3\sigma(v)^3} = \frac{1}{2}\left(\wp'(u) + \wp'(v)\right) \qquad \xi^3 = 1$$

- Methods are being developed for the General Trigonal (3,4)-curve:

$$\begin{aligned} y^3 + (\mu_1 x + \mu_4)y^2 + (\mu_2 x^2 + \mu_5 x + \mu_8)y \\ = x^4 + \mu_3 x^3 + \mu_6 x^2 + \mu_9 x + \mu_{12} \end{aligned}$$

(n,s)-curves

- Work has commenced on the genus 6 cases — (4,5) and (3,7)-curves

# Further Reading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  preprint, 2006, arXiv:math/0610019v1