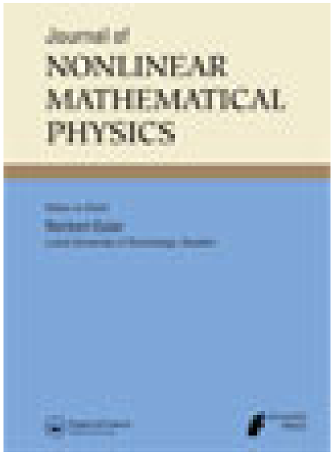

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A Method for q-Calculus

Thomas Ernst [a]

[a] Department of Mathematics , Uppsala University , Box 480 , Uppsala , Sweden E-mail:

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A Method for q-Calculus

Thomas ERNST

Department of Mathematics, Uppsala University, Box 480 Uppsala, Sweden
E-mail: thomas@math.uu.se

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Abstract

We present a notation for q-calculus, which leads to a new method for computations and classifications of q-special functions. With this notation many formulas of q-calculus become very natural, and the q-analogues of many orthogonal polynomials and functions assume a very pleasant form reminding directly of their classical counterparts.

The first main topic of the method is the tilde operator, which is an involution operating on the parameters in a q-hypergeometric series. The second topic is the q-addition, which consists of the Ward–AlSalam q-addition invented by Ward 1936 [102, p. 256] and Al-Salam 1959 [5, p. 240], and the Hahn q-addition.

In contrast to the the Ward–AlSalam q-addition, the Hahn q-addition, compare [57, p. 362] is neither commutative nor associative, but on the other hand, it can be written as a finite product.

We will use the generating function technique by Rainville [76] to prove recurrences for q-Laguerre polynomials, which are q-analogues of results in [76]. We will also find q-analogues of Carlitz' [26] operator expression for Laguerre polynomials. The notation for Cigler's [37] operational calculus will be used when needed. As an application, q-analogues of bilinear generating formulas for Laguerre polynomials of Chatterjea [33, p. 57], [32, p. 88] will be found.

1 Some classical hypergeometric equations

First we collect some well-known hypergeometric formulas in order to prove their q-analogues later. In the whole paper, the symbol $\equiv$ will denote definitions, except when we work with congruences.

Definition 1.1. Let the Pochhammer symbol (or Appell–Pochhammer) $(a)_n$ be defined by

$$(a)_n \equiv \prod_{m=0}^{n-1}(a+m), \qquad (a)_0 \equiv 1. \tag{1.1}$$

Since products of Pochhammer symbols occur so often, to simplify them we shall frequently use the following more compact notation. Let $(a) = (a_1, \ldots, a_A)$ be a vector with A elements. Then

$$((a))_n \equiv (a_1, \ldots, a_A)_n \equiv \prod_{j=1}^{A}(a_j)_n. \tag{1.2}$$

The generalized hypergeometric series, ${}_pF_r$, is given by

$$ {}_pF_r(a_1,\ldots,a_p;b_1,\ldots,b_r;z) \equiv {}_pF_r\left[\begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_r \end{matrix};z\right] \equiv \sum_{n=0}^{\infty} \frac{(a_1,\ldots,a_p)_n}{n!(b_1,\ldots,b_r)_n} z^n. \tag{1.3} $$

An ${}_{r+1}F_r$ series is called *k-balanced* if

$$ b_1+\cdots+b_r = k+a_1+\cdots+a_{r+1}, \tag{1.4} $$

and a 1-balanced series is called balanced or Saalschützian after L Saalschütz (1835–1913).

Remark 1.1. In some books, e.g., [51], a balanced hypergeometric series is defined with the extra condition $z=1$. The above definition is in accordance with [9, p. 475].

The hypergeometric series

$$ {}_{r+1}F_r(a_1,\ldots,a_{r+1};b_1,\ldots,b_r;z) \tag{1.5} $$

is called *well-poised* if its parameters satisfy the relations

$$ 1+a_1 = a_2+b_1 = a_3+b_2 = \cdots = a_{r+1}+b_r. \tag{1.6} $$

The hypergeometric series (1.5) is called *nearly-poised* [105] if its parameters satisfy the relation

$$ 1+a_1 = a_{j+1}+b_j \tag{1.7} $$

for all but one value of j in $1 \le j \le r$. If the series (1.5) is well-poised and $a_2 = 1+\frac{1}{2}a_1$, then it is called a *very-well-poised* series.

The binomial series is defined by

$$ (1-z)^{-\alpha} \equiv \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} z^n, \qquad |z|<1, \quad \alpha \in \mathbb{C}. \tag{1.8} $$

The following three equations are all due to Euler (see book by Gasper and Rahman [51, pp. 10, 19]). The beta integral

$$ \int_0^1 x^{s-1}(1-x)^{t-1}\,dx = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)} \equiv B(s,t), \qquad \mathrm{Re}\,(s)>0, \quad \mathrm{Re}\,(t)>0; \tag{1.9} $$

the integral representation of the hypergeometric series ${}_2F_1(a,b;c;z)$:

$$ {}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a}\,dx, \tag{1.10} $$
$$ |\arg(1-z)|<\pi, \qquad \mathrm{Re}\,(c) > \mathrm{Re}\,(b) > 0; $$

and Euler's transformation formula:

$$ {}_2F_1(a,b;c;z) = (1-z)^{c-a-b}{}_2F_1(c-a,c-b;c;z). \tag{1.11} $$

In 1772 Vandermonde [34, 71, 99] proved the Chu–Vandermonde summation formula

$$ {}_2F_1(-n,b;c;1) \equiv \sum_{k=0}^{n} \frac{(-1)^k (b)_k}{(c)_k} \binom{n}{k} = \frac{(c-b)_n}{(c)_n}, \qquad n = 0,1,\ldots. \tag{1.12} $$

The famous Gauss summation formula from 1812 [54] goes as follows:

$$ {}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \qquad \mathrm{Re}\,(c-a-b) > 0, \tag{1.13} $$

which is a generalization of (1.12).

Gauss [53, p. 127, (2)], [15, p. 23], [45, p. 11, (24)] also found the following formula

$$ (1+x)^n + (1-x)^n = 2\,{}_2F_1\left(\frac{-n}{2}, \frac{1-n}{2}; \frac{1}{2}; x^2\right). \tag{1.14} $$

In 1890 Saalschütz rediscovered the Pfaff–Saalschütz summation formula for a terminating balanced hypergeometric series [73, 81], [51, p. 13]

$$ {}_3F_2(a,b,-n;c,1+a+b-c-n;1) = \frac{(c-a,c-b)_n}{(c,c-a-b)_n}, \qquad n = 0,1,\ldots. \tag{1.15} $$

In 1903 Dixon [38] proved the following summation formula for a so-called well-poised series

$$ \begin{aligned} &{}_3F_2(a,b,c;1+a-b,1+a-c;1) \\ &\quad = \Gamma\left[\begin{matrix} 1+\frac{1}{2}a, 1+a-b, 1+a-c, 1+\frac{1}{2}a-b-c \\ 1+a, 1+\frac{1}{2}a-b, 1+\frac{1}{2}a-c, 1+a-b-c \end{matrix}\right]. \end{aligned} \tag{1.16} $$

Remark 1.2. This equation was also proved independently by Schafheitlin [82, p. 24, (22)] 1912.

In 1923 Whipple [103] showed that by iterating Thomae's ${}_3F_2$ transformation formula [96, eq. (11)], one obtains a set of 120 such series, and he tabulated the parameters of these 120 series.

The very important Whipple formula from 1926 [104, 7.7], [13, p. 145]

$$ \begin{aligned} &{}_7F_6\left[\begin{matrix} a, b, c, d, 1+\frac{1}{2}a, e, -n \\ 1+a-b, 1+a-c, 1+a-d, 1+a-e, \frac{1}{2}a, 1+a+n \end{matrix}; 1\right] \\ &\quad \frac{(1+a, 1+a-d-e)_n}{(1+a-d, 1+a-e)_n}\, {}_4F_3\left[\begin{matrix} d, e, 1+a-b-c, -n \\ 1+a-b, 1+a-c, d+e-n-a \end{matrix}; 1\right] \end{aligned} \tag{1.17} $$

transforms a terminating very-well-poised ${}_7F_6$ series to a Saalschützian ${}_4F_3$ series.

The following important example by Bailey is one of the most general hypergeometric transformations.

Theorem 1.1. *Bailey's 1929 transformation formula for a terminating, 2–balanced, very-well-poised ${}_9F_8$ hypergeometric series.*

Denoting

$$(\alpha) \equiv \left(a, b, c, d, e, f, 1+\tfrac{1}{2}a, \lambda+a+n+1-e-f, -n\right), \tag{1.18}$$

$$(\beta) \equiv \big(1+a-b, 1+a-c, 1+a-d, 1+a-e, \tfrac{1}{2}a, a+1-f, \\ e+f-n-\lambda, a+n+1\big), \tag{1.19}$$

$$(\gamma) \equiv \big(\lambda, \lambda+b-a, \lambda+c-a, \lambda+d-a, e, f, 1+\tfrac{1}{2}\lambda, \\ \lambda+a+n+1-e-f, -n\big) \tag{1.20}$$

and

$$(\delta) \equiv \big(1+a-b, 1+a-c, 1+a-d, 1+\lambda-e, \tfrac{1}{2}\lambda, \\ \lambda+1-f, e+f-n-a, \lambda+n+1\big), \tag{1.21}$$

we find that this formula takes the following form [17]:

$$ {}_9F_8\left[\begin{matrix}(\alpha)\\(\beta)\end{matrix};1\right] = \frac{(1+a, 1+a-e-f, 1+\lambda-e, 1+\lambda-f)_n}{(1+a-e, 1+a-f, 1+\lambda-e-f, 1+\lambda)_n}\, {}_9F_8\left[\begin{matrix}(\gamma)\\(\delta)\end{matrix};1\right] \tag{1.22}$$

when $n = 0, 1, 2, \ldots,$ *and where*

$$2a+1 = \lambda+b+c+d. \tag{1.23}$$

Remark 1.3. In Bailey's paper a slightly different version of this equation was given, but this is equivalent to the above equation.

The theory of multiple hypergeometric series has been developed by the Italian Lauricella and by the Frenchmen Appell and Kampé de Fériet, who 1926 [15] published a standard work on this subject in French. In 1985 Per Karlsson (Copenhagen) and H M Srivastava (Victoria, British Columbia) [91] published another excellent book on this subject, where some new results were presented together with results previously widely scattered in the literature.

2 The tilde operator

q-Calculus has in the last twenty years served as a bridge between mathematics and physics. The majority of scientists in the world who use q-calculus today are physicists.

> The field has expanded explosively, due to the fact that applications of basic hypergeometric series to the diverse subjects of combinatorics, quantum theory, number theory, statistical mechanics, are constantly being uncovered. The subject of q-hypergeometric series is the frog-prince of mathematics, regal inside, warty outside (Jet Wimp) [107].

Furthermore it is said that

> progress in q-calculus is heavily dependent on the use of a proper notation (Per Karlsson, Private communication).

In the last decades q-calculus has developed into an interdisciplinary subject, which is nowadays called q-disease. q-Calculus has found many applications in quantum group theory.

In this context it would be natural to say something about the history of the connection between group representation theory and quantum mechanics, which was initiated by Eugene Wigner (1902–1995) between November 12 and November 26, 1926, when Wigner's first papers on quantum mechanics reached die Zeitschrift der Physik, and both appeared in volume 40.

> It was John von Neumann who first proposed that group representation theory (should) be used in quantum mechanics. Wigner was invited to Göttingen in 1927 as assistant to David Hilbert. Though the new quantum mechanics had been initiated only in 1925, already in 1926–27 the mathematician Hilbert in Göttingen gave lectures on quantum mechanics [106].

Die Gruppenpest [93] (the pest of group theory) would last for three decades [106].

We will give an example of the application of group theory in q-calculus in a moment.

This is a modest attempt to present a new notation for q-calculus and in particular for q-hypergeometric series, which is compatible with the old notation. Also a new method, which follows from this notation is presented. The papers [40, 41, 42, 43, 44, 45] illustrate these ideas. This notation leads to a new method for computations and classifications of q-special functions. With this notation many formulas of q-calculus become very natural, and the q-analogues of many orthogonal polynomials and functions assume a very pleasant form reminding directly of their classical counterparts. This notation will be similar to Gauss' notation for hypergeometric series and in the spirit of Heine [58], Pringsheim [74], Smith [84], Agarwal [1, 2] and Agarwal & Verma [3, 4]. Jackson also used a similar notation in some of his last papers [63, 64]. A similar notation was proposed by Rajeswari V. & Srinivasa Rao K. in 1991 [77] and in 1993 [86, p. 72] in connection with the q-analogues of the 3-j and 6-j coefficients. Compare [87] and [88].

By coincidence, some of these authors were involved in the development of Whipple's work as the following references show. In 1987 Beyer, Louck & Stein [19] and in 1992 Srinivasa Rao, Van der Jeugt, Raynal, Jagannathan & Rajeswari [85] showed that certain two-term transformation formulas between hypergeometric series easily can be described by means of invariance groups. In other words, they explained Whipple's [103] discovery in group language. In 1999 Van der Jeugt & Srinivasa Rao [97] found q-analogues of these results.

These results were extended to double q-hypergeometric series in [69] and [98].

We are now ready for the first definitions in q-calculus.

Definition 2.1. The power function is defined by $q^a \equiv e^{a\log(q)}$. We always use the principal branch of the logarithm.

The q-analogues of a complex number a and of the factorial function are defined by:

$$\{a\}_q \equiv \frac{1-q^a}{1-q}, \qquad q \in \mathbb{C}\backslash\{1\}, \tag{2.1}$$

$$\{n\}_q! \equiv \prod_{k=1}^{n} \{k\}_q, \qquad \{0\}_q! = 1, \qquad q \in \mathbb{C}, \tag{2.2}$$

Let the q-shifted factorial be given by

$$\langle a;q\rangle_n \equiv \begin{cases} 1, & n=0; \\ \prod_{m=0}^{n-1}(1-q^{a+m}), & n=1,2,\ldots. \end{cases} \tag{2.3}$$

Since products of q-shifted factorials occur so often, to simplify them we shall frequently use the following more compact notation. Let $(a)=(a_1,\ldots,a_A)$ be a vector with A elements. Then

$$\langle (a);q\rangle_n \equiv \langle a_1,\ldots,a_A;q\rangle_n \equiv \prod_{j=1}^{A}\langle a_j;q\rangle_n. \tag{2.4}$$

The following operator is one of the main features of the method presented in this paper.

Definition 2.2. In the following, $\frac{\mathbb{C}}{\mathbb{Z}}$ will denote the space of complex numbers mod $\frac{2\pi i}{\log q}$. This is isomorphic to the cylinder $\mathbb{R}\times e^{2\pi i\theta}$, $\theta\in\mathbb{R}$. The operator

$$\widetilde{}: \frac{\mathbb{C}}{\mathbb{Z}} \mapsto \frac{\mathbb{C}}{\mathbb{Z}}$$

is defined by

$$a \mapsto a+\frac{\pi i}{\log q}. \tag{2.5}$$

Furthermore we define

$$\widetilde{\langle a;q\rangle_n} \equiv \langle \widetilde{a};q\rangle_n. \tag{2.6}$$

By (2.5) it follows that

$$\widetilde{\langle a;q\rangle_n} = \prod_{m=0}^{n-1}(1+q^{a+m}), \tag{2.7}$$

where this time the tilde denotes an involution which changes a minus sign to a plus sign in all the n factors of $\langle a;q\rangle_n$.

The following simple rules follow from (2.5)

$$\widetilde{a}\pm b = \widetilde{a\pm b}, \tag{2.8}$$

$$\widetilde{a}\pm\widetilde{b} = a\pm b, \tag{2.9}$$

$$q^{\widetilde{a}} = -q^a, \tag{2.10}$$

where the second equation is a consequence of the fact that we work mod $\frac{2\pi i}{\log q}$.

In a few cases the parameter a in (2.3) will be the real plus infinity $(0<|q|<1)$, they correspond to multiplication by 1.

Eduard Heine (1821–1881) who studied for Gauss, Dirichlet and Jacobi, in 1846 propounded a theory for a so-called q-hypergeometric series, which he formally proved through continued fractions.

Definition 2.3. Generalizing Heine's series, we shall define a q-hypergeometric series by (compare [51, p. 4], [57, p. 345]):

$$
\begin{aligned}
{}_p\phi_r(\hat{a_1},\ldots,\hat{a_p};\hat{b_1},\ldots,\hat{b_r}|q,z) &\equiv {}_p\phi_r\left[\begin{matrix}\hat{a_1},\ldots,\hat{a_p}\\ \hat{b_1},\ldots,\hat{b_r}\end{matrix}\,|q,z\right]\\
&\equiv \sum_{n=0}^{\infty}\frac{\langle\hat{a}_1,\ldots,\hat{a}_p;q\rangle_n}{\langle 1,\hat{b_1},\ldots,\hat{b_r};q\rangle_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+r-p} z^n,
\end{aligned}
\tag{2.11}
$$

where $q \neq 0$ when $p > r+1$, and

$$
\widehat{a} = \begin{cases} a, \\ \widetilde{a}. \end{cases}
\tag{2.12}
$$

Furthermore $a_1,\ldots,a_p,b_1,\ldots,b_r \in \mathbb{C}$,

$$
b_j \neq 0, \quad j=1,\ldots,r, \qquad b_j \neq -m, \quad j=1,\ldots,r, \quad m \in \mathbb{N},
$$

$b_j \neq \frac{2m\pi i}{\log q}$, $j=1,\ldots,r$, $m \in \mathbb{N}$ [84].

The motivation is that we need a q-analogue of

$$
\begin{aligned}
&\lim_{x\to\infty} {}_pF_r(a_1,\ldots,a_p;b_1,\ldots,b_{r-1},x;xz)\\
&\quad = {}_pF_{r-1}(a_1,\ldots,a_p;b_1,\ldots,b_{r-1};z).
\end{aligned}
\tag{2.13}
$$

This q-analogue is given by

$$
\begin{aligned}
&\lim_{x\to-\infty} {}_p\phi_r(a_1,\ldots,a_p;b_1,\ldots,b_{r-1},x|q,zq^x)\\
&\quad = {}_p\phi_{r-1}(a_1,\ldots,a_p;b_1,\ldots,b_{r-1}|q,z), \qquad (0<|q|<1).
\end{aligned}
\tag{2.14}
$$

However, the q-analogue of

$$
\lim_{\epsilon\to 0} {}_pF_r\left(a_1,\ldots,a_{p-1},\frac{1}{\epsilon};b_1,\ldots,b_r;\epsilon z\right) = {}_{p-1}F_r(a_1,\ldots,a_{p-1};b_1,\ldots,b_r;z).
\tag{2.15}
$$

is given by

$$
\begin{aligned}
&\lim_{x\to\infty} {}_p\phi_r(a_1,\ldots,a_{p-1},x;b_1,\ldots,b_r|q,z)\\
&\quad = {}_p\phi_r(a_1,\ldots,a_{p-1},\infty;b_1,\ldots,b_r|q,z), \qquad (0<|q|<1).
\end{aligned}
\tag{2.16}
$$

We have changed the notation for the q-hypergeometric series (2.11) slightly according to the new notation which is introduced in this paper. The terms to the left of | in (2.11) are thought to be exponents, and the terms to the right of | in (2.11) are thought to be ordinary numbers.

The Watson notation will also sometimes be used.

$$
(a;q)_n \equiv \begin{cases} 1, & n=0;\\ \prod_{m=0}^{n-1}(1-aq^m), & n=1,2,\ldots, \end{cases}
\tag{2.17}
$$

Remark 2.1. The relation between the new and the old notation is

$$\langle a; q\rangle_n \equiv (q^a; q)_n. \tag{2.18}$$

Remark 2.2. Also Gelfand [55, p. 38] has used a similar notation in one of his few papers on q-calculus. His comment is the following: Let us assume that at first we use Watson's notation (2.17) for the q-hypergeometric series. If all α_i and β_i are non-zero, it is convenient to pass to the new parameters a_i, b_i, where $\alpha_i = q^{a_i}$, $\beta_i = q^{b_i}$.

It seems that the snag $\alpha_i = 0$ or $\beta_i = 0$ can be evaded by putting $a_i = \infty$ or $b_i = \infty$ as in the present thesis. This was already suggested by Heine in his letter to Dirichlet 1846, which was published in the Crelle journal the same year [58]. Compare [51, p. 3].

Definition 2.4. Generalizing (2.11), we shall define a q-hypergeometric series by

$$\begin{aligned}
&{}_{p+p'}\phi_{r+r'}(a_1,\ldots,a_p;b_1,\ldots,b_r|q,z||s_1,\ldots,s_{p'};t_1,\ldots,t_{r'})\\
&\quad\equiv {}_{p+p'}\phi_{r+r'}\left[\begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_r\end{matrix}|q,z||\begin{matrix} s_1,\ldots,s_{p'} \\ t_1,\ldots,t_{r'}\end{matrix}\right]\\
&\quad\equiv \sum_{n=0}^{\infty}\frac{\langle a_1;q\rangle_n\cdots\langle a_p;q\rangle_n}{\langle 1;q\rangle_n\langle b_1;q\rangle_n\cdots\langle b_r;q\rangle_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+r+r'-p-p'}\\
&\quad\times z^n\prod_{k=1}^{p'}(s_k;q)_n\prod_{k=1}^{r'}(t_k;q)_n^{-1},
\end{aligned} \tag{2.19}$$

where $q \neq 0$ when $p + p' > r + r' + 1$.

Remark 2.3. Equation (2.19) is used in certain special cases when we need factors $(t; q)_n$ in the q-series. One example is the q-analogue of a bilinear generating formula for Laguerre polynomials.

Some of the following definitions were given in another form for bibasic series in [3, pp. 732–733].

Definition 2.5. The series

$${}_{r+1}\phi_r(a_1,\ldots,a_{r+1};b_1,\ldots,b_r\,|\,q,z) \tag{2.20}$$

is called *k-balanced* if

$$b_1+\cdots+b_r = k + a_1+\cdots+a_{r+1}, \tag{2.21}$$

and a 1-balanced series is called balanced (or Saalschützian).

Analogous to the hypergeometric case, we shall call the q-hypergeometric series (2.20) *well-poised* if its parameters satisfy the relations

$$1+a_1 = a_2+b_1 = a_3+b_2 = \cdots = a_{r+1}+b_r. \tag{2.22}$$

The q-hypergeometric series (2.20) is called *nearly-poised* [18] if its parameters satisfy the relation

$$1+a_1 = a_{j+1}+b_j \tag{2.23}$$

for all but one value of j in $1 \le j \le r$.

The q-hypergeometric series (2.20) is called *almost poised* [21] if its parameters satisfy the relation

$$\delta_j + a_1 = a_{j+1} + b_j, \qquad 1 \le j \le r, \tag{2.24}$$

where δ_j is $0, 1$ or 2.

If the series (2.20) is well-poised and if, in addition

$$a_2 = 1 + \tfrac{1}{2}a_1, \qquad a_3 = \widetilde{1 + \tfrac{1}{2}a_1}, \tag{2.25}$$

then it is called a *very-well-poised* series.

The series (2.20) is of *type I* [51] if

$$z = q. \tag{2.26}$$

The series (2.20) is of *type II* [51] if

$$z = q^{b_1 + \cdots + b_r - a_1 - \cdots - a_{r+1}}. \tag{2.27}$$

Remark 2.4. In [20, p. 534] the extra condition $z = q$ in (2.21) is given, compare with the definition of balanced hypergeometric series.

There are several advantages with this new notation:

1. The theory of hypergeometric series and the theory of q-hypergeometric series will be united.
2. We work on a logarithmic scale; i.e., we only have to add and subtract exponents in the calculations. Compare with the 'index calculus' from [14].
3. The conditions for k-balanced hypergeometric series and for k-balanced q-hypergeometric series are the same.
4. The conditions for well-poised and nearly-poised hypergeometric series and for well-poised and nearly-poised q-hypergeometric series are the same. Furthermore the conditions for almost poised q-hypergeometric series are expressed similarly.
5. The conditions for very-well-poised hypergeometric series and for very-well-poised q-hypergeometric series are similar. In fact, the extra condition for a very-well-poised hypergeometric series is $a_2 = 1 + \frac{1}{2}a_1$, and the extra conditions for a very-well-poised q-hypergeometric series are $a_2 = 1 + \frac{1}{2}a_1$ and $a_3 = \widetilde{1 + \frac{1}{2}a_1}$.
6. We don't have to distinguish between the notation for integers and non-integers in the q-case anymore.
7. It is easy to translate to the work of Cigler [37] for q-Laguerre-polynomials.

Furthermore, the method is applicable to the mock theta functions.

Definition 2.6. Let the q-Pochhammer symbol $\{a\}_{n,q}$ be defined by

$$\{a\}_{n,q} \equiv \prod_{m=0}^{n-1} \{a + m\}_q. \tag{2.28}$$

An equivalent symbol is defined in [47, p. 18] and is used throughout that book. This quantity can be very useful in some cases where we are looking for q-analogues and it is included in the new notation.

Since products of q-Pochhammer symbols occur so often, to simplify them we shall frequently use the following more compact notation. Let $(a) = (a_1, \ldots, a_A)$ be a vector with A elements. Then

$$\{(a)\}_{n,q} \equiv \{a_1, \ldots, a_A\}_{n,q} \equiv \prod_{j=1}^{A} \{a_j\}_{n,q}. \tag{2.29}$$

We define a new function, which will be convenient for notational purposes.

$$\mathrm{QE}\,(x) \equiv q^x. \tag{2.30}$$

When there are several q:s, we generalize this to

$$\mathrm{QE}\,(x, q_i) \equiv q_i^x. \tag{2.31}$$

To justify the following three definitions of infinite products we remind the reader of the following well-known theorem from complex analysis, see Rudin [80, p. 300]:

Theorem 2.1. *Let Ω be a region in the complex plane and let $H(\Omega)$ denote the holomorphic functions in Ω. Suppose $f_n \in H(\Omega)$ for $n = 1, 2, 3, \ldots,$ no f_n is identically 0 in any component of Ω, and*

$$\sum_{n=1}^{\infty} |1 - f_n(z)| \tag{2.32}$$

converges uniformly on compact subsets of Ω. Then the product

$$f(z) = \prod_{n=1}^{\infty} f_n(z) \tag{2.33}$$

converges uniformly on compact subsets of Ω. Hence $f \in H(\Omega)$.

Definition 2.7. The following functions are all holomorphic

$$\langle a; q\rangle_\infty \equiv \prod_{m=0}^{\infty} \left(1 - q^{a+m}\right), \qquad 0 < |q| < 1, \tag{2.34}$$

$$\widetilde{\langle a; q\rangle}_\infty \equiv \prod_{m=0}^{\infty} \left(1 + q^{a+m}\right), \qquad 0 < |q| < 1, \tag{2.35}$$

$$(a; q)_\infty \equiv \prod_{m=0}^{\infty} \left(1 - aq^m\right), \qquad 0 < |q| < 1, \tag{2.36}$$

$$\langle (a); q\rangle_\infty \equiv \langle a_1, \ldots, a_A; q\rangle_\infty \equiv \prod_{j=1}^{A} \langle a_j; q\rangle_\infty. \tag{2.37}$$

Remark 2.5. If a in (2.34) is a negative integer the result is zero, and in the following we have to be careful when these infinite products occur in denominators. Sometimes a limit process has to be used when two such factors occur in numerator and denominator.

We shall henceforth assume that $0 < |q| < 1$ whenever $\langle a;q\rangle_\infty$ or $(a;q)_\infty$ appears in a formula, since the infinite product in (2.34) diverges when

$$q^a \neq 0, \qquad |q| \geq 1.$$

Definition 2.8. The following two formulae serve as definitions for $\langle a;q\rangle_\alpha$ and $(a;q)_\alpha$, $\alpha \in \mathbb{C}$. Compare [57, p. 342]

$$\langle a;q\rangle_\alpha \equiv \frac{\langle a;q\rangle_\infty}{\langle a+\alpha;q\rangle_\infty}, \qquad a \neq -m-\alpha, \quad m = 0,1,\ldots, \tag{2.38}$$

$$(a;q)_\alpha \equiv \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty}, \qquad a \neq q^{-m-\alpha}, \quad m = 0,1,\ldots. \tag{2.39}$$

For negative subscripts, the shifted factorial and the q-shifted factorials are defined by

$$(a)_{-n} \equiv \frac{1}{(a-1)(a-2)\cdots(a-n)} \equiv \frac{1}{(a-n)_n} = \frac{(-1)^n}{(1-a)_n}, \tag{2.40}$$

$$\langle a;q\rangle_{-n} \equiv \frac{1}{\langle a-n;q\rangle_n} = \frac{(-1)^n q^{n(1-a)+\binom{n}{2}}}{\langle 1-a;q\rangle_n}. \tag{2.41}$$

Theorem 2.2. *The following formulae hold whenever $q \neq 0$ and $q \neq e^{2\pi i t}$, $t \in \mathbb{Q}$,*

$$\langle -a+1-n;q\rangle_n = \langle a;q\rangle_n (-1)^n q^{-\binom{n}{2}-na}, \tag{2.42}$$

$$\langle a;q\rangle_{n-k} = \frac{\langle a;q\rangle_n}{\langle -a+1-n;q\rangle_k}(-1)^k q^{\binom{k}{2}+k(1-a-n)}, \tag{2.43}$$

$$\langle a+k;q\rangle_{n-k} = \frac{\langle a;q\rangle_n}{\langle a;q\rangle_k}, \tag{2.44}$$

$$\langle a+n_1;q\rangle_{k_1} = \frac{\langle a;q\rangle_{k_2}\langle a+k_2;q\rangle_{n_2}}{\langle a;q\rangle_{n_1}}, \qquad n_1+k_1 = n_2+k_2, \tag{2.45}$$

$$\langle a+2k;q\rangle_{n-k} = \frac{\langle a;q\rangle_n \langle a+n;q\rangle_k}{\langle a;q\rangle_{2k}}, \tag{2.46}$$

$$\langle a;q\rangle_m \langle a-n;q\rangle_{2n} = \langle a;q\rangle_n \langle a-n;q\rangle_m \langle a+m-n;q\rangle_n, \tag{2.47}$$

$$\langle -n;q\rangle_k = \frac{\langle 1;q\rangle_n}{\langle 1;q\rangle_{n-k}}(-1)^k q^{\binom{k}{2}-nk}, \tag{2.48}$$

$$\langle a-n;q\rangle_k = \frac{\langle a;q\rangle_k \langle 1-a;q\rangle_n}{\langle -a+1-k;q\rangle_n q^{nk}}. \tag{2.49}$$

Proof. These identities follow from the definition (2.3). ■

Theorem 2.3. *The following formulae, which are well-known, take the following form in the new notation. They hold whenever $q \neq 0$ and $q \neq e^{2\pi i t}$, $t \in \mathbb{Q}$,*

$$\langle a;q^2\rangle_n = \langle a;q\rangle_n \widetilde{\langle a;q\rangle}_n, \tag{2.50}$$

$$\langle a;q\rangle_{2n} = \langle \frac{a}{2};q^2\rangle_n \langle \frac{a+1}{2};q^2\rangle_n, \tag{2.51}$$

$$\langle a+1;q\rangle_{2n} = \frac{\langle a;q\rangle_{2n}\langle 1+\frac{a}{2};q\rangle_n \langle \widetilde{1+\frac{a}{2}};q\rangle_n}{\langle \frac{a}{2};q\rangle_n \langle \widetilde{\frac{a}{2};q}\rangle_n}, \tag{2.52}$$

$$\frac{\langle a, \frac{2+a}{2}, \widetilde{\frac{2+a}{2}};q\rangle_n}{\langle \frac{a}{2}, \widetilde{\frac{a}{2}};q\rangle_n} = \frac{\langle a+1;q\rangle_{2n}}{\langle a+n;q\rangle_n}. \tag{2.53}$$

Proof. These identities follow from the definition (2.3) and the definition of the tilde operator. ■

Remark 2.6. The first two formulae together form a q-analogue of the very important formula [76, p. 22].

Theorem 2.4. *In 1907 Dougall [39] proved the following summation formula for the very-well-poised* 2*-balanced, i.e.,*

$$1 + 2a + n = b + c + d + e$$

series

$$\begin{aligned} &{}_7F_6\left[\begin{matrix} a, 1+\frac{1}{2}a, b, c, d, e, -n \\ \frac{1}{2}a, 1+a-b, 1+a-c, 1+a-d, 1+a-e, 1+a+n \end{matrix}; 1\right] \\ &\quad = \frac{(1+a, 1+a-b-c, 1+a-b-d, 1+a-c-d)_n}{(1+a-b, 1+a-c, 1+a-d, 1+a-b-c-d)_n}. \end{aligned} \tag{2.54}$$

The following q-analogue of (2.54) was published by Jackson in 1921 [62] (see Gasper and Rahman [51, p. 35], formula (2.6.2)). In the new notation this theorem takes the following form:

Theorem 2.5. *Let the six parameters a, b, c, d, e and n satisfy the relation*

$$1 + 2a + n = b + c + d + e, \tag{2.55}$$

and let

$$(\alpha) \equiv \left(1+a-b, 1+a-c, 1+a-d, 1+a-e, \tfrac{1}{2}a, \widetilde{\tfrac{1}{2}a}, 1+a+n\right). \tag{2.56}$$

Then

$$\begin{aligned} &{}_8\phi_7\left[\begin{matrix} a, b, c, d, 1+\frac{1}{2}a, \widetilde{1+\frac{1}{2}a}, e, -n \\ (\alpha) \end{matrix} | q, q\right] \\ &\quad = \frac{\langle 1+a, 1+a-b-c, 1+a-b-d, 1+a-c-d;q\rangle_n}{\langle 1+a-b, 1+a-c, 1+a-d, 1+a-b-c-d;q\rangle_n}, \end{aligned} \tag{2.57}$$

when $n = 0, 1, 2, \ldots$.

According to Andrews [10] and Slater [83], equation (2.57) was proved by Jackson already in 1905.

We illustrate the use of the new method with the following important example of Bailey.

For brevity, we shall sometimes replace

$$
{}_{r+1}\phi_r\left[\begin{matrix} a_1, 1+\frac{1}{2}a_1, \widetilde{1+\frac{1}{2}a_1}, a_4, a_5, \ldots, a_{r+1} \\ \frac{1}{2}a_1, \widetilde{\frac{1}{2}a_1}, 1+a_1-a_4, 1+a_1-a_5, \ldots, 1+a_1-a_{r+1} \end{matrix}|q, z\right]
$$

by the more compact notation

$$
{}_{r+1}W_r(a_1; a_4, a_5, \ldots, a_{r+1}|q, z). \tag{2.58}
$$

Theorem 2.6. *Bailey's 1929 [16] transformation formula for a terminating balanced, very-well-poised ${}_{10}\phi_9$ q-hypergeometric series. Denoting*

$$
(\alpha') \equiv \big(a, b, c, d, e, f, 1+\tfrac{1}{2}a, \widetilde{1+\tfrac{1}{2}a}, \lambda+a+n+1-e-f, -n\big), \tag{2.59}
$$

$$
\begin{aligned}(\beta') \equiv \big(&1+a-b, 1+a-c, 1+a-d, 1+a-e, \tfrac{1}{2}a, \\ &\widetilde{\tfrac{1}{2}a}, a+1-f, e+f-n-\lambda, a+n+1\big),\end{aligned} \tag{2.60}
$$

$$
\begin{aligned}(\gamma') \equiv \big(&\lambda, \lambda+b-a, \lambda+c-a, \lambda+d-a, e, f, 1+\tfrac{1}{2}\lambda, \\ &\widetilde{1+\tfrac{1}{2}\lambda}, \lambda+a+n+1-e-f, -n\big),\end{aligned} \tag{2.61}
$$

and

$$
\begin{aligned}(\delta') \equiv \big(&1+a-b, 1+a-c, 1+a-d, 1+\lambda-e, \tfrac{1}{2}\lambda, \\ &\widetilde{\tfrac{1}{2}\lambda}, \lambda+1-f, e+f-n-a, \lambda+n+1\big),\end{aligned} \tag{2.62}
$$

we find that this formula takes the following form in the new notation:

$$
\begin{aligned}{}_{10}\phi_9\left[\begin{matrix}(\alpha') \\ (\beta')\end{matrix}|q, q\right] &= \frac{\langle 1+a, 1+a-e-f, 1+\lambda-e, 1+\lambda-f; q\rangle_n}{\langle 1+a-e, 1+a-f, 1+\lambda-e-f, 1+\lambda; q\rangle_n} \\ &\times {}_{10}\phi_9\left[\begin{matrix}(\gamma') \\ (\delta')\end{matrix}|q, q\right],\end{aligned} \tag{2.63}
$$

where $n = 0, 1, 2, \ldots,$ *and where*

$$
2a+1 = \lambda+b+c+d. \tag{2.64}
$$

The concept of separation of a power series into its even and odd parts is at least as old as the series themselves. It has wideranging applications in the theory of generating functions and the MacRobert E-function. The following decomposition of the q-hypergeometric series into even and odd parts is a q-analogue of [90, p. 200–201].

$$
{}_r\phi_s((a); (b)|q, z) = {}_{4r}\phi_{4s+3}\left[\begin{matrix} \frac{(a)}{2}, \widetilde{\frac{(a)}{2}}, \frac{(a+1)}{2}, \widetilde{\frac{(a+1)}{2}} \\ \frac{(b)}{2}, \widetilde{\frac{(b)}{2}}, \frac{(b+1)}{2}, \widetilde{\frac{(b+1)}{2}}, \frac{1}{2}, \widetilde{\frac{1}{2}}, \widetilde{1} \end{matrix}|q, z^2 q^{1+s-r}\right]
$$

$$+(-1)^{1+s-r}\frac{z}{1-q}\frac{\prod_{j=1}^{r}(1-q^{a_j})}{\prod_{j=1}^{s}(1-q^{b_j})}$$

$$\times\,{}_{4r}\phi_{4s+3}\left[\begin{matrix}\frac{(a+1)}{2},\widetilde{\frac{(a+1)}{2}},\frac{(a+2)}{2},\widetilde{\frac{(a+2)}{2}}\\ \frac{(b+1)}{2},\widetilde{\frac{(b+1)}{2}},\frac{(b+2)}{2},\widetilde{\frac{(b+2)}{2}},\frac{3}{2},\widetilde{\frac{3}{2}},\widetilde{1}\end{matrix}\middle|q,z^2q^{3(1+s-r)}\right]. \tag{2.65}$$

3 The Hahn q-addition and q-analogues of the trigonometric functions

The following theorem forms the basis of q-analysis.

According to Ward [102, p. 255] and Kupershmidt [67, p. 244], this theorem was obtained by Euler. It was also obtained by Gauss 1876 [52]. It is proved by induction [41].

Theorem 3.1.

$$\sum_{n=0}^{m}(-1)^n\binom{m}{n}_q q^{\binom{n}{2}}u^n=(u;q)_m. \tag{3.1}$$

Remark 3.1. A number of similar formulas are collected in [92, p. 10].

One of Heine's pupils was Thomae, who together with reverend Jackson would develop the so-called q-integral, the inverse to the q-derivative or q-difference operator. The derivative was invented by Newton and Leibniz. Variants of the q-derivative had been used by Euler and Heine, but a real q-derivative was invented first by Jackson 1908 [60].

Definition 3.1.

$$(D_q\varphi)(x)\equiv\begin{cases}\dfrac{\varphi(x)-\varphi(qx)}{(1-q)x}, & \text{if } q\in\mathbb{C}\backslash\{1\},\quad x\neq 0;\\ \dfrac{d\varphi}{dx}(x), & \text{if } q=1;\\ \dfrac{d\varphi}{dx}(0), & \text{if } x=0.\end{cases} \tag{3.2}$$

If we want to indicate the variable which the q-difference operator is applied to, we write $(D_{q,x}\varphi)(x,y)$ for the operator.

Remark 3.2. The limit as q approaches 1 is the derivative

$$\lim_{q\to 1}(D_q\varphi)(x)=\frac{d\varphi}{dx}, \tag{3.3}$$

if φ is differentiable at x.

The definition (3.2) is more lucid than the one previously given, which was without the condition for $x=0$. It leads to new so-called q-constants, or solutions to $(D_q\varphi)(x)=0$.

We will use a notation introduced by Burchnall and Chaundy.

Definition 3.2.

$$\theta_1 \equiv x D_{q,x}, \qquad \theta_2 \equiv y D_{q,y}. \tag{3.4}$$

There are some interesting q-analogues of the exponential function and of trigonometric functions yet to be defined.

Definition 3.3. If $|q| > 1$, or $0 < |q| < 1$ and $|z| < |1-q|^{-1}$, the q-exponential function $\mathrm{E}_q(z)$ was defined by Jackson [59] 1904, and by Exton [47]

$$\mathrm{E}_q(z) \equiv \sum_{k=0}^{\infty} \frac{1}{\{k\}_q!} z^k. \tag{3.5}$$

It has q-difference

$$D_q \,\mathrm{E}_q(az) = a\,\mathrm{E}_q(az). \tag{3.6}$$

For $0 < |q| < 1$ we can define $\mathrm{E}_q(z)$ for all other values of z by analytic continuation.

Euler [46] found the following two q-analogues of the exponential function:

Definition 3.4.

$$\mathrm{e}_q(z) \equiv {}_1\phi_0(\infty; -|q, z) \equiv \sum_{n=0}^{\infty} \frac{z^n}{\langle 1; q\rangle_n} = \frac{1}{(z;q)_\infty}, \qquad |z| < 1, \quad 0 < |q| < 1, \tag{3.7}$$

$$e_{\frac{1}{q}}(z) \equiv {}_0\phi_0(-; -|q, -z) \equiv \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{\langle 1; q\rangle_n} z^n = (-z;q)_\infty, \qquad 0 < |q| < 1, \tag{3.8}$$

where ${}_0\phi_0$ is defined by (2.11).

The second function is an entire function just as the usual exponential function. The above equations can be generalized to the q-binomial theorem, which was first proved by Cauchy [28] in 1843

$${}_1\phi_0(a; -|q, z) \equiv \sum_{n=0}^{\infty} \frac{\langle a; q\rangle_n}{\langle 1; q\rangle_n} z^n = \frac{(zq^a;q)_\infty}{(z;q)_\infty} \equiv \frac{1}{(z;q)_a}, \qquad |z| < 1, \quad 0 < |q| < 1. \tag{3.9}$$

Two q-trigonometric functions are defined by

$$\mathrm{Sin}_q(x) \equiv \frac{1}{2i}(\mathrm{E}_q(ix) - \mathrm{E}_q(-ix)), \tag{3.10}$$

and

$$\mathrm{Cos}_q(x) \equiv \frac{1}{2}(\mathrm{E}_q(ix) + \mathrm{E}_q(-ix)). \tag{3.11}$$

They have q-difference

$$D_q \,\mathrm{Cos}_q(ax) = -a\,\mathrm{Sin}_q(ax), \tag{3.12}$$

$$D_q \,\mathrm{Sin}_q(bx) = b\,\mathrm{Cos}_q(bx). \tag{3.13}$$

The classical q-oscillator

$$D_q^2 f(x) + \omega^2 f(x) = 0$$

has solution $f(x) = C_1 \mathrm{Sin}_q(\omega x) + C_2 \mathrm{Cos}_q(\omega x)$. This function has an increasing amplitude for $|q| > 1$.

The following equation is easily proved.

$$\mathrm{Cos}_q(x)\mathrm{Cos}_{\frac{1}{q}}(x) + \mathrm{Sin}_q(x)\mathrm{Sin}_{\frac{1}{q}}(x) = 1. \tag{3.14}$$

Definition 3.5. The Hahn q-addition, compare [57, p. 362], is the function $\mathbb{C}^3 \mapsto \mathbb{C}^2$ given by

$$(x, y, q) \mapsto (x, y) \equiv [x + y]_q, \tag{3.15}$$

where

$$[x + y]_q^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q q^{\binom{k}{2}} \, y^k x^{n-k}, \qquad n = 0, 1, 2, \ldots. \tag{3.16}$$

Furthermore,

$$[x - y]_q \equiv [x + (-y)]_q \tag{3.17}$$

By (3.1) we obtain the original definition, which here is

Theorem 3.2 ([57, p. 362]). *Let $f(x) = \sum_{n=-\infty}^{+\infty} A_n x^n$ be a power series in x. Then*

$$f[x \pm y]_q = \sum_{n=-\infty}^{+\infty} A_n x^n \left(\mp \frac{y}{x}; q\right)_n. \tag{3.18}$$

Remark 3.3. Unlike the Ward–AlSalam q-addition, the Hahn q-addition is neither commutative nor associative, but on the other hand, it can be written as the following finite product by (3.1)

$$[x \pm y]_q^n = x^n \left(\mp \frac{y}{x}; q\right)_n, \qquad n = 0, 1, 2, \ldots. \tag{3.19}$$

This equation was generalized to complex n in [7, p. 3 (1.7)].

Remark 3.4. The physicists would probably be more happy with the notation $[x; y]_q$, but as mathematicians we will stick to the given notation.

The following equations obtain:

$$\mathrm{E}_q(x)\mathrm{E}_{\frac{1}{q}}(y) = \mathrm{E}_q[x + y]_q, \tag{3.20}$$

$$\mathrm{e}_q(x)\mathrm{e}_{\frac{1}{q}}(y) = \mathrm{e}_q[x + y]_q. \tag{3.21}$$

Proof. Expand the RHS and use (3.1). ■

The following two addition theorems are proved in the same way [59, p. 32]:

$$\mathrm{Cos}_q(x)\mathrm{Cos}_{\frac{1}{q}}(y) \pm \mathrm{Sin}_q(x)\mathrm{Sin}_{\frac{1}{q}}(y) = \mathrm{Cos}_q[x \mp y]_q, \tag{3.22}$$

$$\mathrm{Sin}_q(x)\mathrm{Cos}_{\frac{1}{q}}(y) \pm \mathrm{Cos}_q(x)\mathrm{Sin}_{\frac{1}{q}}(y) = \mathrm{Sin}_q[x \pm y]_q. \tag{3.23}$$

¿From the q-binomial theorem we obtain

$$\frac{\mathrm{e}_q(y)}{\mathrm{e}_q(x)} = \mathrm{e}_q[y - x]_q. \tag{3.24}$$

The following equations obtain:

$$D_q\mathrm{e}_q(x) = \frac{\mathrm{e}_q(x)}{1-q}, \tag{3.25}$$

$$D_q\mathrm{e}_{\frac{1}{q}}(x) = \frac{\mathrm{e}_{\frac{1}{q}}(qx)}{1-q}. \tag{3.26}$$

Definition 3.6. We can now define four other q-analogues of the trigonometric functions [57]

$$\sin_q(x) \equiv \frac{1}{2i}(\mathrm{e}_q(ix) - \mathrm{e}_q(-ix)) \equiv \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{\langle 1;q\rangle_{2n+1}}, \qquad |x| < 1, \tag{3.27}$$

$$\cos_q(x) \equiv \frac{1}{2}(\mathrm{e}_q(ix) + \mathrm{e}_q(-ix)) \equiv \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{\langle 1;q\rangle_{2n}}, \qquad |x| < 1, \tag{3.28}$$

$$\sin_{\frac{1}{q}}(x) \equiv \frac{1}{2i}(\mathrm{e}_{\frac{1}{q}}(ix) - \mathrm{e}_{\frac{1}{q}}(-ix)) \equiv \sum_{n=0}^{\infty}(-1)^n q^{n(2n+1)} \frac{x^{2n+1}}{\langle 1;q\rangle_{2n+1}}, \tag{3.29}$$

$$\cos_{\frac{1}{q}}(x) \equiv \frac{1}{2}(\mathrm{e}_{\frac{1}{q}}(ix) + \mathrm{e}_{\frac{1}{q}}(-ix)) \equiv \sum_{n=0}^{\infty}(-1)^n q^{n(2n-1)} \frac{x^{2n}}{\langle 1;q\rangle_{2n}}, \tag{3.30}$$

where $x \in \mathbb{C}$ in the last two equations.

The following two addition theorems obtain [59, p. 32], [47, p. 34]:

$$\cos_q(x)\cos_{\frac{1}{q}}(y) \pm \sin_q(x)\sin_{\frac{1}{q}}(y) = \cos_q[x \mp y]_q, \tag{3.31}$$

$$\sin_q(x)\cos_{\frac{1}{q}}(y) \pm \cos_q(x)\sin_{\frac{1}{q}}(y) = \sin_q[x \pm y]_q. \tag{3.32}$$

Remark 3.5. Observe the misprint in [57, p. 363].

The functions $\sin_{\frac{1}{q}}(x)$ and $\cos_{\frac{1}{q}}(x)$ solve the q-difference equation

$$(q-1)^2 D_q^2 f(x) + qf\left(q^2x\right) = 0, \tag{3.33}$$

and the functions $\sin_q(x)$ and $\cos_q(x)$ solve the q-difference equation

$$(q-1)^2 D_q^2 f(x) + f(x) = 0. \tag{3.34}$$

Definition 3.7. The *q-tangent numbers* $T_{2n+1}(q)$ are defined by [11, p. 380]:

$$\frac{\sin_q(x)}{\cos_q(x)} \equiv \tan_q(x) \equiv \sum_{n=0}^{\infty} \frac{T_{2n+1}(q)x^{2n+1}}{\langle 1; q\rangle_{2n+1}}. \tag{3.35}$$

It was proved by Andrews and Gessel [11, p. 380] that the polynomial $T_{2n+1}(q)$ is divisible by $\widetilde{\langle 1; q\rangle_n}$.

Definition 3.8. The *q-Euler numbers* or *q-secant numbers* $\mathcal{S}_{2n}(q)$ are defined by [12, p. 283]:

$$\frac{1}{\cos_q(x)} \equiv \sum_{n=0}^{\infty} \frac{\mathcal{S}_{2n}(q)x^{2n}}{\langle 1; q\rangle_{2n}}. \tag{3.36}$$

The congruence

$$\mathcal{S}_{2n} \equiv 1 \bmod 4 \tag{3.37}$$

was proved by Sylvester (1814–1897) [72, p. 260], [12, p. 283].

It was proved by Andrews and Foata [12, p. 283] that

$$\mathcal{S}_{2n}(q) \equiv q^{2n(n-1)} \bmod (q+1)^2. \tag{3.38}$$

4 The Ward–AlSalam *q*-addition and some variants of the *q*-difference operator

In such an interdisciplinary subject as q-calculus, many different definitions have been used, and in this section we try to collect some of them. In general, physicists tend to use symmetric operators.

Definition 4.1. A *symmetric q-difference operator* is defined by

$$D_{q_1,q_2} f(x) \equiv \frac{f(q_1 x) - f(q_2 x)}{(q_1 - q_2)x}, \tag{4.1}$$

where $q_1 = q_2^{-1}$.

Although the difference operators D_q and D_{q_1,q_2} convey the same idea, it turns out that D_{q_1,q_2} is the proper choice in constructing the Fourier transform between configuration and momentum space [50, p. 1797].

The relation between $D_{q_1,q_1^{-1}}$ and D_q is

$$D_{q_1,q_1^{-1}} = \frac{q_1 - 1}{q_1 - q_1^{-1}} D_{q_1} + \frac{1 - q_1^{-1}}{q_1 - q_1^{-1}} D_{q_1^{-1}}. \tag{4.2}$$

Definition 4.2. Euler used the operator

$$\triangle^+ \varphi(x) \equiv \frac{\varphi(x) - \varphi(qx)}{x}. \tag{4.3}$$

Remark 4.1. The q-Leibniz formula also obtains for $\triangle^+$.

Heine and Thomae used the operator

$$\triangle\varphi(x) \equiv \varphi(qx) - \varphi(x). \tag{4.4}$$

In 1994 [35] Chung K S, Chung W S, Nam S T and Kang H J rediscovered 2 q-operations (q-addition and q-subtraction) which lead to new q-binomial formulas and consequently to a new form of the q-derivative.

We will now invent an operation which will turn out to be the natural way to work with addition for the quantity to the far right of | in the q-hypergeometric series (2.11). More examples will be given in future papers on expansions of q-Appell functions.

Definition 4.3. The Ward–AlSalam q-addition, is the function (compare [102, p. 256], [5, p. 240]) $\mathbb{C}^3 \mapsto \mathbb{C}^2$ given by

$$(x, y, q) \mapsto (x, y) \equiv x \oplus_q y, \tag{4.5}$$

where

$$(x \oplus_q y)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q x^k y^{n-k}, \qquad n = 0, 1, 2, \ldots. \tag{4.6}$$

The q-subtraction is defined by

$$x \ominus_q y \equiv x \oplus_q (-y). \tag{4.7}$$

Remark 4.2. The Rogers-Szegö polynomials [78, 95] are defined in a way equivalent to (4.6).

Theorem 4.1. *This q-addition (4.6) has the following properties, $x, y, z \in \mathbb{C}$:*

$$\begin{aligned}
&(x \oplus_q y) \oplus_q z = x \oplus_q (y \oplus_q z),\\
&x \oplus_q y = y \oplus_q x,\\
&x \oplus_q 0 = 0 \oplus_q x = x,\\
&zx \oplus_q zy = z(x \oplus_q y).
\end{aligned} \tag{4.8}$$

Proof. The first property (associativity) is proved as follows: We must prove that

$$[(x \oplus_q y) \oplus_q z]^n = [x \oplus_q (y \oplus_q z)]^n. \tag{4.9}$$

But this is equivalent to

$$\sum_{k=0}^{n} \binom{n}{k}_q \sum_{l=0}^{k} \binom{k}{l}_q x^l y^{k-l} z^{n-k} = \sum_{k'=0}^{n} \binom{n}{k'}_q x^{k'} \sum_{l'=0}^{n-k'} \binom{n-k'}{l'}_q y^{l'} z^{n-k'-l'}. \tag{4.10}$$

Now put $l = k'$ and $l' = k - l$ to conclude the proof.

The proof of the distributive law is obvious. ∎

Definition 4.4. For $\alpha \in \mathbb{C}$, q-addition is extended to

$$(x \oplus_q y)^\alpha \equiv x^\alpha \sum_{k=0}^{\infty} \binom{\alpha}{k}_q \left(\frac{y}{x}\right)^k, \qquad \left|\frac{y}{x}\right| < 1. \tag{4.11}$$

Remark 4.3. The associative law doesn't hold here.

Remark 4.4. q-Addition is a special case of the so-called Gaussian convolution [56, p. 245] of $\{x_n\}$ and $\{y_n\}$

$$c_n = \sum_{k=0}^{n} \binom{n}{k}_q x_k y_{n-k}, \qquad n = 0, 1, 2, \ldots. \tag{4.12}$$

In 1936 Ward [102, p. 256] proved the following equations for q-subtraction (the original paper seems to contain a misprint of (4.13)):

$$(x \ominus_q y)^{2n+1} = \sum_{k=0}^{n} (-1)^k \binom{2n+1}{k}_q x^k y^k (x^{2n+1-2k} - y^{2n+1-2k}), \tag{4.13}$$

$$(x \ominus_q y)^{2n} = (-1)^n \binom{2n}{n}_q x^n y^n + \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k}_q x^k y^k (x^{2n-2k} + y^{2n-2k}). \tag{4.14}$$

Furthermore [102, p. 262] (the original paper seems to contain a misprint):

$$\mathrm{E}_q(x)\mathrm{E}_q(-x) = \sum_{n=0}^{\infty} \frac{(1 \ominus_q 1)^{2n}}{\{2n\}_q!} x^{2n}, \tag{4.15}$$

$$\mathrm{e}_q(x)\mathrm{e}_q(-x) = \sum_{n=0}^{\infty} \frac{(1 \ominus_q 1)^{2n}}{\langle 1; q\rangle_{2n}} x^{2n}. \tag{4.16}$$

Definition 4.5. Ward [102, p. 258] also showed that q-addition can be a function value, as follows.

If $F(x)$ denotes the formal power series

$$F(x) = \sum_{n=0}^{\infty} c_n x^n, \tag{4.17}$$

we define $F(x \oplus_q y)$ to mean the series

$$\sum_{n=0}^{\infty} c_n (x \oplus_q y)^n \equiv \sum_{n=0}^{\infty} \sum_{k=0}^{n} c_n \binom{n}{k}_q x^{n-k} y^k. \tag{4.18}$$

In like manner

$$F(x_1 \oplus_q x_2 \oplus_q \cdots \oplus_q x_k) = \sum_{n=0}^{\infty} c_n (x_1 \oplus_q x_2 \oplus_q \cdots \oplus_q x_k)^n = \sum_{n=0}^{\infty} c_n P_{kn}(x). \tag{4.19}$$

We immediately obtain the following rules for the product of two q-exponential functions

$$\mathrm{E}_q(x)\mathrm{E}_q(y) = \mathrm{E}_q(x \oplus_q y), \qquad \mathrm{e}_q(x)\mathrm{e}_q(y) = \mathrm{e}_q(x \oplus_q y). \tag{4.20}$$

Compare with the following two expression for the quotient of two q-exponential functions [101], [36, p. 91]

$$\frac{\mathrm{E}_q(y)}{\mathrm{E}_q(x)} = \sum_{k=0}^{\infty} \frac{P_k(y,x)}{\{k\}_q!}, \tag{4.21}$$

[89, p. 71]

$$\frac{\mathrm{e}_q(y)}{\mathrm{e}_q(x)} = \sum_{k=0}^{\infty} \frac{P_k(y,x)}{\langle 1;q\rangle_k}; \tag{4.22}$$

compare (3.24).

We have used the following definition from [41]:

$$P_n(x,a) = \prod_{m=0}^{n-1} (x - aq^m), \qquad n = 1, 2, \ldots. \tag{4.23}$$

In order to present Ward's q-analogue of De Moivre's formula [102] (4.25) and (4.26) we need a new notation.

Definition 4.6. Let

$$(\bar{n}_q)^k \equiv (1 \oplus_q 1 \oplus_q \cdots \oplus_q 1)^k, \qquad n \in \mathbb{N}, \tag{4.24}$$

where the number of 1 in the RHS is n.

Then

$$\mathrm{Cos}_q(\bar{n}_q x) + i\mathrm{Sin}_q(\bar{n}_q x) = (\mathrm{Cos}_q(x) + i\mathrm{Sin}_q(x))^n, \tag{4.25}$$

$$\cos_q(\bar{n}_q x) + i\sin_q(\bar{n}_q x) = (\cos_q(x) + i\sin_q(x))^n. \tag{4.26}$$

Definition 4.7. Furthermore Chung K S, Chung W S, Nam S T and Kang H J [35, p. 2023] defined a new q-derivative as follows:

$$D_{\oplus} f(x) \equiv \lim_{\delta x \to 0} \frac{f(x \oplus_q \delta x) - f(x)}{\delta x}. \tag{4.27}$$

Theorem 4.2. *This q-derivative $D_{\oplus}$ satisfies the following rules:*

$$D_{\oplus}(x \oplus_q a)^n = \{n\}_q (x \oplus_q a)^{n-1}, \tag{4.28}$$

$$D_{\oplus}\mathrm{E}_q(x) = \mathrm{E}_q(x). \tag{4.29}$$

Proof. The first equation is proved as follows:

$$\lim_{\delta x\to 0}\frac{(x\oplus_q \delta x\oplus_q a)^n-(x\oplus_q a)^n}{\delta x} \tag{4.30}$$
$$=\lim_{\delta x\to 0}\frac{\sum_{k=0}^{n}\binom{n}{k}_q\sum_{l=0}^{k}\binom{k}{l}_q x^l\delta x^{k-l}a^{n-k}-\sum_{k'=0}^{n}\binom{n}{k'}_q x^{k'}a^{n-k'}}{\delta x}=\{n\}_q(x\oplus_q a)^{n-1}. \qquad \blacksquare$$

Theorem 4.3.

$$D_\oplus x^\alpha=\{\alpha\}_q x^{\alpha-1}, \tag{4.31}$$

just as for the q-difference operator.

Proof.

$$\lim_{\delta x\to 0}\frac{(x\oplus_q \delta x)^\alpha-x^\alpha}{\delta x}=\lim_{\delta x\to 0}\frac{x^\alpha\sum_{k=0}^{\infty}\binom{\alpha}{k}_q\left(\frac{\delta x}{x}\right)^k-x^\alpha}{\delta x} \tag{4.32}$$
$$=\lim_{\delta x\to 0}x^\alpha\sum_{k=1}^{\infty}\left(\frac{\delta x}{x}\right)^k\frac{\langle-\alpha;q\rangle_k(-1)^k q^{-\binom{k}{2}+k\alpha}}{\langle 1;q\rangle_k\delta x}=\{\alpha\}_q x^{\alpha-1}. \qquad \blacksquare$$

Corollary 4.4. *The two operators D_q and $D_\oplus$ are identical when operating on functions which can be expressed as $x^\alpha\sum_{k=0}^{\infty}a_kx^k$.*

Definition 4.8. The function $\mathrm{Log}_q(x)$ is [35, p. 2025] the inverse function to $\mathrm{E}_q(x)$.

Theorem 4.5. $\mathrm{Log}_q(x)$ *satisfies the following logarithm laws with addition, replaced by q-addition:*

$$\mathrm{Log}_q(ab)=\mathrm{Log}_q(a)\oplus_q\mathrm{Log}_q(b),$$
$$\mathrm{Log}_q\left(\frac{a}{b}\right)=\mathrm{Log}_q(a)\ominus_q\mathrm{Log}_q(b),$$
$$\mathrm{Log}_q(a^n)=n\,\mathrm{Log}_q(a). \tag{4.33}$$

Definition 4.9. The function $\log_q(x)$ is the inverse function to $\mathrm{e}_q(x)$.

Theorem 4.6. $\log_q(x)$ *satisfies the same laws as for* $\mathrm{Log}_q(x)$ *above.*

Definition 4.10. A power function based on q-addition is defined by $a_q^r=\mathrm{E}_q(r\,\mathrm{Log}_q(a))$.

This power function satisfies the following laws:

Theorem 4.7.

$$a_q^xa_q^y=a_q^{x\oplus_q y},\qquad \frac{a_q^x}{a_q^y}=a_q^{x\ominus_q y},\qquad (ab)_q^x=a_q^xb_q^x,\qquad \left(\frac{a}{b}\right)_q^x=\frac{a_q^x}{b_q^x}. \tag{4.34}$$

By (4.20) we obtain the following addition theorems for the q-analogues of the trigonometric functions.

Theorem 4.8.

$$\mathrm{Cos}_q(x)\mathrm{Cos}_q(y)+\mathrm{Sin}_q(x)\mathrm{Sin}_q(y)=\mathrm{Cos}_q(x\ominus_q y), \tag{4.35}$$
$$\mathrm{Cos}_q(x)\mathrm{Cos}_q(y)-\mathrm{Sin}_q(x)\mathrm{Sin}_q(y)=\mathrm{Cos}_q(x\oplus_q y), \tag{4.36}$$
$$\mathrm{Sin}_q(x)\mathrm{Cos}_q(y)+\mathrm{Sin}_q(y)\mathrm{Cos}_q(x)=\mathrm{Sin}_q(x\oplus_q y), \tag{4.37}$$
$$\mathrm{Sin}_q(x)\mathrm{Cos}_q(y)-\mathrm{Sin}_q(y)\mathrm{Cos}_q(x)=\mathrm{Sin}_q(x\ominus_q y), \tag{4.38}$$
$$\cos_q(x)\cos_q(y)+\sin_q(x)\sin_q(y)=\cos_q(x\ominus_q y), \tag{4.39}$$
$$\cos_q(x)\cos_q(y)-\sin_q(x)\sin_q(y)=\cos_q(x\oplus_q y), \tag{4.40}$$
$$\sin_q(x)\cos_q(y)+\sin_q(y)\cos_q(x)=\sin_q(x\oplus_q y), \tag{4.41}$$
$$\sin_q(x)\cos_q(y)-\sin_q(y)\cos_q(x)=\sin_q(x\ominus_q y). \tag{4.42}$$

We also obtain 8 addition theorems for the q-analogues of the hyperbolic functions [108]. A q-analogue of (1.14) is given by

Theorem 4.9.

$$(1\oplus_q x)^n+(1\ominus_q x)^n=2\ {}_4\phi_1\left(-\frac{n}{2},\frac{1-n}{2},\infty,\infty;\frac{1}{2}|q^2,x^2q^{2n-1}\right). \tag{4.43}$$

Proof. There are two cases to consider:

1. n is even.

$$\begin{aligned}\text{LHS}&=2\sum_{k=0}^{\frac{n}{2}}\frac{x^{2k}\langle 1;q\rangle_n}{\langle 1;q\rangle_{2k}\langle 1;q\rangle_{n-2k}}=2\sum_{k=0}^{\frac{n}{2}}\frac{x^{2k}\langle\frac{1}{2},1;q^2\rangle_{\frac{n}{2}}}{\langle\frac{1}{2},1;q^2\rangle_k\langle\frac{1}{2},1;q^2\rangle_{\frac{n}{2}-k}}\\&=2\sum_{k=0}^{\frac{n}{2}}\frac{x^{2k}\langle\frac{-n}{2},\frac{-n+1}{2};q^2\rangle_k q^{2\left(-2\binom{k}{2}+\frac{nk}{2}+\frac{(n-1)k}{2}\right)}}{\langle\frac{1}{2},1;q^2\rangle_k}=\text{RHS}.\end{aligned} \tag{4.44}$$

2. n is odd.

$$\text{LHS}=2\sum_{k=0}^{\frac{n-1}{2}}\frac{x^{2k}\langle 1;q\rangle_1\langle 2;q\rangle_{n-1}}{\langle 1;q\rangle_{2k}\langle 1;q\rangle_1\langle 2;q\rangle_{n-1-2k}}=2\sum_{k=0}^{\frac{n-1}{2}}\frac{x^{2k}\langle\frac{3}{2},1;q^2\rangle_{\frac{n-1}{2}}}{\langle\frac{1}{2},1;q^2\rangle_k\langle\frac{3}{2},1;q^2\rangle_{\frac{n-1}{2}-k}} \tag{4.45}$$

$$=2\sum_{k=0}^{\frac{n}{2}}\frac{x^{2k}\langle\frac{-n}{2},\frac{-n+1}{2};q^2\rangle_k q^{2\left(-2\binom{k}{2}+\frac{nk}{2}+\frac{(n-1)k}{2}\right)}}{\langle\frac{1}{2},1;q^2\rangle_k}=\text{RHS}.\qquad\blacksquare$$

5 Generating functions and recurrences for q-Laguerre polynomials

We will use the generating function technique by Rainville [76] to prove recurrences for q-Laguerre polynomials, which are q-analogues of results in [76]. Some of these recurrences were stated already by Moak [70]. In this paper we will be working with two different q-Laguerre polynomials. The polynomial $L_{n,q,c}^{(\alpha)}(x)$ was used by Cigler [37]

$$L_{n,q,c}^{(\alpha)}(x)\equiv\sum_{k=0}^{n}\binom{n+\alpha}{n-k}_q\frac{\{n\}_q!}{\{k\}_q!}q^{k^2+\alpha k}(-1)^k x^k$$

$$\equiv \sum_{k=0}^{n} \frac{\langle 1+\alpha;q\rangle_n}{\langle 1+\alpha;q\rangle_k} \frac{\langle -n;q\rangle_k}{\langle 1;q\rangle_k} \frac{q^{\frac{k^2+k}{2}+kn+\alpha k}(1-q)^k x^k}{(1-q)^n}$$
$$\equiv \frac{\langle \alpha+1;q\rangle_n}{(1-q)^n} {}_1\phi_1\left(-n;\alpha+1|q,-x(1-q)q^{n+\alpha+1}\right). \tag{5.1}$$

The most common q-Laguerre polynomial $L_{n,q}^{(\alpha)}(x)$ is defined as follows. Except for the notation, this definition is equivalent to [70, 51] and [94]

$$L_{n,q}^{(\alpha)}(x) \equiv \frac{L_{n,q,c}^{(\alpha)}(x)}{\{n\}_q!}. \tag{5.2}$$

In [66] the q-Laguerre polynomial is defined as

$$\frac{\langle \alpha+1;q\rangle_n}{\langle 1;q\rangle_n} {}_1\phi_1\left(-n;\alpha+1|q,-xq^{n+\alpha+1}\right). \tag{5.3}$$

In the literature there are many definitions of q-Laguerre polynomials, but most of them are related to each other by some transformation.

Consider sets $\sigma_n(x)$ defined by

$$E_q(t)\Psi(xt) = \sum_{n=0}^{\infty} \sigma_n(x)t^n. \tag{5.4}$$

Let

$$F = E_q(t)\Psi(xt). \tag{5.5}$$

Then

$$D_{q,x}F = tE_q(t)D_q\Psi, \tag{5.6}$$
$$D_{q,t}F = E_q(t)\Psi + x(1-(1-q)t)E_q(t)D_q\Psi. \tag{5.7}$$

An elimination of Ψ and $D_q\Psi$ from the above equations gives

$$x(1-(1-q)t)D_{q,x}F - tD_{q,t}F = -tF, \tag{5.8}$$

and

$$\sum_{n=0}^{\infty} xD_q\sigma_n(x)t^n - \sum_{n=1}^{\infty} x(1-q)D_q\sigma_{n-1}(x)t^n - \sum_{n=0}^{\infty}\{n\}_q\sigma_n(x)t^n = -\sum_{n=1}^{\infty}\sigma_{n-1}(x)t^n.$$

By equating the coefficients of t^n we obtain the following recurrence ($\sigma_0(x) = \text{const}$):

$$xD_q\sigma_n(x) - x(1-q)D_q\sigma_{n-1}(x) - \{n\}_q\sigma_n(x) = -\sigma_{n-1}(x), \qquad n \geq 1. \tag{5.9}$$

In particular, by (5.24) we obtain the following recurrence for the q-Laguerre polynomials, which is a q-analogue of [76, p. 134]:

$$xD_qL_{n,q}^{(\alpha)}(x) - x(1-q)\{\alpha+n\}_qD_qL_{n-1,q}^{(\alpha)}(x)$$

$$= \{n\}_q L_{n,q}^{(\alpha)}(x) - \{\alpha + n\}_q L_{n-1,q}^{(\alpha)}(x). \tag{5.10}$$

Now let's assume that Ψ has the formal power series expansion

$$\Psi(u) = \sum_{n=0}^{\infty} \gamma_n u^n. \tag{5.11}$$

Then

$$\sum_{n=0}^{\infty} \sigma_n(x) t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{\gamma_k x^k t^n}{\{n-k\}_q!}, \tag{5.12}$$

so that

$$\sigma_n(x) = \sum_{k=0}^{n} \frac{\gamma_k x^k}{\{n-k\}_q!}. \tag{5.13}$$

Now by the q-binomial theorem

$$\begin{aligned} \sum_{n=0}^{\infty} \{c\}_{n,q} \sigma_n(x) t^n &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{\{c\}_{n,q} \gamma_k x^k t^n}{\{n-k\}_q!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{\{c\}_{n+k,q} \gamma_k x^k t^{n+k}}{\{n\}_q!} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{\{c+k\}_{n,q} t^n}{\{n\}_q!} \frac{\{c\}_{k,q} \gamma_k (xt)^k}{1} \\ &= \sum_{k=0}^{\infty} \{c\}_{k,q} \gamma_k (xt)^k \frac{(tq^{c+k};q)_\infty}{(t;q)_\infty} = \sum_{k=0}^{\infty} \frac{\{c\}_{k,q} \gamma_k (xt)^k}{(t;q)_{c+k}}. \end{aligned} \tag{5.14}$$

As a special case we get the following generating function which is a q-analogue of [48, p. 43, (73)], [76, p. 135, (13)]

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{\{c\}_{n,q} L_{n,q}^{(\alpha)}(x) t^n}{\{1+\alpha\}_{n,q}} &= \sum_{n=0}^{\infty} \frac{\{c\}_{n,q} q^{n^2+\alpha n} (-xt)^n}{\{n\}_q! \{1+\alpha\}_{n,q} (t;q)_{c+n}} \\ &\equiv \frac{1}{(t;q)_c} {}_1\phi_2(c; 1+\alpha | q; -xtq^{1+\alpha}(1-q) || -; tq^c). \end{aligned} \tag{5.15}$$

Consider the important case $c = 1 + \alpha$ in (5.15). This is equivalent to [70, p. 29 4.17], [6, p. 132 4.2], [49, p. 120 11′]. Call the RHS $F(x,t,q,\alpha)$. By computing the q-difference of $F(x,t,q,\alpha)$ with respect to x we obtain

$$D_{q,x} F = -tq^{1+\alpha} F(qx, t, q, \alpha+1). \tag{5.16}$$

Equating coefficients of t^n, we obtain the following recurrence relation which is a q-analogue of [76, p. 203]. Also compare with [66, p. 109, 3.21.8] and [68, p. 79]

$$D_q L_{n,q}^{(\alpha)}(x) = -q^{1+\alpha} L_{n-1,q}^{(1+\alpha)}(xq). \tag{5.17}$$

By computing the q-difference of $F(x,t,q,\alpha)$ with respect to t and equating coefficients of t^n, we obtain

$$\{n+1\}_q L_{n+1,q}^{(\alpha)}(x) = \{\alpha+1\}_q L_{n,q}^{(\alpha+1)}(x) + \frac{L_{n+1,q}^{(\alpha+1)}\left(\frac{x}{q}\right) - L_{n+1,q}^{(\alpha+1)}(x)}{1-q}. \tag{5.18}$$

Proof.

$$\begin{aligned}
D_{q,t}F &= \sum_{n=0}^{\infty} \frac{q^{n^2+n\alpha}(-x)^n((t;q)_{\alpha+1+n}\{n\}_q t^{n-1} - t^n D_q(t;q)_{\alpha+1+n})}{(tq;q)_{\alpha+1+n}(t;q)_{\alpha+1+n}\{n\}_q!} \\
&= \sum_{n=0}^{\infty} \frac{q^{n^2+n\alpha}(-x)^n t^n((t;q)_{\alpha+1+n}\{n\}_q t^{-1} + \{\alpha+1+n\}_q(tq;q)_{\alpha+n})}{(tq;q)_{\alpha+1+n}(t;q)_{\alpha+1+n}\{n\}_q!} \\
&= \sum_{n=0}^{\infty} \frac{q^{n^2+n\alpha}(-x)^n t^n(\{n\}_q \frac{1-t}{t} + \{\alpha+1+n\}_q)}{(t;q)_{\alpha+2+n}\{n\}_q!} \\
&= \sum_{n=0}^{\infty} \frac{q^{n^2+n\alpha+n}\{\alpha+1\}_q(-xt)^n}{(t;q)_{\alpha+2+n}\{n\}_q!} + \sum_{n=0}^{\infty} \frac{q^{n^2+n\alpha}\{n\}_q(-x)^n t^{n-1}}{(t;q)_{\alpha+2+n}\{n\}_q!} \\
&= \sum_{n=0}^{\infty} \frac{q^{n^2+n\alpha+n}\{\alpha+1\}_q(-xt)^n}{(t;q)_{\alpha+2+n}\{n\}_q!} + \frac{1}{t(1-q)} \sum_{n=0}^{\infty} \frac{q^{n^2+n\alpha+n}(-xt)^n(\frac{1}{q^n}-1)}{(t;q)_{\alpha+2+n}\{n\}_q!} \\
&= \sum_{n=0}^{\infty} t^n\{\alpha+1\}_q L_{n,q}^{(\alpha+1)}(x) + \frac{1}{1-q}\sum_{n=0}^{\infty} t^{n-1}\left(L_{n,q}^{(\alpha+1)}\left(\frac{x}{q}\right) - L_{n,q}^{(\alpha+1)}(x)\right) \\
&= \sum_{n=0}^{\infty} t^n\{\alpha+1\}_q L_{n,q}^{(\alpha+1)}(x) + \frac{1}{(1-q)}\sum_{n=0}^{\infty} t^{n}\left(L_{n+1,q}^{(\alpha+1)}\left(\frac{x}{q}\right) - L_{n+1,q}^{(\alpha+1)}(x)\right). \qquad (5.19)
\end{aligned}$$

Equating coefficients of t^n we are done. ■

The last equation can be expressed as

$$\{n+1\}_q L_{n+1,q}^{(\alpha)}(x) = \{\alpha+1\}_q L_{n,q}^{(\alpha+1)}(x) - xq^{2+\alpha}L_{n,q}^{(\alpha+2)}(x). \qquad (5.20)$$

Furthermore, the relation $(1-t)F(x,t,q,\alpha+1) = F(x,tq,q,\alpha)$ yields the following mixed recurrence relation, which was already stated in [70, p. 29 4.12]:

$$L_{n,q}^{(\alpha+1)}(x) - L_{n-1,q}^{(\alpha+1)}(x) = q^n L_{n,q}^{(\alpha)}(x). \qquad (5.21)$$

By the q-binomial theorem we obtain the following equation, which is a generalization of [70, p. 29 4.10] and which is a q-analogue of [76, p. 209], [6, p. 131 3.16], [49, p. 130 38]

$$L_{n,q}^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{\langle\alpha-\beta;q\rangle_k}{\langle 1;q\rangle_k} L_{n-k,q}^{(\beta)}(x) q^{(\alpha-\beta)(n-k)}, \qquad \alpha,\beta \in \mathbb{C}. \qquad (5.22)$$

Proof.

$$\begin{aligned}
\sum_{n=0}^{\infty} L_{n,q}^{(\alpha)}(x)t^n &= \sum_{n=0}^{\infty} \frac{q^{n^2+\alpha n}(-xt)^n}{\{n\}_q!(t;q)_{1+\alpha+n}} = \frac{1}{(t;q)_{\alpha-\beta}} \sum_{n=0}^{\infty} \frac{q^{n^2+\beta n}(-xt)^n q^{(\alpha-\beta)n}}{\{n\}_q!(tq^{\alpha-\beta};q)_{1+\beta+n}} \\
&= \sum_{k=0}^{\infty} \frac{\langle\alpha-\beta;q\rangle_k}{\langle 1;q\rangle_k} t^k \sum_{l=0}^{\infty} L_{l,q}^{(\beta)}(x) t^l q^{(\alpha-\beta)l}.
\end{aligned}$$

Equating coefficients of t^n we are done. ■

By (5.21) and (5.17) the following important recurrence obtains:

$$D_q(L_{n,q}^{(\alpha)}(x) - L_{n-1,q}^{(\alpha)}(x)) = -q^{n+\alpha}L_{n-1,q}^{(\alpha)}(xq). \tag{5.23}$$

The following generating function can also be found in [66, p. 109, 3.21.13]. It is a q-analogue of [48, p. 43, (73″)], [76, p. 130], [49, p. 121 12′]

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{L_{n,q}^{(\alpha)}(x)t^n}{\{1+\alpha\}_{n,q}} &= E_q(t)_0\phi_1\left(-;1+\alpha|q,q^{1+\alpha}(1-q)^2(-xt)\right)\\ &= \Gamma_q(1+\alpha)(xt)^{-\frac{\alpha}{2}}E_q(t)J_{\alpha}^{(2)}\left(2(1-q)\sqrt{xt};q\right).\end{aligned} \tag{5.24}$$

Proof. Let $c \to \infty$ in (5.15). ■

Remark 5.1. Another similar generating function is obtained by letting $t \to tq^{-c}$, $c \to -\infty$ in (5.15). These limits are q-analogues of an idea used by Feldheim [48, p. 43], which is not mentioned by Rainville.

Making use of the decomposition of a series into even and odd parts from [90, p. 200, 208], we can rewrite (5.24) in the form

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{L_{2n,q}^{(\alpha)}(x)t^{2n}}{\{1+\alpha\}_{2n,q}} + \frac{t}{\{1+\alpha\}_q}\sum_{n=0}^{\infty}\frac{L_{2n+1,q}^{(\alpha)}(x)t^{2n}}{\{2+\alpha\}_{2n,q}} = E_q(t)\Bigg[{}_0\phi_7\Bigg(-;\frac{1+\alpha}{2},\\ &\quad \frac{\widetilde{1+\alpha}}{2},\frac{2+\alpha}{2},\frac{\widetilde{2+\alpha}}{2},\frac{1}{2},\frac{\widetilde{1}}{2},\widetilde{1}|q,q^{4+2\alpha}(1-q)^4x^2t^2\Bigg) - \frac{q^{1+\alpha}(1-q)xt}{1-q^{1+\alpha}}\\ &\quad \times {}_0\phi_7\Bigg(-;\frac{2+\alpha}{2},\frac{\widetilde{2+\alpha}}{2},\frac{3+\alpha}{2},\frac{\widetilde{3+\alpha}}{2},\frac{3}{2},\frac{\widetilde{3}}{2},\widetilde{1}|q,q^{8+2\alpha}(1-q)^4x^2t^2\Bigg)\Bigg],\end{aligned} \tag{5.25}$$

and replacing t in (5.25) by it, we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{L_{2n,q}^{(\alpha)}(x)(-t^2)^n}{\{1+\alpha\}_{2n,q}} + \frac{it}{\{1+\alpha\}_q}\sum_{n=0}^{\infty}\frac{L_{2n+1,q}^{(\alpha)}(x)(-t^2)^n}{\{2+\alpha\}_{2n,q}}\\ &\quad = (\mathrm{Cos}_q(t)+i\mathrm{Sin}_q(t))_0\phi_7\Bigg(-;\frac{1+\alpha}{2},\frac{\widetilde{1+\alpha}}{2},\frac{2+\alpha}{2},\frac{\widetilde{2+\alpha}}{2},\frac{1}{2},\frac{\widetilde{1}}{2},|q,\widetilde{1}\\ &\quad - q^{4+2\alpha}(1-q)^4x^2t^2\Bigg) + \frac{q^{1+\alpha}(1-q)xt}{1-q^{1+\alpha}}(\mathrm{Sin}_q(t)-i\mathrm{Cos}_q(t))\\ &\quad \times {}_0\phi_7\Bigg(-;\frac{2+\alpha}{2},\frac{\widetilde{2+\alpha}}{2},\frac{3+\alpha}{2},\frac{\widetilde{3+\alpha}}{2},\frac{3}{2},\frac{\widetilde{3}}{2},\widetilde{1}|q,-q^{8+2\alpha}(1-q)^4x^2t^2\Bigg).\end{aligned} \tag{5.26}$$

Next equate real and imaginary parts from both sides to arrive at the generating functions

$$\sum_{n=0}^{\infty}\frac{L_{2n,q}^{(\alpha)}(x)(-t^2)^n}{\{1+\alpha\}_{2n,q}} = \mathrm{Cos}_q(t)_0\phi_7\Bigg(-;\frac{1+\alpha}{2},\frac{\widetilde{1+\alpha}}{2},\frac{2+\alpha}{2},\frac{\widetilde{2+\alpha}}{2},\frac{1}{2},$$

$$\frac{\widetilde{1}}{2}, \widetilde{1}|q, -q^{4+2\alpha}(1-q)^4x^2t^2\Bigg) + \frac{q^{1+\alpha}(1-q)xt}{1-q^{1+\alpha}} \operatorname{Sin}_q(t)$$
$$\times\, {}_0\phi_7\left(-; \frac{2+\alpha}{2}, \widetilde{\frac{2+\alpha}{2}}, \frac{3+\alpha}{2}, \widetilde{\frac{3+\alpha}{2}}, \frac{3}{2}, \frac{\widetilde{3}}{2}, \widetilde{1}|q, -q^{8+2\alpha}(1-q)^4x^2t^2\right) \tag{5.27}$$

and

$$\sum_{n=0}^{\infty} \frac{L^{(\alpha)}_{2n+1,q}(x)(-t^2)^n}{\{2+\alpha\}_{2n,q}} = \frac{\{1+\alpha\}_q \operatorname{Sin}_q(t)}{t} {}_0\phi_7\Bigg(-; \frac{1+\alpha}{2}, \widetilde{\frac{1+\alpha}{2}}, \frac{2+\alpha}{2},$$
$$\widetilde{\frac{2+\alpha}{2}}, \frac{1}{2}, \frac{\widetilde{1}}{2}, \widetilde{1}|q, -q^{4+2\alpha}(1-q)^4x^2t^2\Bigg) - xq^{1+\alpha}\operatorname{Cos}_q(t)$$
$$\times\, {}_0\phi_7\left(-; \frac{2+\alpha}{2}, \widetilde{\frac{2+\alpha}{2}}, \frac{3+\alpha}{2}, \widetilde{\frac{3+\alpha}{2}}, \frac{3}{2}, \frac{\widetilde{3}}{2}, \widetilde{1}|q, -q^{8+2\alpha}(1-q)^4x^2t^2\right). \tag{5.28}$$

The following generating function is a q-analogue of [48, p. 43, (74′)], [27, p. 399], [49, p. 120 11″]

$$\sum_{n=0}^{\infty} L^{(\alpha-n)}_{n,q}(x)t^n q^{\binom{n}{2}-n\alpha} = \frac{E_{\frac{1}{q}}(-xt)}{(-t;q)_{-\alpha}}, \qquad |t|<1, \quad |x|<1. \tag{5.29}$$

Proof.

$$\begin{aligned}
&\sum_{n=0}^{\infty} L^{(\alpha-n)}_{n,q}(x)t^n q^{\binom{n}{2}-n\alpha} \\
&= \sum_{n=0}^{\infty} t^n q^{\binom{n}{2}-n\alpha} \sum_{k=0}^{n} \frac{\langle 1+\alpha-n;q\rangle_n \langle -n;q\rangle_k}{\langle 1+\alpha-n;q\rangle_k \langle 1;q\rangle_k} \frac{q^{-\binom{k}{2}+k^2+k\alpha}(1-q)^k x^k}{\langle 1;q\rangle_n} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} t^{n+k} q^{\frac{n^2+2nk+k^2-n-k}{2}-(n+k)\alpha} \\
&\times \frac{\langle 1+\alpha-n-k;q\rangle_{n+k} \langle -n-k;q\rangle_k}{\langle 1+\alpha-n-k;q\rangle_k \langle 1;q\rangle_k} \frac{q^{-\binom{k}{2}+k^2+k\alpha}(1-q)^k x^k}{\langle 1;q\rangle_{n+k}} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} t^{n+k} q^{\frac{n^2+k^2-n-k}{2}-n\alpha} \frac{\langle 1+\alpha-n;q\rangle_n}{\langle 1;q\rangle_k \langle 1;q\rangle_n} (-1)^k (1-q)^k x^k \qquad (5.30) \\
&= \sum_{n=0}^{\infty} t^n (-1)^n \frac{\langle -\alpha;q\rangle_n}{\langle 1;q\rangle_n} \sum_{k=0}^{\infty} (1-q)^k x^k (-1)^k \frac{t^k q^{\binom{k}{2}}}{\langle 1;q\rangle_k} = \frac{E_{\frac{1}{q}}(-xt)}{(-t;q)_{-\alpha}}. \qquad \blacksquare
\end{aligned}$$

6 Product expansions

The theory of commutative ordinary differential operators was first explored in depth by Burchnall and Chaundy [22, 23, 24]. This technique was then used to find differential equations for hypergeometric functions in many papers, e.g. [25]. Some similar q-analogues of these results already exist in the literature, and we will prove five q-product expansions starting with a q-analogue of Carlitz' result [26, p. 220].

Theorem 6.1. *Let ϵ denote the operator which maps $f(x)$ to $f(qx)$. Then*

$$L_{n,q,c}^{(\alpha)}(x) = \prod_{k=3}^{n} \left(q^k x D_q \epsilon^{-1} - xq^{2k+\alpha-1} + \{\alpha+k\}_q\right) \\ \times \left(qxD_q - xq^{3+\alpha} + \{\alpha+2\}_q\right)\left(xD_q - xq^{1+\alpha} + \{\alpha+1\}_q\right)1, \tag{6.1}$$

where the number of factors to the right is n.

Proof. The theorem is true for $n=0$. Also we find that it's true for $n=1,2$. Assume that it is true for $n-1$, $n \geq 3$. Then we must prove that

$$\sum_{k=0}^{n} \frac{\langle 1+\alpha;q\rangle_n}{\langle 1+\alpha;q\rangle_k} \frac{\langle -n;q\rangle_k}{\langle 1;q\rangle_k} \frac{q^{-\binom{k}{2}+k^2+kn+\alpha k}(1-q)^k x^k}{(1-q)^n} \\ = \left(q^n x D_q \epsilon^{-1} - xq^{2n+\alpha-1} + \{\alpha+n\}_q\right) \\ \times \sum_{k=0}^{n-1} \frac{\langle 1+\alpha;q\rangle_{n-1}}{\langle 1+\alpha;q\rangle_k} \frac{\langle 1-n;q\rangle_k}{\langle 1;q\rangle_k} \frac{q^{\frac{k^2-k}{2}+kn+\alpha k}(1-q)^k x^k}{(1-q)^{n-1}}. \tag{6.2}$$

A calculation shows that

$$\text{RHS} = \sum_{k=0}^{n-1} \frac{\langle 1+\alpha;q\rangle_n}{\langle 1+\alpha;q\rangle_k} \frac{\langle 1-n;q\rangle_k}{\langle 1;q\rangle_k} \frac{q^{\frac{k^2-k}{2}+kn+\alpha k}(1-q)^k x^k}{(1-q)^n} \\ - \sum_{k=0}^{n-1} \frac{\langle 1+\alpha;q\rangle_{n-1}}{\langle 1+\alpha;q\rangle_k} \frac{\langle 1-n;q\rangle_k}{\langle 1;q\rangle_k} \frac{q^{\frac{k^2-k}{2}+kn+\alpha k} q^{2n+\alpha-1}(1-q)^k x^{k+1}}{(1-q)^{n-1}} \\ + q^n \sum_{k=1}^{n-1} \frac{\langle 1+\alpha;q\rangle_{n-1}}{\langle 1+\alpha;q\rangle_k} \frac{\langle 1-n;q\rangle_k}{\langle 1;q\rangle_{k-1}} \frac{q^{\frac{k^2-k}{2}+kn+\alpha k} q^{-k}(1-q)^k x^k}{(1-q)^n}. \tag{6.3}$$

Finally, we must prove that

$$\frac{1-q^{n+\alpha}}{1-q^{k+\alpha}} \frac{1-q^{-n}}{1-q^k} = \frac{q^{-k}(1-q^{n+\alpha})}{1-q^{\alpha+k}} \frac{1-q^{k-n}}{1-q^k} + \frac{q^{n-2k}(1-q^{k-n})}{1-q^{\alpha+k}} - q^{-2k+n}, \tag{6.4}$$

which is easily checked. ∎

The following theorem, which is a q-analogue of [75, p. 374 (2)], [100, p. 5 (31)] is proved in a similar way.

Theorem 6.2.

$$L_{n,q,c}^{(\alpha)}(x) = E_{\frac{1}{q}}(x) \prod_{k=1}^{n} (q^{k+\alpha} x D_q + \{\alpha+k\}_q) E_q(-x). \tag{6.5}$$

Proof. The theorem is true for $n=0$. Assume that it is true for $n-1$. Then we must prove that

$$\sum_{k=0}^{n} \frac{\langle 1+\alpha;q\rangle_n}{\langle 1+\alpha;q\rangle_k} \frac{\langle -n;q\rangle_k}{\langle 1;q\rangle_k} \frac{q^{-\binom{k}{2}+k^2+kn+\alpha k}(1-q)^k x^k}{(1-q)^n}$$

$$
\begin{aligned}
&= E_{\frac{1}{q}}(x)(q^{n+\alpha}xD_q + \{\alpha+n\}_q) \\
&= \sum_{k=0}^{n-1} \frac{\langle 1+\alpha;q\rangle_{n-1}}{\langle 1+\alpha;q\rangle_k} \frac{\langle -n+1;q\rangle_k}{\langle 1;q\rangle_k} \frac{q^{-\binom{k}{2}+k^2+k(n-1)+\alpha k}(1-q)^k x^k}{(1-q)^{n-1}} E_q(-x). \qquad (6.6)
\end{aligned}
$$

A calculation shows that

$$
\begin{aligned}
\text{RHS} = E_{\frac{1}{q}}(x)\Bigg[&\{\alpha+n\}_q \sum_{k=0}^{n-1} \frac{\langle 1+\alpha;q\rangle_{n-1}}{\langle 1+\alpha;q\rangle_k} \frac{\langle -n+1;q\rangle_k}{\langle 1;q\rangle_k} \frac{q^{-\binom{k}{2}+k^2+k(n-1)+\alpha k}(1-q)^k x^k}{(1-q)^{n-1}} \\
&+ q^{n+\alpha}x\left[\sum_{k=1}^{n-1} \frac{\langle 1+\alpha;q\rangle_{n-1}}{\langle 1+\alpha;q\rangle_k} \frac{\langle -n+1;q\rangle_k}{\langle 1;q\rangle_k} \frac{q^{-\binom{k}{2}+k^2+k(n-1)+\alpha k}(1-q)^k(1-q^k)x^{k-1}}{(1-q)^n}\right. \\
&\left. - \sum_{k=0}^{n-1} \frac{\langle 1+\alpha;q\rangle_{n-1}}{\langle 1+\alpha;q\rangle_k} \frac{\langle -n+1;q\rangle_k}{\langle 1;q\rangle_k} \frac{q^{-\binom{k}{2}+k^2+kn+\alpha k}(1-q)^k x^k}{(1-q)^{n-1}}\right]\Bigg] E_q(-x). \qquad (6.7)
\end{aligned}
$$

We must prove that

$$
\begin{aligned}
&\frac{1-q^{n+\alpha}}{\langle 1+\alpha;q\rangle_k} \frac{\langle -n;q\rangle_k}{\langle 1;q\rangle_k} \frac{q^{\frac{k^2}{2}+\frac{k}{2}+kn+\alpha k}(1-q)^k}{(1-q)^n} \\
&\quad = \frac{1-q^{n+\alpha}}{\langle 1+\alpha;q\rangle_k} \frac{\langle 1-n;q\rangle_k}{\langle 1;q\rangle_k} \frac{q^{\frac{k^2}{2}-\frac{k}{2}+kn+\alpha k}(1-q)^k}{(1-q)^n} \\
&\quad + \frac{q^{n+\alpha}}{\langle 1+\alpha;q\rangle_k} \frac{\langle 1-n;q\rangle_k}{\langle 1;q\rangle_k} \frac{q^{\frac{k^2}{2}-\frac{k}{2}+kn+\alpha k}(1-q^k)(1-q)^k}{(1-q)^n} \\
&\quad - \frac{q^{n+\alpha}}{\langle 1+\alpha;q\rangle_{k-1}} \frac{\langle 1-n;q\rangle_{k-1}}{\langle 1;q\rangle_{k-1}} \frac{q^{\frac{k^2}{2}-\frac{k}{2}+kn-n+\alpha k-\alpha}(1-q)^k}{(1-q)^n}, \qquad (6.8)
\end{aligned}
$$

which implies that

$$
\frac{1-q^{n+\alpha}}{1-q^{k+\alpha}} \frac{1-q^{-n}}{1-q^k} = \frac{q^{-k}(1-q^{n+\alpha})}{1-q^{\alpha+k}} \frac{1-q^{k-n}}{1-q^k} + \frac{q^{n+\alpha-k}(1-q^{k-n})}{1-q^{\alpha+k}} - q^{-k}, \qquad (6.9)
$$

which is easily checked. ■

The following theorem is a q-analogue of Chatterjea [30, p. 286 ($k=1$)].

Theorem 6.3.

$$
L_{n,q,c}^{(\alpha)}(x) = x^{-\alpha} E_{\frac{1}{q}}(x)(D_q)^n (x^{\alpha+n} E_q(-x)). \qquad (6.10)
$$

Proof. Just use Leibniz' rule for the n:th q-difference of a product of functions. ■

The following theorem is a q-analogue of Chak [29], see also Chatterjea [33].

Theorem 6.4.

$$
L_{n,q,c}^{(\alpha)}(x) = x^{-\alpha-n-1} E_{\frac{1}{q}}(x)(x^2 D_q)^n (x^{\alpha+1} E_q(-x)). \qquad (6.11)
$$

Proof. The theorem is true for $n=0$. Assume that it is true for $n-1$. Then we must prove that

$$\sum_{k=0}^{n}\frac{\langle 1+\alpha;q\rangle_n}{\langle 1+\alpha;q\rangle_k}\frac{\langle -n;q\rangle_k}{\langle 1;q\rangle_k}\frac{q^{\frac{k^2+k}{2}+kn+\alpha k}(1-q)^k x^k}{(1-q)^n}=x^{-\alpha-n-1}E_{\frac{1}{q}}(x)x^2$$
$$\times D_q\sum_{k=0}^{n-1}\frac{\langle 1+\alpha;q\rangle_{n-1}}{\langle 1+\alpha;q\rangle_k}\frac{\langle -n+1;q\rangle_k}{\langle 1;q\rangle_k}\frac{q^{\frac{k^2+k}{2}+k(n-1)+\alpha k}(1-q)^k x^{k+\alpha+n}}{(1-q)^{n-1}}E_q(-x). \quad (6.12)$$

A calculation shows that

$$\begin{aligned}\text{RHS}&=\sum_{k=0}^{n-1}\frac{\langle 1+\alpha;q\rangle_{n-1}}{\langle 1+\alpha;q\rangle_k}\frac{\langle -n+1;q\rangle_k}{\langle 1;q\rangle_k}\\&\quad\times\frac{q^{\frac{k^2+k}{2}+k(n-1)+\alpha k}(1-q)^k x^k}{(1-q)^{n-1}}(\{k+\alpha+n\}_q(1+(1-q)x)-x)\\&=\frac{\langle 1+\alpha;q\rangle_n}{(1-q)^n}-\frac{\langle -n+1;q\rangle_{n-1}}{\langle 1;q\rangle_{n-1}}q^{\frac{n^2-n}{2}+n^2+\alpha n}x^n\\&\quad+\sum_{k=1}^{n-1}\frac{\langle 1+\alpha;q\rangle_{n-1}}{\langle 1+\alpha;q\rangle_k}\frac{\langle -n+1;q\rangle_k}{\langle 1;q\rangle_k}\frac{q^{\frac{k^2+k}{2}+kn-k+\alpha k}x^k}{(1-q)^{n-k}}(1-q^{k+\alpha+n}) \qquad (6.13)\\&\quad-\sum_{k=1}^{n-1}\frac{\langle 1+\alpha;q\rangle_{n-1}}{\langle 1+\alpha;q\rangle_{k-1}}\frac{\langle -n+1;q\rangle_{k-1}}{\langle 1;q\rangle_{k-1}}\frac{q^{\frac{k^2-k}{2}+kn+\alpha k}x^k}{(1-q)^{n-k}}=\text{LHS}.\end{aligned}$$

■

The following theorem is a q-analogue of Chatterjea [31] and a generalization of (6.11).

Theorem 6.5.

$$L_{n,q,c}^{(\alpha)}(x)=x^{-\alpha-n-k}E_{\frac{1}{q}}(x)\left(\{1-k\}_q x+q^{1-k}x^2D_q\right)^n(x^{\alpha+k}E_q(-x)). \quad (6.14)$$

We will now prove a couple of bilinear generating formulae for q-Laguerre polynomials. There is much more to be proved as can be seen from the corresponding hypergeometric identities in [90, p. 133–135, 245 (18)] and from the paper [65, p. 427, 430–431]. With the help of (6.11) we can prove a q-analogue of Chatterjea [33, p. 57].

Theorem 6.6.

$$\begin{aligned}&\sum_{n=0}^{\infty}\{n\}_q!L_{n,q}^{(\alpha-n)}(x)L_{n,q}^{(\beta-n)}(y)t^n\\&=E_{\frac{1}{q}}(x)E_{\frac{1}{q}}(y)\sum_{r,s=0}^{\infty}\frac{(-1)^{r+s}x^r y^s}{\{r\}_q!\{s\}_q!}{}_3\phi_0\left(\infty,-r-\alpha,-s-\beta;-|q,\frac{tq^{r+\alpha+s+\beta}}{1-q}\right).\end{aligned} \quad (6.15)$$

Proof.

$$\text{LHS}=\sum_{n=0}^{\infty}\frac{x^{-\alpha-1}}{\{n\}_q!}E_{\frac{1}{q}}(x)(x^2D_{q,x})^n x^{\alpha-n+1}E_q(-x)y^{-\beta-1}$$

$$\times E_{\frac{1}{q}}(y)(y^2 D_{q,y})^n y^{\beta-n+1} E_q(-y)t^n = E_{\frac{1}{q}}(x)E_{\frac{1}{q}}(y)x^{-\alpha-1}y^{-\beta-1}$$
$$\times \sum_{n=0}^{\infty} \frac{t^n}{\{n\}_q!}(x\theta_1)^n(y\theta_2)^n x^{\alpha-n+1}y^{\beta-n+1}E_q(-x)E_q(-y)$$
$$= E_{\frac{1}{q}}(x)E_{\frac{1}{q}}(y)x^{-\alpha-1}y^{-\beta-1}\sum_{n=0}^{\infty}\frac{t^n}{\{n\}_q!}(x\theta_1)^n(y\theta_2)^n\sum_{r=0}^{\infty}\frac{(-1)^r}{\{r\}_q!}x^{\alpha+r-n+1}$$
$$\times \sum_{s=0}^{\infty}\frac{(-1)^s}{\{s\}_q!}y^{\beta+s-n+1} = E_{\frac{1}{q}}(x)E_{\frac{1}{q}}(y)x^{-\alpha-1}y^{-\beta-1}$$
$$\times \sum_{n=0}^{\infty}\frac{t^n}{\{n\}_q!}\sum_{r=0}^{\infty}\frac{(-1)^r}{\{r\}_q!}\{r+\alpha-n+1\}_{n,q}x^{\alpha+r+1} \tag{6.16}$$
$$\times \sum_{s=0}^{\infty}\frac{(-1)^s}{\{s\}_q!}\{s+\beta-n+1\}_{n,q}y^{\beta+s+1} = E_{\frac{1}{q}}(x)E_{\frac{1}{q}}(y)\sum_{r,s=0}^{\infty}\frac{(-1)^{r+s}x^r y^s}{\{r\}_q!\{s\}_q!}$$
$$\times \sum_{n=0}^{\infty}\frac{\langle -r-\alpha,-s-\beta;q\rangle_n}{\langle 1;q\rangle_n(1-q)^n}q^{-2\binom{n}{2}+n(\alpha+r+\beta+s)}t^n = \text{RHS}. \blacksquare$$

By the same method, we can find a q-analogue of a bilinear generating formula for Laguerre polynomials of Chatterjea [32, p. 88].

Theorem 6.7.

$$\sum_{n=0}^{\infty}\frac{\langle 1,\gamma;q\rangle_n(xyt)^n}{\langle \alpha+1,\beta+1;q\rangle_n}L_{n,q}^{(\alpha)}(x)L_{n,q}^{(\beta)}(y) = E_{\frac{1}{q}}(x)E_{\frac{1}{q}}(y)$$
$$\times \sum_{r,s=0}^{\infty}\frac{(-1)^{r+s}x^r y^s}{\{r\}_q!\{s\}_q!}\,{}_3\phi_2(\gamma,\alpha+r+1,\beta+s+1;\alpha+1,\beta+1|q,xyt). \tag{6.17}$$

Proof.

$$\text{LHS} = \sum_{n=0}^{\infty}\frac{\langle 1,\gamma;q\rangle_n}{\langle \alpha+1,\beta+1;q\rangle_n}\frac{x^{-\alpha-1}}{(\{n\}_q!)^2}E_{\frac{1}{q}}(x)(x^2D_{q,x})^n x^{\alpha+1}E_q(-x)$$
$$\times y^{-\beta-1}E_{\frac{1}{q}}(y)(y^2D_{q,y})^n y^{\beta+1}E_q(-y)t^n = E_{\frac{1}{q}}(x)E_{\frac{1}{q}}(y)x^{-\alpha-1}y^{-\beta-1}$$
$$\times\sum_{n=0}^{\infty}\frac{\langle 1,\gamma;q\rangle_n t^n}{\langle \alpha+1,\beta+1;q\rangle_n(\{n\}_q!)^2}(x\theta_1)^n\sum_{r=0}^{\infty}\frac{(-1)^r}{\{r\}_q!}x^{\alpha+r+1}(y\theta_2)^n\sum_{s=0}^{\infty}\frac{(-1)^s}{\{s\}_q!}y^{\beta+s+1}$$
$$= E_{\frac{1}{q}}(x)E_{\frac{1}{q}}(y)\sum_{n=0}^{\infty}\frac{\langle 1,\gamma;q\rangle_n t^n}{\langle \alpha+1,\beta+1;q\rangle_n(\{n\}_q!)^2}\sum_{r,s=0}^{\infty}\frac{(-1)^{r+s}x^{n+r}y^{n+s}}{\{r\}_q!\{s\}_q!}$$
$$\times\{r+\alpha+1\}_{n,q}\{s+\beta+1\}_{n,q} = E_{\frac{1}{q}}(x)E_{\frac{1}{q}}(y)\sum_{r,s=0}^{\infty}\frac{(-1)^{r+s}x^r y^s}{\{r\}_q!\{s\}_q!} \tag{6.18}$$
$$\times\sum_{n=0}^{\infty}\frac{\langle \gamma,\alpha+r+1,\beta+s+1;q\rangle_n}{\langle 1,\alpha+1,\beta+1;q\rangle_n}(xyt)^n = \text{RHS}. \blacksquare$$

Put $\gamma = \beta+1$ in (6.17) to obtain

Theorem 6.8.

$$\sum_{n=0}^{\infty}\frac{\langle 1;q\rangle_n t^n}{\langle\alpha+1;q\rangle_n}L_{n,q}^{(\alpha)}(x)L_{n,q}^{(\beta)}(y)=E_{\frac{1}{q}}(y)\frac{1}{(t;q)_{\beta+1}}\sum_{s=0}^{\infty}\frac{(-y)^s}{\{s\}_q!(tq^{\beta+1};q)_s}$$
$$\times\,{}_1\phi_2\left(\beta+s+1;\alpha+1|q,-xt(1-q)q^{1+\alpha}||-;tq^{\beta+s+1}\right). \tag{6.19}$$

Proof.

$$\sum_{n=0}^{\infty}\frac{\langle 1;q\rangle_n t^n}{\langle\alpha+1;q\rangle_n}L_{n,q}^{(\alpha)}(x)L_{n,q}^{(\beta)}(y)=E_{\frac{1}{q}}(x)E_{\frac{1}{q}}(y)\sum_{r,s=0}^{\infty}\frac{(-1)^{r+s}x^r y^s}{\{r\}_q!\{s\}_q!}$$
$$\times\,{}_2\phi_1(\alpha+r+1,\beta+s+1;\alpha+1|q,t)=E_{\frac{1}{q}}(x)E_{\frac{1}{q}}(y)\sum_{r,s=0}^{\infty}\frac{(-1)^{r+s}x^r y^s}{\{r\}_q!\{s\}_q!}$$
$$\times\frac{1}{(t;q)_{\beta+s+1}}{}_2\phi_2(\beta+s+1,-r;\alpha+1|q,tq^{\alpha+r+1}||-;tq^{\beta+s+1})$$
$$=E_{\frac{1}{q}}(x)E_{\frac{1}{q}}(y)\frac{1}{(t;q)_{\beta+1}}\sum_{s=0}^{\infty}\frac{(-y)^s}{\{s\}_q!(tq^{\beta+1};q)_s}\sum_{r=0}^{\infty}\frac{(-x)^r}{\{r\}_q!}$$
$$\times\,{}_2\phi_2(\beta+s+1,-r;\alpha+1|q,tq^{\alpha+r+1}||-;tq^{\beta+s+1})=E_{\frac{1}{q}}(x)E_{\frac{1}{q}}(y)$$
$$\times\frac{1}{(t;q)_{\beta+1}}\sum_{s=0}^{\infty}\frac{(-y)^s}{\{s\}_q!(tq^{\beta+1};q)_s}\sum_{r=0}^{\infty}\frac{(-x)^r}{\{r\}_q!}\sum_{k=0}^{r}\frac{\langle\beta+s+1,-r;q\rangle_k}{\langle 1,\alpha+1;q\rangle_k}$$
$$\times\frac{(-t)^k q^{\binom{k}{2}+k(\alpha+r+1)}}{(tq^{\beta+s+1};q)_k}=E_{\frac{1}{q}}(y)\frac{1}{(t;q)_{\beta+1}}\sum_{s=0}^{\infty}\frac{(-y)^s}{\{s\}_q!(tq^{\beta+1};q)_s} \tag{6.20}$$
$$\times\sum_{k=0}^{\infty}\frac{\langle\beta+s+1;q\rangle_k(-1)^k}{\langle 1,\alpha+1;q\rangle_k}\frac{(xt)^k(1-q)^k q^{k^2+k\alpha}}{(tq^{\beta+s+1};q)_k}=\text{RHS}. \qquad\blacksquare$$

Put $\beta=\alpha$ and $\gamma=\alpha+1$ in (6.17) to obtain the following q-analogue of the Hardy–Hille formula

Theorem 6.9.

$$\sum_{n=0}^{\infty}\frac{\langle 1;q\rangle_n t^n}{\langle\alpha+1;q\rangle_n}L_{n,q}^{(\alpha)}(x)L_{n,q}^{(\alpha)}(y)$$
$$=\frac{E_{\frac{1}{q}}(x)E_{\frac{1}{q}}(y)}{(t;q)_{\alpha+1}}\sum_{s,r,k=0}^{\infty}\frac{(-y)^s}{\{s\}_q!}\frac{(-x)^r}{\{r\}_q!}\frac{(1-q)^{2k}(xyt)^k q^{\alpha k+k^2}}{\langle 1,\alpha+1;q\rangle_k(tq^{\alpha+1};q)_{r+2k+s}}. \tag{6.21}$$

Proof.

$$\sum_{n=0}^{\infty}\frac{\langle 1;q\rangle_n t^n}{\langle\alpha+1;q\rangle_n}L_{n,q}^{(\alpha)}(x)L_{n,q}^{(\alpha)}(y)$$
$$=E_{\frac{1}{q}}(x)E_{\frac{1}{q}}(y)\sum_{r,s=0}^{\infty}\frac{(-1)^{r+s}x^r y^s}{\{r\}_q!\{s\}_q!}{}_2\phi_1(\alpha+r+1,\alpha+s+1;\alpha+1,|q,t)$$

$$
\begin{aligned}
&= E_{\frac{1}{q}}(x)E_{\frac{1}{q}}(y)\sum_{r,s=0}^{\infty}\frac{(-1)^{r+s}x^r y^s}{\{r\}_q!\{s\}_q!}\frac{}{(t;q)_{\alpha+r+s+1}}\\
&\times {}_2\phi_1(-r,-s;\alpha+1,|q,tq^{\alpha+r+s+1}) = \frac{E_{\frac{1}{q}}(x)E_{\frac{1}{q}}(y)}{(t;q)_{\alpha+1}}\sum_{r=0}^{\infty}\frac{(-x)^r}{\{r\}_q!(tq^{\alpha+1};q)_r}\\
&\times \sum_{s=0}^{\infty}\frac{(-y)^s}{\{s\}_q!(tq^{\alpha+1+r};q)_s}{}_2\phi_1(-r,-s;\alpha+1,|q,tq^{\alpha+r+s+1})\\
&= \frac{E_{\frac{1}{q}}(x)E_{\frac{1}{q}}(y)}{(t;q)_{\alpha+1}}\sum_{r=0}^{\infty}\frac{(-x)^r}{\{r\}_q!(tq^{\alpha+1};q)_r}\sum_{s,k=0}^{\infty}\frac{(-y)^{s+k}}{\{s+k\}_q!(tq^{\alpha+1+r};q)_{s+k}}\\
&\times \frac{\langle -s-k,-r;q\rangle_k}{\langle 1,\alpha+1;q\rangle_k}t^k q^{(\alpha+r+s+1)k+k^2} = \frac{E_{\frac{1}{q}}(x)E_{\frac{1}{q}}(y)}{(t;q)_{\alpha+1}}\sum_{r=0}^{\infty}\frac{(-x)^r}{\{r\}_q!(tq^{\alpha+1};q)_r}\\
&\times \sum_{s,k=0}^{\infty}\frac{(-y)^s(yt)^k(1-q)^{s+k}\langle -r;q\rangle_k q^{(\alpha+r)k+\frac{k^2}{2}+\frac{k}{2}}}{\langle 1;q\rangle_s\langle 1,\alpha+1;q\rangle_k(tq^{\alpha+1+r};q)_{s+k}} \qquad (6.22)\\
&= \frac{E_{\frac{1}{q}}(x)E_{\frac{1}{q}}(y)}{(t;q)_{\alpha+1}}\sum_{s,r,k=0}^{\infty}\frac{(-y)^s}{\{s\}_q!}\frac{(-x)^r}{\{r\}_q!}\frac{(1-q)^{2k}(xyt)^k q^{\alpha k+k^2}}{\langle 1,\alpha+1;q\rangle_k(tq^{\alpha+1};q)_{r+2k+s}}. \qquad \blacksquare
\end{aligned}
$$

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